Fourth International Conference on
Geometry, Integrability and Quantization
June 6–15, 2002, Varna, Bulgaria
Ivaïlo M. Mladenov and Gregory L. Naber, Editors
Coral Press, Sofia 2003, pp 235–238

THE CHARGED HEAVY TOP AND THE DIRAC PROBLEM

IONEL MOŞ, SILVANA CHIRICI and MIRCEA PUTA

Seminarul de Geometrie–Topologie, West University of Timişoara
B-dul V. Pârvan no 4, 1900 Timişoara, Romania

Abstract. The Dirac problem for the charged heavy top is discussed and some of its properties are pointed out.

1. Introduction

The notion of charged heavy top was recently introduced by Thiffeault and Morrison [5] and then was recently studied by Puta and Caşu [4]. The goal of our paper is to point out some new properties of this top from the geometric prequantization point of view.

2. The Charged Top

A charged heavy top is by definition a heavy top immersed in a fixed gravitational field and in an electric field. We shall denote by

$$\begin{aligned} m &= (m_1, m_2, m_3) \\ a &= (a_1, a_2, a_3) \\ b &= (b_1, b_2, b_3) \end{aligned}$$

the angular momentum vector, the position of the center of mass, and the position of the center of charge, respectively. The direction and strength of the fixed gravitational and electric forces are given by the vectors

$$\alpha = (\alpha_1, \alpha_2, \alpha_3)$$

and respectively

$$\beta = (\beta_1, \beta_2, \beta_3)\,.$$

The energy of such a top has the following expression:

$$H(m,\alpha,\beta) = \frac{1}{2}\left(\frac{m_1^2}{I_1}+\frac{m_2^2}{I_2}+\frac{m_3^2}{I_3}\right) + \alpha_1 a_1 + \alpha_2 a_2 + \alpha_3 a_3 + \beta_1 b_1 + \beta_2 b_2 + \beta_3 b_3 \tag{1}$$

where I_1, I_2, I_3 are the components of the inertia tensor and we shall suppose as usually that:

$$I_1 > I_2 > I_3 > 0\,.$$

Then the equations of motion have the following form:

$$\begin{aligned}
\dot m_1 &= \left(\frac{1}{I_3}-\frac{1}{I_2}\right)m_2m_3 - \alpha_3a_2 + \alpha_2a_3 - \beta_3b_2 + \beta_2b_3\\
\dot m_2 &= \left(\frac{1}{I_1}-\frac{1}{I_3}\right)m_1m_3 + \alpha_3a_1 - \alpha_1a_3 + \beta_3b_1 - \beta_1b_3\\
\dot m_3 &= \left(\frac{1}{I_2}-\frac{1}{I_1}\right)m_1m_2 - \alpha_2a_1 + \alpha_1a_2 - \beta_2b_1 + \beta_1b_2\\
\dot\alpha_1 &= -\frac{m_2a_3}{I_2} + \frac{m_3a_2}{I_3}\\
\dot\alpha_2 &= \frac{m_1\alpha_3}{I_1} - \frac{m_3\alpha_1}{I_3}\\
\dot\alpha_3 &= -\frac{m_1\alpha_2}{I_1} + \frac{m_2\alpha_1}{I_2}\\
\dot\beta_1 &= -\frac{m_2\beta_3}{I_2} + \frac{m_3\beta_2}{I_3}\\
\dot\beta_2 &= \frac{m_1\beta_3}{I_1} - \frac{m_3\beta_1}{I_3}\\
\dot\beta_3 &= -\frac{m_1\beta_2}{I_1} + \frac{m_2\beta_1}{I_2}\,.
\end{aligned} \tag{2}$$

Let G be the semi-direct product of $SO(3)$ and $\mathbb{R}^3\times\mathbb{R}^3$, $\mathcal{G}$ its Lie algebra and $\mathcal{G}^*\simeq\mathbb{R}^9$ the dual of its Lie algebra.

Theorem 1. ([5]) *The system* (2) has the following Hamilton–Poisson realization:

$$(\mathcal{G}^*\simeq\mathbb{R}^9, \Pi_-, H)$$

where Π_- is the minus-Lie–Poisson structure on $\mathcal{G}^*$ given by the matrix

$$\Pi_- = \begin{pmatrix} 0 & -m_3 & m_2 & 0 & -\alpha_3 & \alpha_2 & 0 & -\beta_3 & \beta_2 \\ m_3 & 0 & -m_1 & \alpha_3 & 0 & -\alpha_1 & \beta_3 & 0 & -\beta_1 \\ -m_2 & m_1 & 0 & -\alpha_2 & \alpha_1 & 0 & -\beta_2 & \beta_1 & 0 \\ 0 & -\alpha_3 & \alpha_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ \alpha_3 & 0 & -\alpha_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\beta_3 & \beta_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ \beta_3 & 0 & -\beta_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\beta_2 & \beta_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and the Hamiltonian H given by (1).

3. Dirac Problem

Let us consider the following diagram:

$$\begin{bmatrix} (M,\omega) \\ \text{respectively} \\ (P,\{\cdot,\cdot\}) \end{bmatrix} \rightarrow \begin{pmatrix} \mathcal{H} \\ \delta \end{pmatrix}$$

where in the left hand side (M,ω) [respectively $(P,\{\cdot,\cdot\})$] is a symplectic (respectively Poisson) manifold and in the right hand $\mathcal{H}$ is a Hilbert space and δ is a map which assigns to each $f \in C^\infty(M,\mathbb{R})$ [respectively $f \in C^\infty(P,\mathbb{R})$] a self adjoint operator

$$\delta_f : \mathcal{H} \rightarrow \mathcal{H}\,.$$

The arrow from left to right is called **prequantization**, i. e. a procedure to derive from classical data (M,ω) [respectively $(P,\{\cdot,\cdot\})$] the quantum data $(\mathcal{H},\delta)$ such that the following conditions called Dirac conditions to be satisfied:

(D_1) $\delta_{f+g} = \delta_f + \delta_g$
(D_2) $\delta_{\lambda f} = \lambda \cdot \delta_f$
(D_3) $\delta_{\mathrm{id}_M} = \mathrm{Id}_{\mathcal{H}}$
(D_4) $[\delta_f, \delta_g] = \mathrm{i}\hbar\delta_{\{f,g\}_\omega}$

for each $f, g \in C^\infty(M,\mathbb{R})$ and for each $\lambda \in \mathbb{R}$ respectively

(D_1) $\delta_{f+g} = \delta_f + \delta_g$
(D_2) $\delta_{\lambda f} = \lambda \cdot \delta_f$
(D_3) $\delta_{\mathrm{id}_p} = \mathrm{Id}_{\mathcal{H}}$
(D_4) $[\delta_f, \delta_g] = \mathrm{i}\hbar\delta_{\{f,g\}}$

for each $f, g \in C^\infty(P,\mathbb{R})$ and for each $\lambda \in \mathbb{R}$.
The existence of such a prequatization is usually called **Dirac problem**.

4. The Charged Heavy Top and the Dirac Problem

Let

$$R\colon G \times G \to G$$

be the action of G on itself by right translations and R^{T^*} its lift to T^*G. This action has the momentum map

$$J : T^*G \to \mathcal{G}^*$$

given by

$$(J(\alpha_g))(\xi) = \alpha_g(TL_g(\xi))$$

which is a Poisson map, see [2].

On the other hand, it is well known that $(T^*G, \omega = \mathrm{d}\theta)$ is a quantizable manifold and let δ^ω be its prequantum operator, see [3]. If we take now:

$$\mathcal{H} = L^2(T^*G, \mathbb{C}), \qquad \delta_f = \delta^\omega_{f \circ J}$$

for each $f \in C^\infty(\mathcal{G}^*, \mathbb{R})$, then an easy computation leads us to

Theorem 2. *The pair $(\mathcal{H}, \delta)$ gives rise to a prequantization of the Poisson manifold*

$$(\mathcal{G}^*, \Pi_-) .$$

Using now the same technique as in Chernoff [1] with obvious modifications we can prove:

Theorem 3. *Let $\mathcal{O}(L^2(T^*G, \mathbb{C}))$ be the space of self-adjoint operators on the Hilbert space $L^2(T^*G, \mathbb{C})$. Then the map*

$$f \in C^\infty(\mathcal{G}^*, \mathbb{R}) \mapsto \delta_f \in \mathcal{O}(L^2(T^*G, \mathbb{C}))$$

gives rise to an irreducible representation of $C^\infty(\mathcal{G}^*, \mathbb{R})$ onto the space

$$\mathcal{O}(L^2(T^*G, \mathbb{C})) .$$

References

[1] Chernoff P., *Irreducible Representation of Infinite Dimensional Groups and Lie Algebras I*, J. of Funct. Analysis **130** (1995) 225–282.

[2] Marsden J. and Raţiu T., *Introduction to Mechanics and Symmetry*, Vol. 17, Springer 1994.

[3] Puta M., *Hamiltonian Mechanical Systems and Geometric Quantization*, Mathematics and Applications, Vol. 260, Kluwer 1993.

[4] Puta M. and Caşu I., *Some Remarks on the Charged Top* (to appear).

[5] Thiffeault J. and Morrison P., *The Twisted Top*, Physics Letters A **283** (2001) 335–341.