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# SOME ESTIMATES FOR THE CURVATURES OF SPACELIKE HYPERSURFACES IN DE SITTER SPACE

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**Abstract**. In this paper we will report on our recent studies on curvature properties of space-like hypersurfaces in de Sitter space. In particular, we will state certain estimates for the higher order mean curvatures, the scalar curvature and the Ricci curvature of complete space-like hypersurfaces in de Sitter space. We will also establish a sufficient condition for a compact space-like hypersurface in de Sitter space to be spherical in terms of a lower bound for the square of its mean curvature.

## 1. Introduction and Statement of the Main Results

The study of space-like hypersurfaces in de Sitter space $\mathbb{S}_1^{n+1}$ has been of increasing interest in the last years, because of their nice Bernstein-type properties. Since Goddard [7] conjectured in 1977 that the only complete space-like hypersurfaces in $\mathbb{S}_1^{n+1}$ with constant mean curvature $H$ should be the totally umbilical ones (which is clearly false), many authors have worked on the problem of finding global rigidity theorems for space-like hypersurfaces in de Sitter space [1, 5, 10, 12, 14, 17, 18].

In this paper we will report on our recent studies on curvature properties of space-like hypersurfaces in de Sitter space. By curvatures here we mean the higher order mean curvatures of the hypersurface, as well as its scalar and Ricci curvatures. For further details we refer the reader to the original papers [2] and [3]. In particular, for the case of complete space-like hypersurfaces we have obtained the following (Theorem 1 in [2]).

**Theorem 1.** *Let* $\psi\colon M^n \to \mathbb{S}_1^{n+1} \subset \mathbb{L}^{n+2}$ *be a complete space-like hypersurface in de Sitter space whose sectional curvatures are bounded away from* $-\infty$. *If* $\psi(M)$ *is contained in the region*

$$\Omega(a,r) = \{x \in \mathbb{S}_1^{n+1} \colon \langle a, x\rangle \leq -\sinh(r) < 0\}$$

*for a time-like direction* $a \in \mathbb{L}^{n+2}$ *and a positive real number* $r > 0$, *then for each* $j = 1, \ldots, n$, *the* $j$*-th mean curvature* $H_j$ *of* $M$ *satisfies*

$$\sup H_j \geq \tanh^j(r)\,.$$

Here the sign of the extrinsic $j$-th mean curvatures (when $j$ is odd) is the one given by the orientation of $M$ determined by the Gauss map $N$ which is in the same time-orientation that the time-like direction $a$.

Our study in [2] was motivated by the recent papers of Leung [9] and Erdoĝan [6], where they established interesting estimates for the Ricci curvature of a complete hypersurface contained in a geodesic ball of a Riemannian space form. In our case, the unbounded regions $\Omega(a,r)$ play, in some sense, the same role as the geodesic balls in the Riemannian space forms. In fact, $\Omega(a,r)$ can be thought as a Lorentzian geodesic ball centered at infinity, which is bounded by a totally umbilical round sphere of radius $\cosh r$ and constant $j$-th mean curvature $h_j(r)$ given by

$$h_j(r) = \tanh^j(r)\,.$$

In particular, the bound for $H_j$ is precisely the value of $h_j(r)$. Since the scalar curvature $S$ and the second mean curvature $H_2$ are related by $S = n(n-1)(1-H_2)$, it follows that under the assumptions of the theorem above

$$\inf S \leq \frac{n(n-1)}{\cosh^2(r)}\,.$$

Even more, we were also able to estimate the Ricci curvature of the hypersurface as follows (Theorem 2 in [2]).

**Theorem 2.** *Let* $\psi\colon M^n \to \mathbb{S}_1^{n+1} \subset \mathbb{L}^{n+2}$ *be a complete space-like hypersurface in de Sitter space whose sectional curvatures are bounded away from* $-\infty$. *If* $\psi(M)$ *is contained in the region*

$$\Omega(a,r) = \{x \in \mathbb{S}_1^{n+1} \colon \langle a, x\rangle \leq -\sinh(r) < 0\}$$

*for a time-like direction* $a \in \mathbb{L}^{n+2}$ *and a positive real number* $r > 0$, *then*

$$\inf \mathrm{Ric} = \inf_{\substack{p \in M\\ v \in T_pM\\ |v|=1}} \mathrm{Ric}_p(v,v) \leq \frac{n-1}{\cosh^2(r)}\,.$$

Observe again that the bounds for the scalar and Ricci curvatures of the hypersurface $M$ coincide with the values of the corresponding curvatures for the boundary of the region $\Omega(a, r)$ where $\psi(M)$ is contained.

On the other hand, for the case of compact space-like hypersurfaces in $\mathbb{S}_1^{n+1}$, we have characterized the totally umbilical round spheres in terms of some appropriate bounding conditions for their curvatures. Recall that the Gauss map of a space-like hypersurface in de Sitter space can be seen as a map $N\colon M^n \to \mathbb{H}^{n+1}$, where $\mathbb{H}^{n+1}$ denotes the $(n+1)$-dimensional hyperbolic space, that is

$$\mathbb{H}^{n+1} = \{x \in \mathbb{L}^{n+2} \colon \langle x, x \rangle = -1\}.$$

The image $N(M)$ is then called the **hyperbolic image** of $M$.

In [4] the second author established a sufficient condition for a compact space-like hypersurface $M$ in de Sitter space to be spherical in terms of a pinching condition for the Ricci curvature, involving also the size of its hyperbolic image. Specifically, he proved that if the hyperbolic image of $M$ is contained in a geodesic ball in $\mathbb{H}^{n+1}$ of radius $\varrho$ and its Ricci curvature satisfies $\mathrm{Ric} \leq (n-1)/\cosh^2(\varrho)$, then $M$ must be a round sphere of radius $\cosh(\varrho)$. As for the case of the mean curvature, we have obtained the following result (Theorem 1 in [3]).

**Theorem 3.** *Let $\psi\colon M^n \to \mathbb{S}_1^{n+1} \subset \mathbb{L}^{n+2}$ be a compact space-like hypersurface in de Sitter space such that its hyperbolic image is contained in a geodesic ball in $\mathbb{H}^{n+1}$ of radius $\varrho \geq 0$. If the mean curvature $H$ of $M$ satisfies*

$$H^2 \geq \tanh^2(\varrho),$$

*then $M$ must be a totally umbilical round sphere of radius $\cosh(\varrho)$.*

As an application of this, we can extend the main result in [4] as follows.

**Theorem 4.** *Let $\psi\colon M^n \to \mathbb{S}_1^{n+1} \subset \mathbb{L}^{n+2}$ be a compact space-like hypersurface in de Sitter space such that its hyperbolic image is contained in a geodesic ball in $\mathbb{H}^{n+1}$ of radius $\varrho \geq 0$. If the scalar curvature $S$ of $M$ satisfies*

$$S \leq \frac{n(n-1)}{\cosh^2(\varrho)},$$

*then $M$ must be a totally umbilical round sphere of radius $\cosh(\varrho)$.*

For further applications and related results, we refer the reader to [3].

## 2. Preliminaries

Let $\mathbb{L}^{n+2}$ be the $(n+2)$-dimensional Lorentz–Minkowski space, that is, the real vector space $\mathbb{R}^{n+2}$ endowed with the Lorentzian metric tensor $\langle , \rangle$ given by

$$\langle v, w \rangle = \sum_{i=1}^{n+1} v_i w_i - v_{n+2} w_{n+2} \,,$$

and let $\mathbb{S}_1^{n+1} \subset \mathbb{L}^{n+2}$ be the $(n+1)$-dimensional unitary de Sitter space, that is,

$$\mathbb{S}_1^{n+1} = \{ x \in \mathbb{L}^{n+2} \colon \langle x, x \rangle = 1 \} \,.$$

As is well-known, for $n \geq 2$ the de Sitter space $\mathbb{S}_1^{n+1}$ is the standard simply connected Lorentzian space form of positive constant sectional curvature. A smooth immersion $\psi \colon M^n \to \mathbb{S}_1^{n+1} \subset \mathbb{L}^{n+2}$ of an $n$-dimensional connected manifold $M$ is said to be a **space-like hypersurface** if the induced metric via $\psi$, which we will also denote by $\langle , \rangle$, is a Riemannian metric on $M$. As is usual, the space-like hypersurface is said to be **complete** if the Riemannian induced metric is a complete metric on $M$.

First of all, let us remark that every space-like hypersurface in de Sitter space is orientable, so that there exists a time-like unit normal field $N$ globally defined on $M$. Actually, if $a \in \mathbb{L}^{n+2}$ is a constant unit time-like vector on $\mathbb{L}^{n+2}$, then there exists a unique time-like unit normal field $N$ on $M$ which in the same time-cone as $a$. We will refer to $N$ as the **Gauss map** of the immersion in the time-orientation given by $a$, and we will say that $M$ is oriented by $N$.

Associated to $N$ there is the shape operator of the hypersurface, $A \colon \mathcal{X}(M) \to \mathcal{X}(M)$, given by $A = -dN$. We will denote by $k_1, \ldots, k_n$ the principal curvatures of $M$, and by

$$\sigma_j(k_1, \ldots, k_n) = \sum_{i_1 < \cdots < i_j} k_{i_1} \cdots k_{i_j} \,, \quad 1 \leq j \leq n \,,$$

their elementary symmetric functions. Then we define the **$j$-th mean curvature** $H_j$ of the space-like hypersurface by

$$\binom{n}{j} H_j = (-1)^j \sigma_j(k_1, \ldots, k_n) = \sigma_j(-k_1, \ldots, -k_n) \,. \tag{1}$$

When $j = 1$, $H_1 = -(1/n)\,\mathrm{Tr}(A) = H$ is the mean curvature of $M$. The choice of the sign $(-1)^j$ in our definition of $H_j$ is motivated by the fact that in that case the mean curvature vector is given by $\vec{H} = HN$. Therefore, $H(p) > 0$ at a point $p \in M$ if and only if $\vec{H}(p)$ is in the time-orientation determined by $N(p)$, that is, if $\langle \vec{H}, N \rangle < 0$.

On the other hand, when $j = n$, $H_n = (-1)^n \det(A)$ defines the **Gauss–Kronecker curvature** of the space-like hypersurface, and for $j = 2$, $H_2$ is, up to a constant, the scalar curvature $S$ of $M$. Indeed, the Ricci curvature of $M$ is given by

$$\mathrm{Ric}(X,Y) = (n-1)\langle X,Y\rangle - \mathrm{Tr}(A)\langle A(X),Y\rangle + \langle A(X),A(Y)\rangle\,, \tag{2}$$

for $X, Y \in \mathcal{X}(M)$, so that its scalar curvature is

$$S = \mathrm{Tr}(\mathrm{Ric}) = n(n-1) - \mathrm{Tr}(A)^2 + \mathrm{Tr}(A^2) = n(n-1)(1-H_2)\,. \tag{3}$$

## 3. Complete Space-like Hypersurfaces

Our results for the case of complete hypersurfaces will be an application of the following generalized maximum principle for Riemannian manifolds given by Omori [13] (see also Yau's paper [16]).

A **generalized maximum principle.** *Let $M$ be a complete Riemannian manifold whose sectional curvatures are bounded away from $-\infty$ and let $u\colon M \to \mathbb{R}$ be a smooth function bounded from above. Then, for each $\varepsilon > 0$ there exists a point $p_\varepsilon \in M$ such that*

i) $|\nabla u(p_\varepsilon)| < \varepsilon$,
ii) $\nabla^2 u_{p_\varepsilon}(v,v) < \varepsilon$, *for all tangent vector* $v \in T_pM$, $|v| = 1$,
iii) $\sup u - \varepsilon < u(p_\varepsilon) \leq \sup u$,

*where $\nabla u$ and $\nabla^2 u$ denote, respectively, the gradient and the Hessian of $u$.*

Indeed, the idea of the proof of our Theorems 1 and 2 is to apply this generalized maximum principle to the function $u = \langle a, \psi\rangle$, which is bounded from above on $M$ by $-\sinh(r)$. Then, for each $\varepsilon_m = 1/m$ we find a point $p_m \in M$ such that

$$|\nabla u(p_m)| < \varepsilon_m, \quad \text{and} \quad \nabla^2 u_{p_m}(v,v) < \varepsilon_m\,, \tag{4}$$

for all unit tangent vector $v \in T_{p_m}M$, and satisfying

$$\sup u - \varepsilon_m < u(p_m) \leq \sup u\,. \tag{5}$$

It is easy to see that the gradient and the Hessian of $u$ are respectively given by

$$\nabla u = a^T = a + \langle a, N\rangle N - \langle a,\psi\rangle\psi\,, \tag{6}$$

and

$$\nabla^2 u(X,X) = -\langle a,N\rangle\langle A(X),X\rangle - \langle a,\psi\rangle\langle X,X\rangle\,.$$

Therefore, if $\{e_i^m\}_{i=1,\ldots,n}$ is an orthonormal basis of principal directions at the point $p_m$ satisfying $A_{p_m}(e_i^m) = k_i(p_m)e_i^m$, then one obtains that

$$\nabla^2 u(e_i^m, e_i^m) = -\langle a, N(p_m)\rangle k_i(p_m) - u(p_m) < \varepsilon_m \,,$$

and, since $\langle a, N(p_m)\rangle < 0$, it follows that

$$k_i(p_m) < \frac{u(p_m) + \varepsilon_m}{-\langle a, N(p_m)\rangle} \,.$$

From (5) we know that $\lim_{m\to\infty} u(p_m) + \varepsilon_m = \sup u \leq -\sinh(r) < 0$, so that $k_i(p_m)$ is negative for sufficiently large $m$. Let us assume from now on that $m$ is large enough such that $k_i(p_m) < 0$. On the other hand, from (6) we get that

$$-1 = \langle a, a\rangle = |\nabla u|^2 - \langle a, N\rangle^2 + u^2 \,,$$

and using (4), we can deduce that

$$-\langle a, N(p_m)\rangle = \sqrt{1 + u^2(p_m) + |\nabla u(p_m)|^2} < \sqrt{1 + u^2(p_m) + \varepsilon_m^2} \,.$$

From here, one concludes that

$$k_i(p_m) < \frac{u(p_m) + \varepsilon_m}{\sqrt{1 + u^2(p_m) + \varepsilon_m^2}} < 0 \,.$$

In particular,

$$\binom{n}{j} H_j(p_m) \geq \binom{n}{j} \left( \frac{-u(p_m) - \varepsilon_m}{\sqrt{1 + u^2(p_m) + \varepsilon_m^2}} \right)^j ,$$

and

$$\begin{aligned}
\mathrm{Ric}_{p_m}(e_j^m, e_j^m) &= (n-1) - \sum_{i=1}^{n} k_i(p_m)k_j(p_m) + k_j^2(p_m) \\
&= (n-1) - \sum_{i\neq j} k_i(p_m)k_j(p_m) \\
&\leq (n-1) - (n-1) \left( \frac{u(p_m) + \varepsilon_m}{\sqrt{1 + u^2(p_m) + \varepsilon_m^2}} \right)^2 .
\end{aligned}$$

Finally, the results follow letting $m \to \infty$, since $\lim\limits_{m\to\infty} u(p_m) = \sup(u) \leq -\sinh(r)$.

## 4. Compact Space-like Hypersurfaces

In the case of compact hypersurfaces, our Theorems 3 and 4 are a consequence of the following stronger result.

**Theorem 5.** *Let* $\psi\colon M^n \to \mathbb{S}_1^{n+1} \subset \mathbb{L}^{n+2}$ *be a compact space-like hypersurface in de Sitter space,* $a \in \mathbb{L}^{n+2}$ *a time-like direction and* $r \geq 0$*, verifying that* $\langle a, \psi\rangle^2 \leq \sinh^2(r)$*. If the mean curvature* $H$ *of* $M$ *satisfies*

$$H^2 \geq \tanh^2(r)\,,$$

*then* $M$ *must be a totally umbilical round sphere of radius* $\cosh(r)$*.*

In other words, if $\psi(M)$ is contained in the region

$$B(a,r) = \{x \in \mathbb{S}_1^{n+1};\ -\sinh(r) \leq \langle a, x\rangle \leq \sinh(r)\}\,,$$

whose boundaries are two totally umbilical round spheres with radius $\cosh(r)$ and constant mean curvature $h^2(r) = \tanh^2(r)$, and the mean curvature of $M$ satisfies $H^2 \geq h^2(r)$, then $M$ must be one of these two round spheres.

Similar results were obtained by Markvorsen in [11] for the mean curvature of compact hypersurfaces bounded in a geodesic ball of a Riemannian space, generalizing the classical result of Koutroufiotis [8] on surfaces in Euclidean 3-space. More recently, Vlachos [15] has obtain similar results for the case of the higher order mean curvatures of compact hypersurfaces bounded in a geodesic ball of a Riemannian space.

The proof of Theorem 5 has two main steps. The first step consists on proving the result under the additional hypothesis that the function $\langle a, \psi\rangle$ does not vanish on $M$. Assume, for instance, that $\langle a, \psi\rangle$ is negative on $M$. Then, using some analysis we conclude that $\Delta\langle a, \psi\rangle \leq 0$ on $M$, which by Hopf–Bochner theorem implies that $\langle a, \psi\rangle$ is constant on $M$, yielding the result (for the details, see the proof of Proposition 6 and Corollary 7 in [3]). The second step consists on proving that (in the non-trivial case, that is, when $r \neq 0$) the function $\langle a, \psi\rangle$ actually does not vanish on $M$. Actually, if this is not the case then there exist non-negative real numbers $r_1, r_2$, not both vanishing, such that

$$-r \leq -r_1 \leq 0 \leq r_2 \leq r\,,$$

and there exist points $p_{\max}, p_{\min} \in M$ satisfying

$$\max_{p\in M} \langle a, \psi(p)\rangle = \langle a, \psi(p_{\max})\rangle = \sinh(r_1)$$
$$\min_{p\in M} \langle a, \psi(p)\rangle = \langle a, \psi(p_{\min})\rangle = -\sinh(r_2)\,.$$

Besides, we also know from Lemma 5 in [3] that

$$\min H \leq \tanh(r_2) \tag{7}$$

and

$$\max H \geq -\tanh(r_1)\,. \tag{8}$$

On the other hand, the hypothesis on $H$ implies that either

$$H \geq \tanh(r) \tag{9}$$

or

$$H \leq -\tanh(r)\,. \tag{10}$$

We may assume without loss of generality that $r_1 \geq r_2$, that is, $r \geq r_1 \geq r_2 \geq 0$, and moreover $r_1 > 0$. Otherwise, just replace $a$ by $-a$. If (9) holds, then from equation (7) it follows that $r_2 = r = r_1 > 0$. Since $\langle a, \psi(p_{\min})\rangle = -\sinh(r) < 0$, then there exists a neighbourhood $\mathcal{U}$ of $p_{\min}$ where $\langle a, \psi\rangle$ is negative. Using that $-\sinh(r) \leq \langle a, \psi\rangle$ on $M$, at each point $p \in M$ we have from (9)

$$H(p) \geq \tanh(r) \geq \frac{-\langle a, \psi(p)\rangle}{\sqrt{1 + \langle a, \psi(p)\rangle^2}}\,,$$

which is strictly positive on $\mathcal{U}$. Since $-\langle a, N\rangle \geq \sqrt{1 + \langle a, \psi\rangle^2}$ on $M$ (see equation (6)), then

$$\Delta\langle a, \psi\rangle \leq -n\langle a, \psi\rangle \left(1 + \frac{\langle a, N\rangle}{\sqrt{1 + \langle a, \psi\rangle^2}}\right) \leq 0 \quad \text{on} \quad \mathcal{U}\,.$$

Therefore, by the maximum principle, $\langle a, \psi\rangle$ is constant on $\mathcal{U}$. Since $M$ is connected, this implies that $\langle a, \psi\rangle$ is constant on $M$, in contradiction with the fact that $\min_{p\in M}\langle a, \psi\rangle = -\max_{p\in M}\langle a, \psi(p)\rangle < 0$.

On the other hand, if (10) holds, then from (8) it follows that $r_1 = r$. Moreover, since $\langle a, \psi(p_{\max})\rangle = \sinh(r) > 0$, there exists a neighbourhood $\mathcal{V}$ of $p_{\max}$ where $\langle a, \psi\rangle$ is positive. Using that $\langle a, \psi\rangle \leq \sinh(r)$ on $M$, we conclude from (10) that

$$H \leq -\tanh(r) \leq \frac{-\langle a, \psi\rangle}{\sqrt{1 + \langle a, \psi\rangle^2}} < 0 \quad \text{on} \quad \mathcal{V}\,.$$

Reasoning as above we obtain now that $\Delta\langle a, \psi\rangle \geq 0$ on $\mathcal{V}$, and that the function $\langle a, \psi\rangle$ is constant on $M$ and equal to $\sinh(r) > 0$, in contradiction to the fact that its minimum is non-positive. This finishes the proof of Theorem 5.

Once we have got Theorem 5, we can easily derive Theorems 3 and 4. Indeed, assume the hyperbolic image of $M$ is contained in a geodesic ball $\tilde{B}(a,\varrho)$ in $\mathbb{H}^{n+1}$ of radius $\varrho \geq 0$ centered at $a \in \mathbb{H}^{n+1}$, $\langle a,a\rangle = -1$. Recall that

$$\tilde{B}(a,\varrho) = \{q \in \mathbb{H}^{n+1};\, 1 \leq \langle a,q\rangle^2 \leq \cosh^2(\varrho)\}\,,$$

so that $\langle a,N(p)\rangle^2 \leq \cosh^2(\varrho)$ at each $p \in M$, which gives

$$\langle a,\psi\rangle^2 \leq \langle a,N\rangle^2 - 1 \leq \sinh^2(\varrho)$$

on $M$. Therefore, Theorem 3 follows directly from Theorem 5. To obtain Theorem 4, simply observe that by the Cauchy–Schwarz inequality we have that $H^2 \geq H_2$, so that

$$S \geq n(n-1)(1-H^2)\,,$$

since $S = n(n-1)(1-H_2)$. Thus, if $S \leq n(n-1)/\cosh^2(\varrho)$ then $H^2 \geq \tanh^2(\varrho)$ and we can apply Theorem 3.

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