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ON THE REDUCTIONS AND SCATTERING DATA FOR THE CBC SYSTEM

GEORGI GRAHOVSKI

Institute for Nuclear Research and Nuclear Energy, Bulgarian Academy of Sciences
72 Tsarigradsko Chaussée Blvd, 1784 Sofia, Bulgaria

Abstract. The reductions for the first order linear systems of the type:

$$L\psi(x,\lambda) \equiv \left(\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}x} + q(x) - \lambda J\right)\psi(x,\lambda) = 0\,, \quad J \in \mathfrak{h}\,, \quad q(x) \in \mathfrak{g}_J$$

are studied. This system generalizes the Zakharov–Shabat system and the systems studied by Caudrey, Beals and Coifman (CBC systems). Here J is a regular complex constant element of the Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ of the simple Lie algebra $\mathfrak{g}$ and the potential $q(x)$ takes values in the image $\mathfrak{g}_J$ of ad_J. Special attention is paid to the scattering data of CBC systems and their behaviour under the Weyl group reductions. The analytical properties of the generating functional of the integrals of motion and their reduced analogs are studied. These results are demonstrated on an example of N-wave type equations.

1. Introduction

The idea that the inverse scattering method (ISM) is a generalized Fourier transform has appeared as early as 1974 in [1]. In the class of nonlinear evolution equations (NLEE) related to the Zakharov–Shabat (ZS) system [26, 24] related to $sl(2)$ algebra was studied. This class of NLEE contains such physically important equations as the nonlinear Schrödinger equation (NLS), the sin-Gordon and modified Korteweg–de Vries (mKdV) equations.

The multicomponent ZS system leads to such important systems like the multicomponent NLS, the N-wave type equations, etc.

The classical results of [26, 24] have been generalized in several directions.

One can consider systems with polynomial dependence on λ and $1/\lambda$:

$$L\Psi(x,t,\lambda) = \left(\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}x} + \sum_{k=-M}^{N-1} q_k(x,t)\lambda^k - \lambda^N \sigma_3 \right) \Psi(x,t,\lambda) = 0\,, \qquad (1)$$

with $N \geq 2$, $M \geq 0$. Since this is again 2×2 matrix-valued problem it is also related to the $sl(2)$ algebra.

Another possibility was to generalize to $n\times n$ system [5, 7, 10]:

$$L\Psi(x,t,\lambda) = \left(\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}x} + q(x,t) - \lambda J \right) \Psi(x,t,\lambda) \qquad (2)$$

where $q(x,t)$ and J take values in the semisimple Lie algebra $\mathfrak{g}$ [21, 13, 25, 11]:

$$\begin{gathered} q(x,t) = \sum_{\alpha\in\Delta_+} (q_\alpha(x,t)E_\alpha + q_{-\alpha}(x,t)E_{-\alpha}) \in \mathfrak{g}_J \\ J = \sum_{j=1}^{r} a_j H_j \in \mathfrak{h}\,. \end{gathered} \qquad (3)$$

Here J is a regular element in the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$, $\mathfrak{g}_J$ is the image of ad_J, $\{E_\alpha, H_i\}$ form the Cartan-Weyl basis in $\mathfrak{g}$, Δ_+ is the set of positive roots of the algerbra, $r = \operatorname{rank}\mathfrak{g} = \dim\mathfrak{h}$. For more details see the Section 2 below. The regularity of the Cartan elements means that $\mathfrak{g}_J$ is spanned by all root vectors E_α of $\mathfrak{g}$, i. e. $\alpha(J) \neq 0$ for any root α of $\mathfrak{g}$.

Indeed the given NLEE as well as the other members of its hierarchy posses Lax representation of the form (according to (2))

$$[L(\lambda), M_P(\lambda)] = 0$$

where

$$M_P\Psi(x,t,\lambda) = \left(\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t} + \sum_{k=-S}^{P-1} V_k(x,t) - \lambda^P f_P I \right) \Psi(x,t,\lambda) = 0\,, \quad I \in \mathfrak{h}\,, \qquad (4)$$

which must hold identically with respect to λ. A standard procedure generalizing the AKNS one [1] allows us to evaluate $V_k(x,t)$ in terms of $q(x,t)$ and its x-derivatives. Here and below we consider only the class of potentials $q(x,t)$ vanishing fast enough for $|x| \to \infty$. Then one may also check that the asymptotic value of the potential in $M_P(\lambda)$ namely $f^{(P)}(\lambda) = f_P\lambda^P I$ may be understood as the dispersion law of the corresponding NLEE. For example, the N-wave type equations are obtained in the simplest nontrivial case with $P = 1$, $f_P = 1$, $q(x,t) = [J, Q(x,t)]$ and $V_0(x,t) = [I, Q(x,t)]$, $Q(x,t) \in \mathfrak{g}/h$.

The number of roots $|\Delta|$ of the simple Lie algebras given on the table below

$\mathfrak{g}$	$\mathbf{A}_r$	$\mathbf{B}_r, \mathbf{C}_r$	$\mathbf{D}_r$	$\mathbf{G}_2$	$\mathbf{F}_4$	$\mathbf{E}_6$	$\mathbf{E}_7$	$\mathbf{E}_8$
$\lvert\Delta\rvert$	$r(r+1)$	$2r^2$	$2r(r-1)$	12	48	72	126	240

grows rather quickly with the rank of the algebra r. The corresponding generic N-wave equations are systems of $|\Delta|$ equations for $|\Delta|$ independent complex-valued functions. They are solvable for any r but their applications to physics for large r become doubtful. Such is the situation and for any other types NLEE.

Thus another important trend was developing in the framework of the ISM, initialized by the introduction of the so-called reduction group introduced by Mikhailov [20], and further developed in [10, 11, 25, 21]. As reductions we mean algebraic restrictions on the potentitial matrix $q(x,t)$ which will diminish the number of equations and/or the number of the independent functions in them. Of course such restrictions must be compatible with the dynamics of the NLEE. This procedure allowed one to prove that some of the well known models in the field theory [20] and also a number of new interesting NLEE [20, 10, 21] are integrable by the ISM and posses special symmetry properties. As a result its potential $q(x,t)$ has a very special form and J can no longer be chosen real.

This problem of constructing the spectral theory for (2) in the most general case when J has an arbitrary complex eigenvalues was initialized by Beals, Coifman and Caudrey [2–4, 7] and continued by Zhou [27] in the case when the algebra $\mathfrak{g}$ is $sl(n)$, $q(x,t)$ vanishing fast enough for $|x| \to \infty$ and no a priori symmetry conditions are imposed on $q(x,t)$. This has been done later for any semi-simple Lie algebras by Gerdjikov and Yanovski [16]. there the minimal sets of scattering data of (2) for complex J are constructed and the properties of the generating functions of the integrals of motion are studied.

Our basic aim in this paper is to study the effect of the action of the reduction group G_R on the scattering data of (2) for any semisimple Lie algebra in the case of complex J. A special attention will be paid to the sectors in the complex plane of the spectral parameter and their reductions. In particular the author's interest of this problem is inspired of [16].

In Section 2 we summarize some basic fact about the reduction group and Lie algebraic details.

The construction of the fundamental analytic solutions (FAS) is sketched in Section 3 which is done separately for the case of real Cartan elements (Section 3.1) and for complex ones (Section 3.2).

The reductions of the scattering data of CBC system is described in Section 4. These results are demonstrated on examples of reductions from the Weyl group of $so(5)$ algebra in Section 5.

We finish this report with several conclusive remarks and a brief outlook in Section 6.

2. Preliminaries and General Approach

The main idea underlying Mikhailov's reduction group [20] is to impose algebraic restrictions on the Lax operators L and M which will be automatically compatible with the corresponding equations of motion. Due to the purely Lie-algebraic nature of the Lax representation this is most naturally done by imbedding the reduction group as a subgroup of $\mathrm{Aut}\,\mathfrak{g}$ — the group of automorphisms of $\mathfrak{g}$. Obviously to each reduction imposed on L and M there will correspond a reduction of the space of fundamental solutions $\mathbf{S}_\Psi \equiv \{\Psi(x,t,\lambda)\}$ of (2).

Some of the simplest $\mathbb{Z}_p$-reductions of Zakharov–Shabat systems have been known for a long time (see [20]) and are related to outer automorphisms of $\mathfrak{g}$ and $\mathfrak{G}$, namely:

$$C_1\left(\Psi(x,t,\lambda)\right) = A_1\Psi^\dagger(x,t,\kappa_1(\lambda))A_1^{-1} = \tilde{\Psi}^{-1}(x,t,\lambda)\,, \quad \kappa_1(\lambda) = \pm\lambda^* \tag{5}$$

where A_1 belongs to the Cartan subgroup of the group $\mathfrak{G}$:

$$A_1 = \exp\left(\pi \mathrm{i} H_1\right)\,, \tag{6}$$

and $H_1 \in \mathfrak{h}$ is such that $\alpha(H_1) \in \mathbb{Z}$ for all roots $\alpha \in \Delta$ in the root system Δ of $\mathfrak{g}$. Note that the reduction condition relates the fundamental solution $\Psi(x,t,\lambda) \in \mathfrak{G}$ to a fundamental solution $\tilde{\Psi}(x,t,\lambda)$ of (2) which in general differs from $\Psi(x,t,\lambda)$.

Another class of $\mathbb{Z}_p$ reductions are related to outer automorphisms of the type:

$$C_2\left(\Psi(x,t,\lambda)\right) = A_2\Psi^T(x,t,\kappa_2(\lambda))A_2^{-1} = \tilde{\Psi}^{-1}(x,t,\lambda)\,, \quad \kappa_2(\lambda) = \pm\lambda \tag{7}$$

where A_2 is again of the form (6). The best known examples of NLEE obtained with the $\mathbb{Z}_2$ reduction (7) are the sin-Gordon and the MKdV equations which are related to $\mathfrak{g} \simeq sl(2)$.

In fact the reductions (5) and (7) provide us examples when the reduction is obtained with the combined use of outer and inner automorphisms.

Along with (6), (5) one may use also reductions with inner automorphisms:

$$C_3\left(\Psi(x,t,\lambda)\right) = A_3\Psi^*(x,t,\kappa_1(\lambda))A_3^{-1} = \tilde{\Psi}(x,t,\lambda)\,, \tag{8}$$

and

$$C_4\left(\Psi(x,t,\lambda)\right) = A_4\Psi(x,t,\kappa_2(\lambda))A_4^{-1} = \tilde{\Psi}(x,t,\lambda)\,. \tag{9}$$

Since our aim is to preserve the form of the Lax pair we limit ourselves by automorphisms preserving the Cartan subalgebra $\mathfrak{h}$. This conditions is obviously fulfilled if A_k, $k = 1, \dots, 4$ is in the form (6). Another possibility is to choose $A_1, \dots, A_4$ so that they correspond to a Weyl group automorphisms.

In fact (5) is related to outer automorphisms only if $\mathfrak{g}$ is from the $\mathbf{A}_r$ and $\mathbf{D}_r$ series. For the $\mathbf{B}_r$ and $\mathbf{C}_r$ series (6) is equivalent to an inner automorphism (8) with the special choice for the Weyl group element w_0 which maps all highest weight vectors into the corresponding lowest weight vectors. $\mathbb{Z}_2$ reductions of the form (5) in fact restrict us to the corresponding real form of the algebra $\mathfrak{g}$.

Here we fix up the notations and the normalization conditions for the Cartan–Weyl generators of $\mathfrak{g}$ [18]. We introduce $h_k \in \mathfrak{h}$, $k = 1, \dots, r$ and E_α, $\alpha \in \Delta$ where $\{h_k\}$ are the Cartan elements dual to the orthonormal basis $\{e_k\}$ in the root space $\mathbb{E}^r$. Along with h_k we introduce also

$$H_\alpha = \frac{2}{(\alpha, \alpha)} \sum_{k=1}^{r} (\alpha, e_k) h_k \,, \qquad \alpha \in \Delta \tag{10}$$

where (α, e_k) is the scalar product in the root space $\mathbb{E}^r$ between the root α and e_k. The commutation relations are given by:

$$\begin{aligned} &[h_k, E_\alpha] = (\alpha, e_k) E_\alpha \,, \quad [E_\alpha, E_{-\alpha}] = H_\alpha \,, \\ &[E_\alpha, E_\beta] = \begin{cases} N_{\alpha,\beta} E_{\alpha+\beta} & \text{for } \alpha + \beta \in \Delta \\ 0 & \text{for } \alpha + \beta \notin \Delta \cup \{0\} \,. \end{cases} \end{aligned} \tag{11}$$

We will denote by $\vec{a} = \sum_{k=1}^{r} a_k e_k$ the r-dimensional vector dual to $J \in \mathfrak{h}$; obviously $J = \sum_{k=1}^{r} a_k h_k$. If J is a regular real element in $\mathfrak{h}$ then without restrictions we may use it to introduce an ordering in Δ. Namely we will say that the root $\alpha \in \Delta_+$ is positive (negative) if $(\alpha, \vec{a}) > 0$ ($(\alpha, \vec{a}) < 0$ respectively). The normalization of the basis is determined by:

$$\begin{aligned} &E_{-\alpha} = E_\alpha^T, \quad \langle E_{-\alpha}, E_\alpha \rangle = \frac{2}{(\alpha, \alpha)} \,, \\ &N_{-\alpha,-\beta} = -N_{\alpha,\beta}, \quad N_{\alpha,\beta} = \pm (p+1) \end{aligned} \tag{12}$$

where the integer $p \geq 0$ is such that $\alpha + s\beta \in \Delta$ for all $s = 1, \dots, p$, $\alpha + (p+1)\beta \notin \Delta$ and $\langle \cdot, \cdot \rangle$ is the Killing form of $\mathfrak{g}$. The root system Δ of $\mathfrak{g}$ is invariant with respect to the Weyl reflections A_α^*; on the vectors $\vec{y} \in \mathbb{E}^r$ they act as

$$A_\alpha^* \vec{y} = \vec{y} - \frac{2(\alpha, \vec{y})}{(\alpha, \alpha)} \alpha \,, \qquad \alpha \in \Delta \,. \tag{13}$$

All Weyl reflections A_α^* form a finite group $W_\mathfrak{g}$ known as the Weyl group. One may introduce in a natural way an action of the Weyl group on the Cartan–Weyl basis, namely:

$$\begin{aligned} A_\alpha^*(H_\beta) &\equiv A_\alpha H_\beta A_\alpha^{-1} = H_{A_\alpha^*\beta}\,, \\ A_\alpha^*(E_\beta) &\equiv A_\alpha E_\beta A_\alpha^{-1} = n_{\alpha,\beta} E_{A_\alpha^*\beta}\,, \qquad n_{\alpha,\beta} = \pm 1\,. \end{aligned} \tag{14}$$

It is also well known that the matrices A_α are given (up to a factor from the Cartan subgroup) by

$$A_\alpha = \mathrm{e}^{E_\alpha}\,\mathrm{e}^{-E_{-\alpha}}\,\mathrm{e}^{E_\alpha} H_A \tag{15}$$

where H_A is a conveniently chosen element from the Cartan subgroup such that $H_A^2 = 1\!\!1$. The formula (15) and the explicit form of the Cartan–Weyl basis in the typical representation will be used in calculating the reduction condition following from (16).

The reduction group G_R is a finite group which preserves the Lax representation, i. e. it ensures that the reduction constraints are automatically compatible with the evolution. G_R must have two realizations:

i) $G_R \subset \operatorname{Aut}\mathfrak{g}$

ii) $G_R \subset \operatorname{Conf}\mathbb{C}$, i. e. as conformal mappings of the complex λ-plane.

To each $g_k \in G_R$ we relate a reduction condition for the Lax pair as follows [20]:

$$C_k(U(\Gamma_k(\lambda))) = \eta_k U(\lambda) \tag{16}$$

where $U(x,\lambda) = q(x) - \lambda J$, $C_k \in \operatorname{Aut}\mathfrak{g}$ and $\Gamma_k(\lambda)$ are the images of g_k and $\eta_k = 1$ or -1 depending on the choice of C_k. Since G_R is a finite group then for each g_k there exist an integer N_k such that $g_k^{N_k} = 1\!\!1$. In all the examples below $N_k = 2$ and the reduction group is isomorphic to $\mathbb{Z}_2$.

The condition (16) is obviously compatible with the group action. Therefore it is enough to ensure that (16) is fulfilled for the generating elements of G_R.

In fact (see [8]) every finite group G is determined uniquely by its generating elements g_k and genetic code, e. g.:

$$g_k^{N_k} = 1\!\!1\,, \qquad (g_j g_k)^{N_{jk}} = 1\!\!1\,, \qquad N_k, N_{jk} \in \mathbb{Z}\,. \tag{17}$$

For example the cyclic $\mathbb{Z}_N$ and the dihedral $\mathbb{D}_N$ groups have as genetic codes

$$g^N = 1\!\!1\,, \qquad N \geq 2 \ \text{ for } \ \mathbb{Z}_N\,, \tag{18}$$

and

$$g_1^2 = g_2^2 = (g_1 g_2)^N = 1\!\!1\,, \qquad N \geq 2 \ \text{ for } \ \mathbb{D}_N\,. \tag{19}$$

It is well known that $\operatorname{Aut}\mathfrak{g} \equiv V \otimes \operatorname{Aut}_0\mathfrak{g}$ where V is the group of outer automorphisms (the symmetry group of the Dynkin diagram) and $\operatorname{Aut}_0\mathfrak{g}$ is the group of inner automorphisms. Since we start with $I, J \in \mathfrak{h}$ it is natural to consider only those inner automorphisms that preserve the Cartan subalgebra $\mathfrak{h}$. Then $\operatorname{Aut}_0\mathfrak{g} \simeq \operatorname{Ad}_H \otimes W$ where Ad_H is the group of similarity transformations with elements from the Cartan subgroup:

$$\operatorname{Ad}_C X = CXC^{-1}\,, \qquad C = \exp\left(\frac{2\pi\mathrm{i}H_c}{N}\right), \qquad X \in \mathfrak{g}\,, \tag{20}$$

and W is the Weyl group of $\mathfrak{g}$. Its action on the Cartan–Weyl basis was described in (14) above. From (11) one easily finds

$$CH_\alpha C^{-1} = H_\alpha\,, \qquad CE_\alpha C^{-1} = \mathrm{e}^{2\pi\mathrm{i}(\alpha,\vec{c})/N} E_\alpha \tag{21}$$

where $\vec{c} \in \mathbb{E}^r$ is the vector corresponding to $H_c \in \mathfrak{h}$ in (20). Then the condition $C^N = \mathbb{1}$ means that $(\alpha, \vec{c}) \in \mathbb{Z}$ for all $\alpha \in \Delta$. Obviously H_c must be chosen so that $\vec{c} = \sum_{k=1}^{r} 2c_k\omega_k/(\alpha_k, \alpha_k)$ where ω_k are the fundamental weights of $\mathfrak{g}$ and c_k are integer. In the examples below we will use several possibilities by choosing C_k as appropriate compositions of elements from V, $\operatorname{Ad}_\mathfrak{h}$ and W. In fact if $\mathfrak{g}$ belongs to $\mathbf{B}_r$ or $\mathbf{C}_r$ series then $V \equiv \mathbb{1}$.

Generically each element $g_k \in G$ maps λ into a fraction-linear function of λ. Such action however is appropriate for a more general class of Lax operators which are fraction linear functions of λ.

3. The Caudrey–Beals–Coifman Systems

3.1. Fundamental Analytical Solutions and Scattering Data for Real J

The direct scattering problem for the Lax operator (2) is based on the Jost solutions:

$$\lim_{x\to\infty} \psi(x,\lambda)\,\mathrm{e}^{\mathrm{i}\lambda Jx} = \mathbb{1}\,, \qquad \lim_{x\to-\infty} \phi(x,\lambda)\,\mathrm{e}^{\mathrm{i}\lambda Jx} = \mathbb{1}\,, \tag{22}$$

and the scattering matrix

$$T(\lambda) = (\psi(x,\lambda))^{-1}\phi(x,\lambda)\,. \tag{23}$$

The fundamental analytic solutions (FAS) $\chi^\pm(x,\lambda)$ of $L(\lambda)$ are analytic functions of λ for $\operatorname{Im}\lambda \gtrless 0$ and are related to the Jost solutions by [13]

$$\chi^\pm(x,\lambda) = \phi(x,\lambda)S^\pm(\lambda) = \psi^\pm(x,\lambda)T^\mp(\lambda)D^\pm(\lambda) \tag{24}$$

where $T^{\pm}(\lambda)$, $S^{\pm}(\lambda)$ and $D^{\pm}(\lambda)$ are the factors of the Gauss decomposition of the scattering matrix:

$$T(\lambda) = T^{-}(\lambda)D^{+}(\lambda)\hat{S}^{+}(\lambda) = T^{+}(\lambda)D^{-}(\lambda)\hat{S}^{-}(\lambda) \tag{25}$$

$$T^{\pm}(\lambda) = \exp\left(\sum_{\alpha>0} t^{\pm}_{\pm\alpha}(\lambda)E_{\alpha}\right), \qquad S^{\pm}(\lambda) = \exp\left(\sum_{\alpha>0} s^{\pm}_{\pm\alpha}(\lambda)E_{\alpha}\right),$$

$$D^{+}(\lambda) = I\exp\left(\sum_{j=1}^{r} \frac{2d^{+}(\lambda)}{(\alpha_j,\alpha_j)}H_j\right), \qquad D^{-}(\lambda) = I\exp\left(\sum_{j=1}^{r} \frac{2d^{-}(\lambda)}{(\alpha_j,\alpha_j)}H_j^{-}\right).$$

Here $H_j = H_{\alpha_j}$, $H_j^{-} = w_0(H_j)$, $\hat{S} \equiv S^{-1}$, I is an element from the universal centre of the corresponding Lie group $\mathfrak{g}$ and the superscript $+$ (or $-$) in the Gauss factors means (upper-) or lower-trianguality for $T^{\pm}(\lambda)$, $S^{\pm}(\lambda)$ and shows that $D^{+}(\lambda)$ (or $D^{-}(\lambda)$) are analytic functions with respect to λ for $\operatorname{Im}\lambda > 0$ (or $\operatorname{Im}\lambda < 0$ respectively).

On the real axis $\chi^{+}(x,\lambda)$ and $\chi^{-}(x,\lambda)$ are linearly related by:

$$\chi^{+}(x,\lambda) = \chi^{-}(x,\lambda)G_0(\lambda), \qquad G_0(\lambda) = S^{+}(\lambda)\hat{S}^{-}(\lambda), \tag{26}$$

and the sewing function $G_0(\lambda)$ may be considered as a minimal system of scattering data provided the Lax operator (2) has no discrete eigenvalues [13].

3.2. The CBC Construction for Semisimple Lie Algebras

Here we will sketch the construction of the FAS for the case of complex-valued regular Cartan element J: $\alpha(\psi) \neq 0$, following the general ideas of Beals and Coifmal [2] for the $sl(n)$ algebras and [16] for the orthogonal and symplectic algebras. These ideas consist of the following:

1. For potentials $q(x)$ with small norm $\|q(x)\|_{L^1} < 1$ one can divide the complex λ-plane into sectors and then construct an unique FAS $m_{\nu}(x,\lambda)$ which is analytic in each of these sectors Ω_{ν};
2. For these FAS in each sector there is a certain Gauss decomposition problem for the scattering matrix $T(\lambda)$ which has an unique solution in the case of absence of discrete eigenvalues.

The main difference between the cases of real-valued and complex-valued J consists in the fact that for complex J the Jost solutions and the scattering data exist only for the potentials on compact support.

We define the regions (sectors) Ω_{ν} as consisting of those λ's for which $\operatorname{Im}(\lambda\alpha(J)) \neq 0$ for any $\alpha \in \Delta$. Thus the boundaries of the Ω_{ν}'s consist of the set of straight lines:

$$l_{\alpha} \equiv \{\lambda;\ \operatorname{Im}\lambda\alpha(J) = 0\,,\ \alpha \in \Delta\}, \tag{27}$$

where in the first equality we take $\lambda = \mu\,\mathrm{e}^{\mathrm{i}0}$ and for the second — $\lambda = \mu\,\mathrm{e}^{-\mathrm{i}0}$ with $\mu \in l_\nu$. The corresponding expressions for the Gauss factors have the form:

$$
\begin{aligned}
S_\nu^+(\lambda) &= \exp\left(\sum_{\alpha\in\Delta_\nu^+} s_{\nu,\alpha}^+(\lambda)E_\alpha\right), & S_\nu^-(\lambda) &= \exp\left(\sum_{\alpha\in\Delta_{\nu-1}^+} s_{\nu,\alpha}^-(\lambda)E_{-\alpha}\right),\\
T_\nu^+(\lambda) &= \exp\left(\sum_{\alpha\in\Delta_{\nu-1}^+} t_{\nu,\alpha}^+(\lambda)E_\alpha\right), & T_\nu^-(\lambda) &= \exp\left(\sum_{\alpha\in\Delta_\nu^+} t_{\nu,\alpha}^-(\lambda)E_{-\alpha}\right),\\
D_\nu^+(\lambda) &= \exp(\mathbf{d}_\nu^+(\lambda)\cdot\mathbf{H}_\nu)\,, & D_\nu^-(\lambda) &= \exp(\mathbf{d}_\nu^-(\lambda)\cdot\mathbf{H}_{\nu-1})\,.
\end{aligned}
\tag{35}
$$

Here $\mathbf{d}_\nu^\pm(\lambda) = (d_{\nu,1}^\pm,\dots,d_{\nu,r}^\pm)$ is a vector in the root space and

$$
\begin{aligned}
\mathbf{H}_\eta &= \left(\frac{2H_{\eta,1}}{(\alpha_{\eta,1},\alpha_{\eta,1})},\dots,\frac{2H_{\eta,r}}{(\alpha_{\eta,r},\alpha_{\eta,r})}\right),\\
(\mathbf{d}_\nu^\pm(\lambda),\mathbf{H}_\eta) &= \sum_{k=1}^{r}\frac{2d_{\nu,k}^\pm(\lambda)H_{\eta,k}}{(\alpha_{\eta,k},\alpha_{\eta,k})}
\end{aligned}
\tag{36}
$$

where $\alpha_{\eta,k}$ is the k-th simple root of $\mathfrak{g}$ with respect to the ordering Δ_η^+ and $H_{\eta,k}$ are their dual elements in the Cartan subalgebra $\mathfrak{h}$.

We skip the details about CBC construction which can be founded in [16] and go to the minimal set of scattering data for the case of complex J which are defined by the sets $\mathcal{F}_1$ and $\mathcal{F}_2$ as follows:

$$
\begin{aligned}
\mathcal{F}_1 &= \bigcup_{\nu=1}^{2M}\mathcal{F}_{1,\nu}\,, \qquad \mathcal{F}_2 = \bigcup_{\nu=1}^{2M}\mathcal{F}_{2,\nu}\,,\\
\mathcal{F}_{1,\nu} &= \{\rho_{B,\nu,\alpha}^\pm(\lambda)\,,\ \alpha\in\delta_\nu^+\,,\ \lambda\in l_\nu\}\,,\\
\mathcal{F}_{2,\nu} &= \{\tau_{B,\nu,\alpha}^\pm(\lambda)\,,\ \alpha\in\delta_\nu^+\,,\ \lambda\in l_\nu\}
\end{aligned}
\tag{37}
$$

where

$$
\rho_{B,\nu,\alpha}^\pm(\lambda) = \langle S_\nu^\pm(\lambda)B\hat{S}_\nu^\pm(\lambda),E_{\mp\alpha}\rangle\,,
\tag{38}
$$

$$
\tau_{B,\nu,\alpha}^\pm(\lambda) = \langle T_\nu^\pm(\lambda)B\hat{T}_\nu^\pm(\lambda),E_{\mp\alpha}\rangle\,,
\tag{39}
$$

with $\alpha\in\delta_\nu^+$, $\lambda\in l_\nu$ and B is a properly chosen regular element of the Cartan subalgebra $\mathfrak{h}$. Without loose of generality we can take in (38) $B = H_\alpha$.

Note that the functions $\rho_{B,\nu,\alpha}^\pm(\lambda)$ and $\tau_{B,\nu,\alpha}^\pm(\lambda)$ are continuous functions of λ for $\lambda\in l_\nu$.

If we choose J in such way that $2M = |\Delta|$ — the number of the roots of $\mathfrak{g}$. then to each pair of roots $\{\alpha,-\alpha\}$ one can relate a separate pair of rays

$\{l_\alpha, l_{\alpha+M}\}$, and $l_\alpha \neq l_\beta$ if $\alpha \neq \pm\beta$. In this case each of the subalgebras $\mathfrak{g}_\alpha$ will be isomorphic to $sl(2)$.

4. The Scattering Data of CBS Systems and Reductions

Now we pass to the case when the linear problem L has a symmetry. Under symmetry we will understand that an invariance condition on $L(\lambda)$ with respect to some automorphism of the algebra is imposed. To be more specific we consider two important $\mathbb{Z}_2$-reductions:

$$1) \qquad A_1 U^\dagger(\pm\lambda^*) A_1^{-1} = U(\lambda)\,, \qquad A_1^2 = \mathbb{1} \tag{40}$$

$$2) \qquad A_2 U(\pm\lambda) A_2^{-1} = U(\lambda)\,, \qquad A_2^2 = \mathbb{1}\,. \tag{41}$$

This leads in natural way to the following reduction conditions for the potential matrix $q(x)$ and the Cartan elements J (according to (40) and (41)):

$$1) \qquad A_1 q^\dagger(x) A_1^{-1} = q(x)\,, \qquad A_1 J^* A_1^{-1} = \pm J, \tag{42}$$

$$2) \qquad A_2 q(x) A_2^{-1} = q(x)\,, \qquad A_1 J A_1^{-1} = \pm J\,. \tag{43}$$

In such way we decrease the number of independent functions $q_\alpha(x)$ and $q_{-\alpha}(x)$ in the potential matrix $q(x)$ and the eigenvalues of the Cartan elements J. As we mentioned above in order to obtain dynamics of NLEE compatible with the evolution we must restrict ourselves considering only the group of automorphisms of $\mathfrak{g}$ which preserve the Cartan subalgebra $\mathfrak{h}$: $\mathrm{Aut}_0\,\mathfrak{g} = \mathrm{Ad}_{\mathfrak{h}}\,\mathfrak{g} \otimes \mathfrak{W}(\mathfrak{g})$. the case with Cartan group automorphisms is discussed in [14].

We remind also that the isometry between the root space $\mathbb{E}^r \supset \Delta$ and the Cartan subalgebra $\mathfrak{h}$ is given by [18]:

$$\langle H_i, H_j \rangle = (\alpha_i, \alpha_j) = \alpha_i(H_j)\,. \tag{44}$$

As a consequence we have:

$$\mathrm{Im}(A^*(\alpha(J))\lambda) = \mathrm{Im}(\alpha(AJA^{-1})\lambda) = \mathrm{Im}(\alpha(J)\omega\lambda) \tag{45}$$

where A^* is the co-adjoint action of A on the root system:

$$A^*\alpha(H) = \alpha(AHA^{-1})\,, \qquad H \in \mathfrak{h}\,. \tag{46}$$

and $\omega = \pm 1$. Clearly if π_α is the set of simple roots of the subalgebra $\mathfrak{g}_\alpha$ on the ray l_α then

$$l_{A^*\alpha} = \{\lambda;\ \mathrm{Im}(\alpha(J)\omega^{-1}\lambda) = 0\}\,, \quad \alpha \in \pi_\alpha\,. \tag{47}$$

Generally this means that $l_{A^*\alpha}$ is obtained from l_α through the rotation $\lambda \to \omega\lambda$, $\omega = \exp(2\pi\mathrm{i}/p)$ in the complex λ-plane and therefore the set of lines $\{l_\alpha\}_{\alpha\in\Delta}$

is invariant under the action of the group $\mathbb{Z}_p$ generated by ω. Clearly $\mathbb{Z}_p$ maps sectors into sectors. Without loss of generality we can assume that Ω_1, $\Omega_2, \dots, \Omega_{2M}$ come one after the other in positive direction.

The symmetry of the Lax operator $L(\lambda)$ is of course reflected on the properties of the FAS by:

$$1) \quad A_1 m_\nu^\dagger(x, \omega\lambda^*) A_1^{-1} = m_{2M-A^*(\nu)+1}(x, \lambda), \qquad \lambda \in \Omega_\nu, \tag{48}$$

$$2) \quad A_2 m_\nu(x, \omega\lambda) A_2^{-1} = m_{M+A^*(\nu)}(x, \lambda), \qquad \lambda \in \Omega_\nu \tag{49}$$

where $A^*(\nu)$ is the image of the ν-th ray l_ν under the reflection $A^*(\alpha)$.

Thus for the Gauss factors $S_\nu^\pm(\lambda)$, $T_\nu^\pm(\lambda)$ and $D_\nu^\pm(\lambda)$ we obtain:

$$1) \quad \begin{aligned} S_\nu^+(\lambda) &= A_1 (S_{2M-A^*(\nu)+1}^-(\omega\lambda^*))^\dagger A_1^{-1}, \\ T_\nu^+(\lambda) &= A_1 (T_{2M-A^*(\nu)+1}^-(\omega\lambda^*))^\dagger A_1^{-1}, \\ D_\nu^+(\lambda) &= A_1 (D_{2M-A^*(\nu)+1}^-(\omega\lambda^*))^* A_1^{-1}, \end{aligned} \tag{50}$$

$$2) \quad \begin{aligned} S_\nu^+(\lambda) &= A_2 S_{M+A^*(\nu)}^-(\omega\lambda) A_2^{-1}, \\ T_\nu^+(\lambda) &= A_2 T_{M+A^*(\nu)}^-(\omega\lambda) A_2^{-1}, \\ D_\nu^+(\lambda) &= A_2 D_{M+A^*(\nu)}^-(\omega\lambda) A_2^{-1}, \end{aligned} \tag{51}$$

and as a consequence for the minimal set of scattering data (38):

$$1) \quad \begin{aligned} \rho_{B,\nu,\alpha}^\pm(\lambda) &= (\rho_{B,2M-A^*(\nu)+1,A^*(\alpha)}^\mp(\omega\lambda^*))^*, \qquad & \alpha \in \delta_{+,\nu}^0, \\ \rho_{B,\nu,\alpha}^\pm(\lambda) &= (\rho_{B,2M-A^*(\nu)+1,A^*(\alpha)}^\pm(\omega\lambda^*)^*, \qquad & \alpha \in \delta_{+,\nu}^1, \end{aligned} \tag{52}$$

$$2) \quad \begin{aligned} \rho_{B,\nu,\alpha}^\pm(\lambda) &= \rho_{B,M+A^*(\nu),A^*(\alpha)}^\pm(\omega\lambda^*), \qquad & \alpha \in \delta_{+,\nu}^0, \\ \rho_{B,\nu,\alpha}^\pm(\lambda) &= \rho_{B,M+A^*(\nu),A^*(\alpha)}^\mp(\omega\lambda^*), \qquad & \alpha \in \delta_{+,\nu}^1 \end{aligned} \tag{53}$$

where the sets $\delta_{+,\nu}^0$ and $\delta_{+,\nu}^1$ are introduced as follows:

$$\delta_{+,\nu}^+ = \delta_{+,\nu}^0 \cup \delta_{+,\nu}^1 \tag{54}$$

and

$$A^*(\delta_{+,\nu}^0) = \delta_{+,A^*(\nu)}^0, \qquad A^*(\delta_{+,\nu}^1) = -\delta_{+,A^*(\nu)}^1.$$

In the next section we consider an nontrivial example of $\mathbb{Z}_2$ reductions of type 1) on the $so(5)$ algebra.

5. Example: $\mathfrak{g} \simeq \mathbf{B}_2$

This algebra has 4 positive roots: $\Delta^+_{\mathbf{B}_2} = \{e_1 - e_2, e_2, e_1, e_1 + e_2\}$. We will limit ourselves with a particular case choosing the Cartan element in the form:

$$J = \mathrm{diag}(a, a^*, 0, -a^*, -a) . \tag{55}$$

The continuous spectrum of the Lax operator (2) fills up a bunch of 8 rays l_α intersecting at the origin defined by (27):

1) l_{12} coincides with the real axis in the complex λ-plane (see Fig. 1);
2) without loss of generality we can choose $l_{01} = l_2$ and $l_{11} = l_8$ (see the notations below) to close with the real axis an opposite angles $-\phi$ and ϕ respectively;
3) l_{10} goes through the imaginary axis.

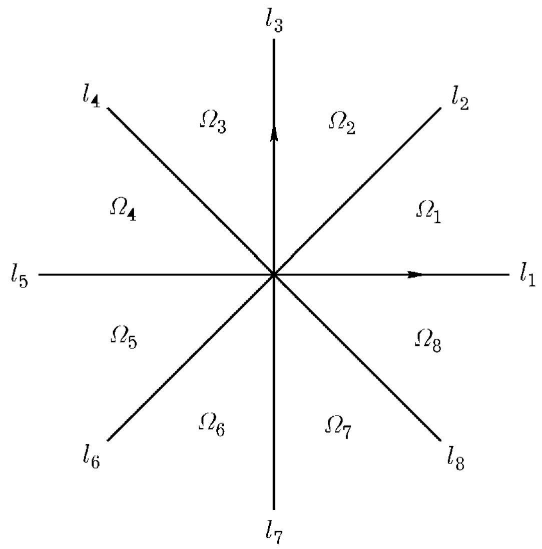

Figure 1. The continuous spectrum of the Lax operator $L(\lambda)$ related to Eq. (55)

Here we use the notations $(mn) = m\alpha_1 + n\alpha_2$, where $\alpha_1 = e_1 - e_2$, $\alpha_2 = e_2$ are the simle roots of $\mathbf{B}_2$-algebra, $m, n \geq 0$ and $(\overline{mn}) = -m\alpha_1 - n\alpha_2$. In Fig. 1 we note $l_1 = l_{12}$, $l_2 = l_{01}$, $l_3 = l_{10}$, $l_8 = l_{11}$ and by l_4, l_5, l_6, l_7 their opposite rays respectivelly. These rays devide the complex plane into 8 sectors $\Omega_1, \ldots, \Omega_8$ and each sector Ω_k is bounded by the rays l_k and l_{k+1}, $k = 1, \ldots, 8$. To each ray l_α corresponds $sl(2)$ subalgebra generated by the roots $\{\pm\alpha\}$. The minimal set of scattering data is

$$\begin{aligned}\mathcal{F} = \{&\rho^{\pm}_{12,1}(\lambda), \rho^{\pm}_{11,8}(\lambda), \rho^{\pm}_{10,3}(\lambda), \rho^{\pm}_{01,2}(\lambda), \\ &\rho^{\pm}_{\overline{12},5}(\lambda), \rho^{\pm}_{\overline{11},4}(\lambda), \rho^{\pm}_{\overline{10},7}(\lambda), \rho^{\pm}_{\overline{01},6}(\lambda)\} .\end{aligned} \tag{56}$$

Let us now impose the following reduction:

$$A^*_{e_1-e_2}(U^\dagger(x,\lambda^*)) = U(\lambda), \qquad U(x,\lambda) = q(x) - \lambda J\,. \tag{57}$$

The orbits of the authomorphism (57) are as follows:

$$\{l_2,l_4\},\{l_1,l_5\},\{l_6,l_8\},\{l_3\},\{l_7\}; \tag{58}$$

$$\{\Omega_1,\Omega_4\},\{\Omega_2,\Omega_3\},\{\Omega_5,\Omega_8\},\{\Omega_6,\Omega_7\}, \tag{59}$$

and the reduction conditions on the scattering data are:

$$\begin{aligned}
&\rho^{\pm}_{10,3}(\lambda) = (\rho^{\pm}_{\bar{1}0,7}(\lambda^*))^*, &&\rho^{\pm}_{\bar{1}0,7}(\lambda) = (\rho^{\pm}_{10,3}(\lambda^*))^*,\\
&\rho^{\pm}_{12,1}(\lambda) = (\rho^{\mp}_{\bar{1}2,5}(\lambda^*))^*, &&\rho^{\pm}_{\bar{1}2,5}(\lambda) = (\rho^{\mp}_{12,1}(\lambda^*))^*,\\
&\rho^{\pm}_{01,2}(\lambda) = (\rho^{\mp}_{1\bar{1},4}(\lambda^*))^*, &&\rho^{\pm}_{1\bar{1},4}(\lambda) = (\rho^{\mp}_{01,2}(\lambda^*))^*,\\
&\rho^{\pm}_{11,8}(\lambda) = (\rho^{\mp}_{0\bar{1},6}(\lambda^*))^*, &&\rho^{\pm}_{0\bar{1},6}(\lambda) = (\rho^{\mp}_{11,8}(\lambda^*))^*.
\end{aligned} \tag{60}$$

Thus the reduced minimal set of scattering data becomes:

$$\mathcal{F}_{1,\mathrm{red}} = \{\rho^{+}_{10,3}(\lambda),\rho^{-}_{\bar{1}0,7}(\lambda),\rho^{\pm}_{11,8}(\lambda),\rho^{\pm}_{1\bar{1},4}(\lambda),\rho^{\pm}_{12,1}(\lambda)\}\,. \tag{61}$$

6. Conclusions

We announced here some preliminary results about the classification of CBC systems related to (semi-) simple Lie algebras. More detailed article about this problem containing wider class of reductions is in preparation.

We finish this report with several remarks:

- Note that in the case of reduction 2) it may happen that two or more of the eigenvalues of J become equal. Then the construction of the FAS requires the use of the generalized Gauss decomposition in which the factors $D^{\pm}_\nu(\lambda)$ are block-diagonal matrices while $T^{\pm}_\nu(\lambda)$ and $S^{\pm}_\nu(\lambda)$ are block-triangular matrices, see [12].
- The $\mathbb{Z}_2$-reductions which map λ to λ^* combined with Cartan involution on $\mathfrak{g}$ lead in fact to restrictions of the system to a specific real form of the algebra $\mathfrak{g}$ i. e. lead to the case of real J. A classification of such reductions applied to N-wave type systems is done in [14];
- To all reduced CBC systems one can apply the analysis [16] and derive the completeness relations for the corresponding system of "squared" solutions. Such analysis will allow one to prove the pair-wise compatibility of the Hamiltonian structures and eventually to derive their action-angle variables, see e. g. [23, 5] for the $\mathbf{A}_r$-series.

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