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TOPOLOGICAL PROPERTIES OF SOME COHOMOGENEITY ON RIEMANNIAN MANIFOLDS OF NONPOSITIVE CURVATURE

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Abstract. In this paper we study some non-positively curved Riemannian manifolds acted on by a Lie group of isometries with principal orbits of codimension one. Among other results it is proved that if the universal covering manifold satisfies some conditions then every non-exceptional singular orbit is a totally geodesic submanifold. When M is flat and is not toruslike, it is proved that either each orbit is isometric to $\mathbb{R}^k \times \mathbb{T}^m$ or there is a singular orbit. If the singular orbit is uniqe and non-exceptional, then it is isometric to $\mathbb{R}^k \times \mathbb{T}^m$.

1. Introduction

Recently, cohomogeneity one Riemannian manifolds have been studied from different points of view. A. Alekseevsky and D. Alekseevsky in [1] and [2] gave a description of such manifolds in terms of Lie subgroups of a Lie group G, Podesta and Spiro in [13] got some nice results in negatively curved case, Searle in [14] provided a complete classification of such manifolds in dimensions less than 6 when they are compact and of positive curvature. The aim of this paper is to deal with some non-positively curved cohomogeneity one Riemannian manifolds. We generalize some of the theorems of [13] to the case where M is a product of negatively curved manifolds. Also in Section 4 we study some cohomogeneity one flat Riemannian manifolds. Our main results are Theorems 3.5, 3.7, 3.10, and 4.4.

2. Preliminaries

Definition 2.0. *Let M be a complete Riemannian manifold and G a Lie group of isometries which is closed in the full group of isometries of M. We say* *that M is of cohomogeneity one under the action of G if G has an orbit of codimension one.*

It is known (see [1] and [4, 11]) that the orbit space $\Omega = M/G$ is a topological Hausdorff space homeomorphic to one of the following spaces: $\mathbb{R}$, S^1, $\mathbb{R}^+ = [0, +\infty)$ and $[0, 1]$. In the following we will indicate by $k : M \to \Omega$ the projection to the orbit space. Given a point $x \in M$, the orbit $D = Gx$ is called **principal** (resp. **singular**) if the corresponding image in the orbit space is an **internal** (resp. **boundary**) point of Ω, and the point x is called a **regular** (resp. **singular**) point. We say that a singular orbit is exceptional if it has codimension one. Also note that the principal orbits are diffeomorphic to each other and M is diffeomorphic to $\Omega \times D$ if $M/G = R$.
If G_p is the isotropy subgroup of G at $p, (p \in M)$, then G_x and G_y are conjugate if both x, y are regular, while G_x is conjugate to a subgroup of G_y if x is regular and y is singular.

Definition 2.1. *A (complete) geodesic γ on a Riemannian manifold of cohomogeneity one is called a normal geodesic if it crosses each orbit orthogonally.*

We know (see [2]) that a geodesic γ is a normal geodesic if and only if it is orthogonal to each orbit Gx at one point $x \in \gamma$, and that each regular point belongs to a unique normal geodesic.

Definition 2.2. *A differentiable real valued function F on a complete Riemannian manifold M is said to be convex (resp. strictly convex) if for each geodesic $\gamma\colon \mathbb{R} \to M$ the composed function $F \circ \gamma\colon \mathbb{R} \to \mathbb{R}$ is convex (resp. strictly convex), that is $(F \circ \gamma)'' \geq 0$ (resp. $(F \circ \gamma)'' > 0$).*

Let φ be an isometry of a simply connected Riemannian manifold M, the squared **displacement function** of φ is the function defined by $\mathrm{d}^2_\varphi(p) = \mathrm{d}^2(p, \varphi(p)), p \in M$, where d denotes the distance on M.
In the next proposition we list some known properties of cohomogeneity one Riemannian manifolds, which we will use in the sequel.

Proposition 2.3. ([4], [8] and [13]) *Let M be a cohomogeneity one Riemannian manifold under the action of a connected Lie group G which is closed in the full isometry goup of M, then*

a) *If M is simply connected with nonpositive curvature, there is at most one singular orbit;*
b) *If M has nonpositive curvature and B is the unique singular orbit of M, $\pi_1(M) = \pi_1(B)$;*
c) *If M is simply connected no exceptional orbit may exist;*

d) *If M is simply connected and without singular orbit then $\Omega \neq S^1$, i. e. $\Omega = \mathbb{R}$;*

e) *No exceptional orbit is simply connected;*

f) *If γ is a normal geodesic then the map $k\colon \gamma \to \Omega$ is surjective and it defines a covering over the set Ω^0 of internal points of Ω. When $\Omega = \mathbb{R}^+$ or $\mathbb{R}$, we can endow Ω with the metric given by the covering k.*

The following proposition and theorems will be needed later.

Proposition 2.4. (see [3]) *Let M be a simply connected and complete Riemannian manifold of nonpositive curvature, then*

a) *If the minimum point set C of a real valued convex function F defined on M is a submanifold of M then C is totally geodesic in M, and each critical point of F belongs to C;*

b) d^2_φ *is a convex function for each isometry φ of M and if M has negative curvature it is strictly convex except at the minimum point set C which is at most the image of a geodesic.*

Theorem 2.5. ([15]) *Let M be a connected homogeneous Riemannian manifold with nonpositive curvature, then M is diffeomorphic to the product of a torus and a Euclidean space.*

Theorem 2.6. ([9]) *Let M be a homogeneous Riemannian manifold with nonpositive curvature and negative definite Ricci tensor then M is simply connected.*

3. Cohomogeneity on UND Manifolds

Throught the following M will denote a complete Riemannian manifold of dimension n with nonpositive curvature and of cohomogeneity one under the action of G, a connected Lie group which is closed in the full group of isometries of M. If M is not simply connected then $\tilde{M}$ will denote the universal Riemannian covering manifold of M endowed with the pulled back metric and $\pi\colon \tilde{M} \to M$ will be the covering projection, with the symbol Δ we will denote the deck transformation group of the universal covering of M. We know (see [4] page 63) that the group G always admits a connected covering group $\tilde{G}$ which acts on $\tilde{M}$ by isometries and of cohomogeniety one, the projection $\tilde{\pi}\colon \tilde{G} \to G$ is such that $\tilde{\pi}(\tilde{g})(x) = \pi(\tilde{g}(y))$ for all $\tilde{g} \in \tilde{G}$, $x \in M$ and $y \in \pi^{-1}(x)$. Moreover Δ centeralizes $\tilde{G}$ so that it maps $\tilde{G}$-orbits onto $\tilde{G}$-orbits, so for each $\varphi \in \Delta$, d^2_φ is constant along orbits.

Definition 3.0. *We say that a Riemannian manifold M is universally and negatively decomposable (UND) when its universal covering manifold $\tilde{M}$ decomposes as $\tilde{M} = \tilde{M}_1 \times \tilde{M}_2 \times \cdots \times \tilde{M}_k$ and for each i, $\tilde{M}_i$ has negative curvature and each $\varphi \in \Delta$ decomposes as $\varphi = \varphi_1 \times \varphi_2 \times \cdots \times \varphi_k$ where φ_i is an isometry of $\tilde{M}_i$.*

Lemma 3.1. *If $M = M_1 \times M_2$ is a complete simply connected Riemannian manifold of nonpositive curvature such that for a geodesic $\gamma(t) = (\gamma_1(t), \gamma_2(t))$ and for an isometry $\varphi = \varphi_1 \times \varphi_2$, $\mathrm{d}^2_{1\varphi_1} \circ \gamma_1 \colon \mathbb{R} \to \mathbb{R}$ is strictly convex, then $\mathrm{d}^2_{\varphi} \circ \gamma \colon \mathbb{R} \to \mathbb{R}$ is a strictly convex function.*

Lemma 3.2. *If $\varphi \in \Delta$ is nontrivial and for a normal geodesic γ, $\mathrm{d}^2_{\varphi} \circ \gamma \colon \mathbb{R} \to \mathbb{R}$ does not have any minimum point then, φ maps each orbit $\tilde{D}$ onto itself.*

Lemma 3.3. *Let M be a UND cohomogeneity one Riemannian manifold and let $\varphi \in \Delta$ be nontrivial, then there exists a normal geodesic γ on $\tilde{M}$ such that $\mathrm{d}^2_{\varphi} \circ \gamma \colon \mathbb{R} \to \mathbb{R}$ is a strictly convex function.*

Lemma 3.4. *Let γ be a normal geodesic in $\tilde{M}$ and $\varphi \in \Delta$ be such that $\mathrm{d}^2_{\varphi} \circ \gamma \colon \mathbb{R} \to \mathbb{R}$ is strictly convex and $t_1 \in \mathbb{R}$ is not a minimum point of the function $F(t) = \mathrm{d}^2_{\varphi} \circ \gamma(t)$, then the orbit $\tilde{B} = \tilde{G}\gamma(t_1)$ is a hypersurface in $\tilde{M}$.*

Theorem 3.5. *If M is a non-simply connected UND cohomogeneity one Riemannian manifold with only one singular orbit B, and B is not exceptional, then it is a totally geodesic submanifold of M diffeomorphic to $\mathbb{R}^k \times \mathbb{T}^m$ and $\pi_1(M) = \mathbb{Z}^m$.*

Proof: First note that since $\dim \pi^{-1}(B) = \dim B < n-1$, each component of $\pi^{-1}(B)$ must be a non-exceptional singular orbit in $\tilde{M}$. Therefore by 2.3(a), $\pi^{-1}(B)$ has only one component $\tilde{B}$. Now let $\varphi \in \Delta$ be a nontrivial deck transformation and γ a normal geodesic in $\tilde{M}$ such that $F = \mathrm{d}^2_{\varphi} \circ \gamma \colon \mathbb{R} \to \mathbb{R}$ is a strictly convex function (see 3.3), then we have two cases.

Case 1: F has only one minimum point $t_0 \in \mathbb{R}$.

In this case since d^2_{φ} is constant along orbits, we get that $\tilde{G}\gamma(t_0)$ is the minimum point set of d^2_{φ}, so by 2.4(a) it is a totally geodesic submanifold of $\tilde{M}$. We show that $\tilde{B} = \tilde{G}.\gamma(t_0)$. If not, then $\tilde{B} = \tilde{G}\gamma(t_1)$, $t_1 \neq t_0$, so by 3.4 $\tilde{B}$ must be a hypersurface in $\tilde{M}$, since $\dim \tilde{B} < n-1$ this is a contradiction, therefore $\tilde{B} = \tilde{G}\gamma(t_0)$ and $\tilde{B}$ is a totally geodsic submanifold of $\tilde{M}$. Consequently $B = \pi(\tilde{B})$ is totally geodesic in M, so is of nonpositive curvature. Since B is homogeneous we get by 2.5 that B is diffeomorphic to $\mathbb{R}^k \times \mathbb{T}^m$ and by 2.3(b) we have $\pi_1(M) = \pi_1(B) = \mathbb{Z}^m$.

Case 2: F has not any minimum point.

This case can not occur because by 3.4 each orbit of $\tilde{M}$ must be a hypersurface, so $\tilde{B}$ is a hypersurface, which is in contrast with the fact $\dim \tilde{B} < n-1$. □

Lemma 3.6. *If for each deck transformation $\varphi \in \Delta$ and each orbit $\tilde{D}$ in $\tilde{M}$, φ maps $\tilde{D}$ onto itself and if there is no singular orbit in M, then each orbit D in M is diffeomorphic to $\mathbb{R}^{k_1} \times \mathbb{T}^{m_1}$.*

Proof: The proof of this lemma in given in a portion of the proof of Theorem 3.7 in [4] and the sketch of the proof is as follows: for an orbit D in M, $\pi^{-1}(D)$ has only one component $\tilde{D}$ and $\tilde{D} = \tilde{G}/\tilde{K}$ with $\tilde{K}$ maximal compact in $\tilde{G}$. So there is a solvable subgroup H acting transitively on $\tilde{D}$. Since $D = \tilde{D}/\Delta$ and Δ centeralizes $\tilde{G}$ (and hence H too). we obtain that H acts transitively on D, so D is a solvmanifold and diffeomorphic to a product $\mathbb{R}^{k_1} \times \mathbb{T}^{m_1}$ (see [13], p. 76 and [16]). □

Theorem 3.7. *If M is a non-simply connected UND cohomogeneity one Riemannian manifold without any singular orbit, then each orbit is diffeomorphic to $\mathbb{R}^{k_1} \times \mathbb{T}^{m_1}$. In this case if $M/G = \mathbb{R}$, then M is diffeomorphic to $\mathbb{R}^{k} \times \mathbb{T}^{m}$, $k = k_1 + 1$.*

Proof: By 3.3 for each nontrivial $\varphi \in \Delta$, there is a normal geodesic γ (related to φ) such that $\mathrm{d}^2_\varphi \circ \gamma$ is a strictly convex function. We have two cases.

Case 1: There exists a $\varphi \in \Delta$ such that $\mathrm{d}^2_\varphi \circ \gamma$ has a minimum point $t_0 \in \mathbb{R}$.

In this case the orbit $\tilde{B} = \tilde{G}\gamma(t_0)$ is the minimum point set of the function d^2_φ. Therefore by 2.4(a) it is totally geodesic and so $B = \pi(\tilde{B})$ is totally geodesic in M, hence is of nonpositive curvature. Since B is homogeneous, it is diffeomorphic to $\mathbb{R}^{k_1} \times \mathbb{T}^{m_1}$ by 2.5. From the fact that the (principal) orbits are diffeomorphic we get that each orbit is diffeomorphic to $\mathbb{R}^{k_1} \times \mathbb{T}^{m_1}$.

Case 2: For each nontrivial $\varphi \in \Delta$, $\mathrm{d}^2_\varphi \circ \gamma$ does not have any minimum point.

In this case by 3.2, φ maps each orbit $\tilde{D}$ onto itself. Therefore by 3.6 each orbit D in M is diffeomorphic to $\mathbb{R}^{k_1} \times \mathbb{T}^{m_1}$.

If $M/G = \mathbb{R}$, from the fact that M is diffeomorphic to $M/G \times D$ we get that M is diffeomorphic to $\mathbb{R} \times \mathbb{R}^{k_1} \times \mathbb{T}^{m_1} = \mathbb{R}^{k} \times \mathbb{T}^{m_1}$. □

Lemma 3.8. *Let $M = M_1 \times M_2$ and $X = X_1 + X_2$, Z be two vectors at the point $p = (p_1, p_2)$ such that X_1, Z are tangent to M_1 and X_2 is tangent to M_2, then $K_M(X, Z) = K_{M_1}(X_1, Z)$.*

Lemma 3.9. *If $\tilde{M} = \tilde{M}_1 \times \tilde{M}_2 \times \cdots \times \tilde{M}_k$, where for each i, $\tilde{M}_i$ is negatively curved with $\dim \tilde{M}_i \geq 3$, then each totally geodesic hypersurface S of $\tilde{M}$ has negative definite Ricci tensor.*

Theorem 3.10. *If M is a nonsimply connected UND cohomogeneity one Riemannian manifold and $\tilde{M} = \tilde{M}_1 \times \tilde{M}_2 \times \cdots \times \tilde{M}_k$, where for each i, $\dim \tilde{M}_i \geq 3$, then*

a) *There is at most one singular orbit;*
b) *If there is a singular orbit B, it is non-exceptional and diffeomorphic to $\mathbb{R}^{K_1} \times \mathbb{T}^{m_1}$ and $\pi_1(M) = \mathbb{Z}^{m_1}$.*

Proof: We prove the theorem in two steps.

Step 1: M does not have two exceptional singular orbits.

If M has two exceptional singualr orbits, then the dimension of each orbit of M (and so the dimension of each orbit of $\tilde{M}$) is $n-1$, so by 2.3(c, e), $\tilde{M}$ does not have any singualr orbit and $\tilde{M}/\tilde{G} = \mathbb{R}$. Therefore each normal geodesic γ in $\tilde{M}$ intersects an orbit $\tilde{D}$ exactly once. But since $M/G = [0,1]$, the normal geodesic $\pi \circ \gamma$ intersects a principal orbit D in M infinitely many times, so $\pi^{-1}(D)$ has more than one connected component. Therefore if $\tilde{D}$ is a component of $\pi^{-1}(D)$, there exist a nontrivial $\varphi \in \Delta$ such that $\varphi(\tilde{D}) \neq \tilde{D}$ thus by Lemmas 3.2, 3.3, for a normal geodesic γ, $\mathrm{d}^2_\varphi \circ \gamma$ is strictly convex with a minimum point $t_0 \in \mathbb{R}$, and since d^2_φ is constant along orbits, $\tilde{B} = \tilde{G}\gamma(t_0)$ is the minimum point set of d^2_φ. So it is totally geodesic by 2.4(a). Now since each factor of the decomposition of $\tilde{M}$ is negatively curved with $\dim \tilde{M}_i \geq 3$, we get by 3.9 that every totally geodesic hypersurface of $\tilde{M}$ has negative definite Ricci tensor, so $\tilde{B}$ (hence $B = \pi(\tilde{B})$) has negative definite Ricci tensor, thus by 2.6, B is simply connected. Since $\dim B = n-1$, we get by 2.3(d) that B is not a singular orbit. As B is simply connected, $B = G/K$ ($K = G_x$, $x \in B$), where K is maximal compact subgroup of G (see [10] , Vol II, p. 112), which is in contrast with the fact that there exists singular orbit.

Step 2: M does not have two singular orbits, at least one orbit non-exceptional.

Let B_1 be a non-exceptional singular orbit of M then $\tilde{B} = \pi^{-1}(B_1)$ is the unique singular orbirt of $\tilde{M}$. Because of dimensional reasons for each $\varphi \in \Delta$ we have $\varphi(\tilde{B}) = \tilde{B}$. The isometry φ induces an isometry φ^* on the orbis pace $\mathbb{R}^+$ of $\tilde{M}$ such that for each orbit $\tilde{D}$ we have $\varphi^*(k(\tilde{D}) = k(\varphi(\tilde{D}))$. Since $\varphi(\tilde{B}) = \tilde{B}$, we get that $\varphi^*(0) = \varphi^*(k(\tilde{B})) = k\varphi(\tilde{B}) = k(\tilde{B}) = 0$, so for each $t \in \mathbb{R}^+$ we have $\varphi^*(t) = t$. Thus $\varphi(\tilde{D}) = \tilde{D}$. Now we have a contradiction because a normal geodesic γ in $\tilde{M}$ intersects each principal orbit in two points ($\tilde{M}/\tilde{G} = \mathbb{R}^+$) while $\pi \circ \gamma$ intersects a principal orbit infinitely many times ($M/G = [0,1]$). So there exists $\varphi \in \Delta$ such that $\varphi(\tilde{D}) \neq \tilde{D}$.

We need only to show that B can not be an exceptional orbit, the other parts of the claim is a simple consequence of Theorem 3.5. To prove the claim observe that if it were the case, $\tilde{M}$ would admit only principal orbits and a normal geodesic intersects each orbit in $\tilde{M}$ exactly in one point while since

$M/G = \mathbb{R}^+$, a normal geodesic in M intersects each principal orbit in two points, and a contradiction arises as in the Step 1. □

4. Cohomogeneity One Flat Manifolds

In this section we study cohomogeneity one flat Riemannian manifolds which are not toruslike.

It is known that every isometry $\varphi \in \mathrm{Iso}(\mathbb{R}^n)$ is of the form $\varphi = (A, b)$, $A \in O(n)$, $b \in \mathbb{R}^n$ that is, $\varphi(x) = Ax + b$, $x \in \mathbb{R}^n$. We say that φ is an ordinary translation when $A = \mathrm{Id}$ (Id is the identity map on $\mathbb{R}^n$).

Note that $\mathbb{R}^n$ is the universal Riemannian covering manifold of each flat manifold M of dimension n.

Definition 4.1. *We say that a flat Riemannian manifold M is "toruslike" if each deck transformation of the universal covering manifold of M is an ordinary translation.*

In the following $V.W$ denotes the inner product of the vectors V and W in $\mathbb{R}^n$ and $|V|$ is the length of V.

Lemma 4.2. *Let $\mathbb{R}^n$ be of cohomogeneity one under the action of a closed Lie subgroup $G \subset \mathrm{Iso}(\mathbb{R}^n)$ and let $\varphi = (A, b) \in G$, $A \neq \mathrm{Id}$. Then there is a normal geodesic γ on $\mathbb{R}^n$ such that the function $F(t) = \mathrm{d}_\varphi^2 \circ \gamma(t)$ is a strictly convex function with the minimum point $t_0 \in R$.*

Lemma 4.3. *If $\mathbb{R}^n$ is of cohomogeneity one under the action of a closed Lie subgroup G of $Iso(\mathbb{R}^n)$ and if all the orbits are regular and one orbit is isometric to $\mathbb{R}^{n-1}$, then other orbits are isometric to $\mathbb{R}^{n-1}$.*

Theorem 4.4. *If M is a flat cohomogeneity one Riemannian manifold under the action of a closed Lie group $G \subset \mathrm{Iso}(M)$ and M is not torus-like, then*

a) *Either each orbit D of M is isometric to $\mathbb{R}^k \times \mathbb{T}^m$ for some $m, k, m + k = n - 1$, or there is a singular orbit B in M;*

b) *If there is a unique singular orbit B which is non-exceptional, then B is isometric to $\mathbb{R}^k \times \mathbb{T}^m$ for some m, k and $\pi_1(M) = \mathbb{Z}^m$.*

Proof: Let $\tilde{M} = \mathbb{R}^n$ be the universal covering manifold of M and let $\tilde{G}$ be the corresponding covering Lie group of G, which acts on $\tilde{M} = \mathbb{R}^n$ by cohomogeneity one.

(a): Since M is not toruslike there is a deck transformation φ such that $\varphi = (A, b), A \neq \mathrm{Id}$. By Lemma 4.2 there is a normal geodesic γ in $\tilde{M}$ such that the function $F(t) = \mathrm{d}_\varphi^2 \circ \gamma(t)$ is a strictly convex function with a minimum point t_0. Since d_φ^2 is constant along orbits we get that the orbit $\tilde{D}_0 = \tilde{G}\gamma(t_0)$ is the minimum point set of d_φ^2 Thus by 2.4(a) it is totally geodesic in $\tilde{M} = \mathbb{R}^n$, so it is flat and therefore isometric to $\mathbb{R}^r$ for some r. Now let there is not any singular orbit in M. So $\tilde{M} = \mathbb{R}^n$ does not have any singular orbit, therefore $r = n - 1$ and $\tilde{D}_0$ is isometric to $\mathbb{R}^{n-1}$, so by Lemma 4.3 we get that each orbit $\tilde{D}$ of $\tilde{M}$ is isometric to $\mathbb{R}^{n-1}$, therefore each orbit D $(= \pi(\tilde{D}))$ of M is flat, and since it is homogeneous we get by Theorem 2.5 that D is isometric to $\mathbb{R}^k \times \mathbb{T}^m$, for some m, k, $m + k = n - 1$. This proves the part (a).

(b): Let B be the unique non-exceptional singular orbit of M and $\tilde{B} = \pi^{-1}(B)$ and let $F(t)$ be the function obtained in the proof of part (a) with the minimum point t_0. For each $t \in R$ we have $\tilde{G}\gamma(t) = g^{-1}(F(t))$, where $g = \mathrm{d}_\varphi^2$. If c and b are regular values of g then $g^{-1}(c)$ and $g^{-1}(b)$ are diffeomorphic (see [3], p. 10, Corollary 3.11), from these facts we get that $\tilde{B} = g^{-1}(F(t_0))$ (because if not, then $\tilde{B} = g^{-1}(b)$ where b is a regular value of g, and so $\tilde{B}$ must be diffeomorphic to principal orbits which is a contradiction). So $\tilde{B}$ is the minimum point set of g and therefore by 2.4(a) it is totally geodesic in M and is flat, thus B is flat. Since it is homogeneous we get by 2.5 that B is diffeomorphic to $\mathbb{R}^k \times \mathbb{T}^m$ and by 2.3(b) we have $\pi_1(M) = \pi_1(B) = \mathbb{Z}^m$. $\square$

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