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DIAGONALIZATION OF HAMILTONIANS, UNCERTAINTY MATRICES AND ROBERTSON INEQUALITY

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Abstract. The problem of diagonalization of Hamiltonians of N-dimensional boson systems by means of time-dependent canonical transformations (CT) is considered, the case of quadratic Hamiltonians being treated in greater detail. The unitary generator of time-dependent CT which can transform any Hamiltonian to that of a system of uncoupled stationary oscillators is constructed. The close relationship between methods of canonical transformations, time-dependent integrals of motion and dynamical symmetry is noted.

The diagonalization and symplectic properties of the uncertainty matrix for $2N$ canonical observables are studied. It is shown that the normalized uncertainty matrix is symplectic for the squeezed multimode Glauber coherent states and for the squeezed Fock states with equal photon numbers in each mode. The Robertson uncertainty relation for the dispersion matrix of canonical observables is shown to be minimized in squeezed coherent states only.

1. Introduction

The method of canonical transformations (CT) proved to be a fruitful approach in treating quantum systems. It is most efficient for systems that are described by Hamiltonians, that are quadratic in coordinates and momenta, or equivalently in boson creation and annihilation operators (quadratic Hamiltonians). The main advantage of the method of CT consists in reducing the Hamiltonian H of the treated system $\mathcal{S}$ to a Hamiltonian H' of some simple system $\mathcal{S}'$ with known solutions. The well known example (and probably the first one) of such an application is the diagonalization of the model quadratic Hamiltonians in superfluidity and superconductivity theory by means of linear time-independent

transformations of boson or fermion operators (the Bogolyubov transforms) [1]. In [2] time-dependent CT for quadratic systems were used (probably for the first time) in construction of integrals of motion that are linear in coordinates and momenta.

Quadratic Hamiltonians model many quantum (and classical) systems: from free particle and free electromagnetic field to the waves in nonlinear media, molecular dynamics and gravitational waveguides [3–6]. A considerable attention to quadratic classical and/or quantum systems is paid in the literature for a long period of time (see, for example, [3, 6–9] and references therein).

Diagonalization problem of quadratic Hamiltonians is considered in a number of papers [8, 10–14]. In general, quadratic Hamiltonians can not be diagonalized by means of time-independent CT, even in the one-dimensional case [11, 13]. In the one-dimensional case the term proportional to the product of coordinate and momentum can be eliminated by a time-dependent CT only. For this purpose a time-dependent point transformation (i. e. scale or squeeze transformation) is sufficient [11]. Time-dependent CT are very powerful. Seleznyova [8] has shown that the Hamiltonian of a nonstationary quantum oscillator can always be brought to the diagonal form of that of the stationary harmonic oscillator by means of linear time-dependent CT.

The aim of the present paper is to establish the canonical equivalence of N-dimensional quantum systems and to perform it explicitly in the case of systems with quadratic Hamiltonians. Two systems are called **canonically equivalent** if their Hamiltonians can be related by means of a CT. Due to the well-known von Neumann theorem CT in quantum mechanics are generated by unitary operators. Therefore canonical equivalence is in fact unitary one. A second aim of the present paper is to consider the symplectic properties of the uncertainty matrix for canonical observables and its diagonalization using linear CT [15, 16].

The paper is organized as follows. In Section 2 we show that any two N-dimensional quantum Hamiltonians (time-dependent, in general) $H(t)$ and $H'(t)$ can be canonically related via time-dependent unitary operator $U(t)$. The group of CT which leave H invariant (i. e. $H = H'$) is shown to coincide with the dynamical symmetry group of the system. In the case of two quadratic Hamiltonians the operator $U(t)$ is an exponent of a quadratic form in coordinates and momenta (that is, an element the metaplectic group $Mp(N, \mathbb{R})$). In particular, with such operators one can diagonalize any quadratic Hamiltonian. We note that there are two types of diagonalizations depending on the type of the canonical variables in which the target Hamiltonian is diagonal.

In Section 3 we perform the diagonalization of N-dimensional quadratic Hamiltonian, expressing the parameters of the corresponding linear CT in terms of the solutions of linear first order differential equations. For $N = 1$ these equa-

tions are reduced to the equation $\ddot{z} + \Omega^2(t)z = 0$ of the classical oscillator with varying frequency. The relation of CT to the linear integrals of motion is briefly discussed.

In Section 4 the main properties of the uncertainty matrix σ for N observables are considered. It is shown that for canonical observables the uncertainty matrix is positive definite and thus (due to the known theorem by Williamson [17, 12]) can be diagonalized by means of linear CT. For squeezed canonical coherent states (CS) [4] and for squeezed Fock states with equal boson/photon numbers in every mode the matrix σ (when normalized to unity) is found to be symplectic itself. The symplectic character of the normalized uncertainty matrix in squeezed CS can also be inferred from the results of paper [18].

2. Unitary Equivalence of Quantum Systems

The principal aim in the method of CT is to reduce the Hamiltonian H of the treated system $\mathcal{S}$ to a Hamiltonian H' of some simple system $\mathcal{S}'$ with known solutions. The CT in quantum theory is generated by an unitary operator U, which is called the generator of CT. If CT is *time-independent* then H and H' are unitary equivalent and their spectra are the same. However not any pair H and H' can be related by means of time-independent CT. In particular, not any quadratic Hamiltonian can be reduced to that of a harmonic oscillator by means of time-independent CT [11–14], even in the one-dimensional case [11]. The *time-dependent* CT are much more powerful as we shall see below.

Let $|\Psi(t)\rangle$ be a solution of the Schrödinger equation $[\mathrm{i}\hbar\partial/\partial t - H]|\Psi(t)\rangle = 0$. Then for any unitary operator $U(t)$ the transformed state $|\Psi'(t)\rangle$, $|\Psi'(t)\rangle = U(t)|\Psi(t)\rangle$, obeys the equation $[\mathrm{i}\hbar\partial/\partial t - H']|\Psi'(t)\rangle = 0$ with the new Hamiltonian H',

$$H' = U(t)HU^\dagger(t) - \mathrm{i}\hbar U(t)\partial U^\dagger(t)/\partial t\,. \tag{2.1}$$

Conversely, if two Hamiltonians H and H' are related by means of an (unitary) operator $U(t)$ in accordance with Eq. (2.1) then any solution $|\Psi(t)\rangle$ of the system $\mathcal{S}$ is mapped into a solution $|\Psi'(t)\rangle$ of the system $\mathcal{S}'$. However, not any two given solutions $|\Psi(t)\rangle$ and $|\Psi'(t)\rangle$ of the two systems could be mapped into each other by means of $U(t)$ since $U(t)$ in general cannot act transitively in the Hilbert space. A more compact form of the relation (2.1) is $D'(t) = U(t)D(t)U^\dagger(t)$, where $D(t) = \mathrm{i}\hbar\partial/\partial t - H(t)$. $U(t)$ is interwinding operator for $D(t)$ and $D'(t)$. When $D(t)$ and $D'(t)$ act in the same Hilbert space one says that $D(t)$ and $D'(t)$ are unitary equivalent. $D(t)$ is often called Schrödinger operator.

From the requirement for the mean values of the "old" operator A and the "new" one A',

$$\langle \Psi(t) \,|A|\, \Psi(t) \rangle = \langle \Psi'(t) \,|A'|\, \Psi'(t) \rangle \,,$$

it follows that the operators A and A' are related as $A' = U(t)AU^{\dagger}(t)$. Therefore the new canonical operators of the coordinates and momenta q'_k and $p'_k, k = 1, \ldots, N$ are related to the old ones as

$$q'_k = U(t)q_kU^{\dagger}(t), \quad p'_k = U(t)p_kU^{\dagger}(t) \,. \tag{2.2}$$

Two quantum systems should be called *canonically* or **unitary equivalent** if their Schrödinger operators are unitary equivalent. The corresponding Hamiltonian operators H and H', related in accordance with Eq. (2.1), should be called *canonically equivalent* with respect to $U(t)$. Let us note the main three advantages of establishing unitary equivalence of two systems (see also [8], where in fact canonical equivalence of one dimensional oscillators with constant and time-dependent frequencies was considered):

a) If we know solutions $|\Psi\rangle$ for one of the two canonically related systems we can obtain solutions for the other one as $U(t)|\Psi\rangle$.
b) If a time-dependent state $|\Psi'(t)\rangle$ of the system $\mathcal{S}'$ is an eigenstate of an operator A' then its $U(t)$-partner $|\Psi(t)\rangle = U^{\dagger}(t)|\Psi'(t)\rangle$ in the system $\mathcal{S}$ is an eigenstate of the operator $A = U^{\dagger}(t)A'U(t)$.
c) If the operator A' is an integral of motion for $\mathcal{S}'$, i. e. if A' commutes with the Schrödinger operator, $\partial A'/\partial t - (\mathrm{i}/\hbar)[A', H'] = 0$, then the operator $A = U^{\dagger}(t)A'U(t)$ is an integral of motion for the old system $\mathcal{S}$,

$$\partial A/\partial t - (\mathrm{i}/\hbar)[A, H] = 0 \,. \tag{2.3}$$

This property is very important since if we know one solution for a given system $\mathcal{S}$ we can construct new solutions acting by the invariant operators on the known solution.

Proposition 2.1. *Any two N-dimensional quantum Hamiltonians H and H' are canonically equivalent. The unitary operator $U(t)$, that relates H and H' takes the form*

$$\begin{aligned} U(t) &= T \exp\left[-\frac{\mathrm{i}}{\hbar} \int_{t_0}^{t} H'(t)\,\mathrm{d}t \right] U_0 \tilde{T} \exp\left[\frac{\mathrm{i}}{\hbar} \int_{t_0}^{t} H(t)\,\mathrm{d}t \right] \\ &\equiv S'(t)U_0S^{\dagger}(t) \,, \end{aligned} \tag{2.4}$$

where U_0 is constant unitary operator and T and $\tilde{T}$ stand for the chronological and anti-chronological product. The solution (2.4) *is unique for any initial condition $U(0) = U_0$.*

Proof: Let us perform two successive time-dependent CTs by means of $U_1 = U_0 S^\dagger(t)$ and $U_2 = S'(t)$,

$$S^\dagger(t) = \tilde{T} \exp\left[\frac{\mathrm{i}}{\hbar}\int^t H(t)\,\mathrm{d}t\right]. \tag{2.5}$$

From Eq. (2.1) (and taking into account $\partial U_1^\dagger(t)/\partial t = (-\mathrm{i}/\hbar)HU_1^\dagger$) we easily get $H_1 = 0$ for any U_0. The second transformation by means of $U_2 = S'(t)$,

$$S'(t) = T \exp\left[-\frac{\mathrm{i}}{\hbar}\int^t H'(t)\,\mathrm{d}t\right], \tag{2.6}$$

then yields the required result $\left(\partial U_2^\dagger/\partial t = \partial S'^\dagger/\partial t = (\mathrm{i}/\hbar)S'H'\right)$:

$$H_2 = U_2 H_1 U_2^\dagger - \mathrm{i}\hbar U_2 \partial U_2^\dagger/\partial t = -\mathrm{i}\hbar S' \partial S'^\dagger/\partial t = H'. \tag{2.7}$$

Now we see that the direct CT: $H \to H'$ is performed by the unitary operator (2.4).

For a given H and H' the interwinding operator $U(t)$ is not unique. However the time-dependence of $U(t)$ is uniquely determined by any initial condition $U(0) = U_0$. Indeed, suppose that there is another unitary operator $\tilde{U}(t)$, which also relates H and H' canonically and $\tilde{U}(0) = U_0$. Now we note (which is easily derived from (2.1)) that if $\tilde{U}$ transforms H into H' then $\tilde{U}^\dagger$ transforms H' back into H and therefore the product $V \equiv \tilde{U}^\dagger U$ keeps H invariant:

$$H = VHV^\dagger + \frac{\mathrm{i}}{\hbar}[\partial V/\partial t]V^\dagger \quad \text{and} \quad V(0) = 1\,.$$

On the other hand, by using Eq. (2.1) for U and $\tilde{U}$, one obtains the equality $\partial V/\partial t - (\mathrm{i}/\hbar)[V, H] = 0$, which means that V is an integral of motion for the system $\mathcal{S}$. Any invariant operator for H has the form (note that $S(t)$ is the evolution operator for $\mathcal{S}$) $V(t) = S(t)V(0)S^\dagger(t)$, and since $V(0) = 1$ we have $V(t) = \tilde{U}^\dagger U = S(t)S^\dagger(t) = 1$. In a similar way one can get $U(t)\tilde{U}^\dagger(t) = 1$. And if $\tilde{U}^\dagger U = 1 = U\tilde{U}^\dagger$, then $U = \tilde{U}$ (because of the uniqueness of the inverse U^{-1}). □

Let us note that $H(0) = H'(0)$ is not necessarily true as we have

$$H'(0) = U_0 H U_0^\dagger - \mathrm{i}\hbar U(0)[\partial U^\dagger/\partial t]_{t=0}\,.$$

Suppose now that $H(t)$ and $H'(t)$ are elements of the Lie algebra $\mathcal{L}$. Then $S \in G \ni S'$, where G is the Lie group generated by $\mathcal{L}$. Thus, the CT generator $U(t) \in G$ (for $U_0 = 1$ and for $U_0 \in G$ as well) and one can use the known properties of G to represent $U(t)$ in other factorized forms.

The operator (2.4) converts canonically any N-dimensional H into any desired N-dimensional H'. In particular H can be converted into H' for a system of N free particles or for a system of uncoupled harmonic oscillators (N-mode free boson field). In the latter case if H is a quadratic form in terms of N canonical operators q_k and p_j the operator (2.4) solves the *diagonalization problem* for quadratic Hamiltonians.

A CT will be called diagonalizing if the new Hamiltonian H' in terms of the coordinates and momenta is diagonal quadratic form with *constant* coefficients, i. e. H' is a Hamiltonian for a system of uncoupled harmonic oscillators $H_{\rm Ho}$. One has to distinguish between two different kinds of diagonalization of H:

First kind diagonalization: H' is diagonal in terms of the new variables q'_j, p'_k,

Second kind diagonalization: H' is diagonal in terms of the old variables q_j, p_k.

In the first case the two systems $\mathcal{S}$ and $\mathcal{S}'$ are treated in two different (q- and q'-) coordinate representations (wave functions $\Psi(q,t) = \langle q|\Psi(t)\rangle$ and $\Psi'(q',t) = \langle q'|\Psi'(t)\rangle$), whereas in the second case one can work in the same q-representation (wave functions $\Psi(q,t) = \langle q|\Psi(t)\rangle$ and $\Psi'(q,t) = \langle q|\Psi'(t)\rangle$). The second kind diagonalization is achieved by means of operator $U(t)$, Eq. (2.4), with H' of the form of Hamiltonian of N uncoupled stationary oscillators (in terms of old variables),

$$H' = \frac{1}{2}\sum_k^N \left[\frac{1}{m_k}p_k^2 + m_k\omega_k^2 q_k^2\right] \equiv H_{\rm Ho}(p,q)\,, \tag{2.8}$$

The target Hamiltonian H' may also be taken as a sum of stationary oscillators $H_{\rm Ho}$ in terms of the intermediate variables $q_k^{(1)}, p_k^{(1)}$ as well. In the latter case the second CT $(q_k^{(1)}, p_k^{(1)}) \to (q'_k, p'_k)$, generated by $U_2(t) = \exp[-(\mathrm{i}/\hbar)H_{\rm Ho}(q^{(1)}, p^{(1)})t] = S_{\rm Ho}(t)$, takes the explicit form of rotations

$$\begin{aligned} q'_k &= q_k^{(1)}\cos(\omega_k t) + \frac{1}{m_k\omega_k}p_k^{(1)}\sin(\omega_k t)\,,\\ p'_k &= -m_k\omega_k q_k^{(1)}\sin(\omega_k t) + p_k^{(1)}\cos(\omega_k t)\,. \end{aligned} \tag{2.9}$$

Let us briefly elucidate the two CT involved into the Proposition 2.1. The first one, generated by $U_1 = U_0 S^\dagger(t)$, brings H to zero, therefore the new states $|\Psi\rangle_1$ are time-independent. This is because $S^\dagger(t)$ is an evolution operator for the $\mathcal{S}$ backward in time. After the first CT (generated by U_1) the new canonical variables $q_k^{(1)} = U_1(t)q_k U_1^\dagger(t)$ and $p_j^{(1)} = U_1(t)p_j U_1^\dagger(t)$ obey the equations ($\partial U_1/\partial t = \mathrm{i}U_1 H$, $\partial U_1^\dagger/\partial t = -\mathrm{i}HU_1^\dagger$)

$$\frac{\partial q_k^{(1)}}{\partial t} = \frac{\mathrm{i}}{\hbar}[U_1 H U_1^\dagger, q_k^{(1)}]\,, \qquad \frac{\partial p_k^{(1)}}{\partial t} = \frac{\mathrm{i}}{\hbar}[U_1 H U_1^\dagger, p_k^{(1)}]\,, \tag{2.10}$$

i. e., $q_k^{(1)}, p_k^{(1)}$ are Heisenberg operators for the old system $\mathcal{S}$.

The generator of the second CT $U_2(t) = S'(t)$ is recognized as the evolution operator forward in time for the target system $\mathcal{S}'$. In the construction (2.4) U_2 is applied to the intermediate Hamiltonian H_1.

It is worth noting at this point the case of CT for the system $\mathcal{S}$, generated by its own evolution operator $S(t)$. This CT converts $H(t)$ into Hamiltonian $H''(t) = S(t)H(t)S^\dagger(t) + H(t)$. If H is time-independent then $S(t)HS^\dagger(t) = H$ and $H'' = 2H$. From $\langle\Psi(t)|A|\Psi(t)\rangle = \langle\Psi|S^\dagger(t)AS(t)|\Psi\rangle$ we derive that the new canonical variables in this case,

$$q_k'' = S(t)q_kS^\dagger(t) \equiv q_k^0, \qquad p_k'' = S(t)p_kS^\dagger(t) \equiv p_k^0\,, \tag{2.11}$$

when expressed in terms of the old ones, q_k, p_j, are **integrals of motion** of $\mathcal{S}$, satisfying Eq. (2.3). Such integrals of motions for quadratic systems $H(t)$ have been constructed in [2] and intensively used later [3, 6, 8, 9].

Consider the symmetry of H under CT. We want to specify the set of CT for which H', defined in (2.4), coincides with H, i. e. we look for CT that keep H invariant (and thus keep the Schrödinger equation invariant),

$$H' \equiv U(t)HU^\dagger(t) - \mathrm{i}\hbar U(t)\frac{\partial U^\dagger(t)}{\partial t} = H\,. \tag{2.12}$$

For time-independent U Eq. (2.12) reduces to $H = UHU^\dagger$. From (2.4) for $H' = H$ the CT generator is (see (2.4)) $U(t) = S(t)U_0S^\dagger(t)$, where U_0 is arbitrary unitary operator. Then $\partial U^\dagger(t)/\partial t = (\mathrm{i}/\hbar)[U^\dagger(t), H]$ and we see that the equality in (2.12) is identically satisfied. Thus, the CT generators $U(t)$ for which $H' = H$ have the form $S(t)U_0S^\dagger(t)$, i. e. $U(t)$ are integrals of motion for the system: $[U(t), D(t)] = 0$, where $D(t)$ is the Schrödinger operator, $D(t) = \mathrm{i}\hbar\partial/\partial t - H$. In the first paper of refs. [7] the dynamical symmetry group for the system $\mathcal{S}$ has been defined as the group of unitary operators, that commute with $D(t)$ and act irreducibly in the Hilbert space. Now we see that this symmetry group leaves $H' = H$ and is highly nonunique, since the unitary operator U_0 in $U(t)$ is arbitrary — one can take U_0 from irreducible representations of any Lie group. Then the set of invariants $S(t)U_0S^\dagger(t)$ realize an equivalent representation of the same group. For example, by means of the invariants q_k^0 and p_k^0 one can construct an irreducible representation of the Lie algebra of the Heisenberg–Weyl group $H_W(N)$ and the quasi unitary group $SU(N, 1)$ as well [7]. This means that the groups $H_W(N)$ and $SU(N, 1)$ can be considered on equal as dynamical symmetry groups of any N-dimensional system.

In the next section we consider the above described unitary (canonical) equivalence approach in greater detail for quadratic quantum systems, for which some explicit solutions can be obtained.

3. Canonical Transformations of Quadratic Systems and Diagonalization

We consider the general N-dimensional nonstationary quantum system with Hamiltonian $H(t)$, that is a homogeneous quadratic form of coordinates and momenta,

$$H(t) = \mathcal{A}_{jk}(t)p_j p_k + \mathcal{B}_{jk}(t)p_j q_k + \tilde{\mathcal{B}}_{jk}(t)q_j p_k + \mathcal{C}_{jk}(t)q_j q_k\,, \qquad (3.1)$$

where the coefficients $\mathcal{A}_{jk}(t) = \mathcal{A}_{kj}(t)$, $\mathcal{B}_{jk}(t), \tilde{\mathcal{B}}_{jk}(t)$ and $\mathcal{C}_{jk}(t) = \mathcal{C}_{kj}(t)$ are arbitrary functions of time. From $H^\dagger = H$ it follows that $\mathcal{A}_{jk}(t)$ and $\mathcal{C}_{jk}(t)$ are real, and $\mathcal{B}_{jk}(t) = \tilde{\mathcal{B}}^*_{kj}(t)$. It is not a significant restriction to take $\mathcal{B}_{jk}$ real and put $\mathcal{B}_{jk}(t) = \tilde{\mathcal{B}}_{kj}(t)$ (the imaginary parts of $\mathcal{B}_{jk}$ can be eliminated by adding a non-operator term to H). In (3.1) the summation over the repeated indices is adopted. We can introduce N-component vectors $\vec{q} = (q_1, q_2, \ldots, q_N)$, $\vec{p} = (p_1, p_2, \ldots, p_N)$, $N \times N$ real matrices $\mathcal{A}(t)$, $\mathcal{B}(t)$, $\mathcal{C}(t)$ (where $\mathcal{A}(t)$ and $\mathcal{C}(t)$ are symmetric) and rewrite the Hamiltonian (3.1) in a more compact form

$$H(t) = \vec{p}\mathcal{A}(t)\vec{p} + \vec{p}\mathcal{B}(t)\vec{q} + \vec{q}\mathcal{B}(t)^{\mathsf{T}}\vec{p} + \vec{q}\mathcal{C}(t)\vec{q}\,,$$

where $\mathcal{B}^{\mathsf{T}}$ is the transposed of $\mathcal{B}$. To shorten the notation it is convenient to introduce the $2N$-component vector $\vec{Q} = (\vec{p}, \vec{q})$, $2N \times 2N$ matrix $\mathcal{H}$ (the grand matrix) and rewrite the Hamiltonian (3.1) as

$$H(t) = \mathcal{H}_{\mu\nu}Q_\mu Q_\nu \equiv \vec{Q}\mathcal{H}(t)\vec{Q}\,, \qquad \mathcal{H} = \begin{pmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{B}^{\mathsf{T}} & \mathcal{C} \end{pmatrix}, \qquad (3.2)$$
$$\mu, \nu = 1, 2, \ldots, 2N\,.$$

We note that non-homogeneous quadratic Hamiltonians (i. e., Hamiltonians of the form (3.1), (3.2) with linear terms added) can be easily reduced to the forms (3.1), (3.2) by means of simple time dependent displacement transformations.

Let H' is another quadratic Hamiltonian

$$H'(t) = \vec{Q}\mathcal{H}'(t)\vec{Q}, \qquad \mathcal{H}' = \begin{pmatrix} \mathcal{A}' & \mathcal{B}' \\ \mathcal{B}'^{\mathsf{T}} & \mathcal{C}' \end{pmatrix}. \qquad (3.3)$$

Then the unitary operator $U(t)$, Eq. (2.4), which relates canonically Hamiltonians (3.2) and (3.3), is an exponent of a quadratic in $\vec{q}$ and $\vec{p}$ form (we take

$U_0 \in Mp(N,\mathbb{R})$),

$$U(t) = S'(t)U_0S^{\dagger}(t) = \exp\left[\frac{\mathrm{i}}{\hbar}\vec{Q}\tilde{\mathcal{H}}(t)\vec{Q}\right], \tag{3.4}$$

where $\tilde{\mathcal{H}}(t)$ is a new grand matrix of the form (3.2) and (3.3). $\tilde{\mathcal{H}}(t)$ can be expressed in terms of the Hamiltonian matrices $\mathcal{H}(t)$ and $\mathcal{H}'(t)$ using the Baker-Campbell-Hausdorff formula. In this case the operator (3.4) generates linear transformation of coordinates and momenta (we write it in $N\times N$ and $2N\times 2N$ matrix forms),

$$\vec{Q}' = \Lambda(t)\vec{Q} \quad \text{or} \quad \begin{pmatrix}\vec{p}'\\ \vec{q}'\end{pmatrix} = \begin{pmatrix}\lambda_{pp} & \lambda_{pq}\\ \lambda_{qp} & \lambda_{qq}\end{pmatrix}\begin{pmatrix}\vec{p}\\ \vec{q}\end{pmatrix}, \tag{3.5}$$

where $\lambda_{pp}, \lambda_{pq}, \lambda_{qp}$ and λ_{qq} are $N\times N$ submatrices of $\Lambda(t)$.

From Eqs (2.1), (3.2), (3.3) and (3.5) we obtain the following relation between the symmetric matrices $\mathcal{H}$, $\mathcal{H}'$ and $\tilde{\mathcal{H}}$ (3.4) and the symplectic matrix Λ,

$$\frac{\mathrm{d}}{\mathrm{d}t}\tilde{\mathcal{H}}(t) = -\mathcal{H}'(t) + \Lambda^{\mathsf{T}}\mathcal{H}(t)\Lambda\,. \tag{3.6}$$

We see that for a given $\tilde{\mathcal{H}}(t)$ and $\mathcal{H}(t)$ this is a simple linear equation for $\mathcal{H}'(t)$. However for a given Hamiltonian matrices $\mathcal{H}$ and $\mathcal{H}'$ this is highly nonlinear equation for $\tilde{\mathcal{H}}(t)$ since the matrix $\Lambda(t)$ is to be expressed in terms of $\tilde{\mathcal{H}}(t)$ again: $\Lambda\vec{Q} = U(t)\vec{Q}U^{\dagger}(t)$. Nevertheless for any given (differentiable with respect to t) matrices $\mathcal{H}(t)$ and $\mathcal{H}'(t)$) and for a given initial condition $\tilde{\mathcal{H}}_0$ the above system of equations has unique solution $\tilde{\mathcal{H}}(t)$, since the expression of Λ in terms of $\tilde{\mathcal{H}}$ is also differentiable and Peano theorem could be applied [27].

In this scheme $\Lambda(t)$ is naturally represented as a product of two other $2N\times 2N$ matrices $\Lambda^{(1)}$ and $\Lambda^{(2)}$ of the form (3.5) corresponding to the two successive CT generated by $U_1(t)$ and $U_2(t)$:

$$\Lambda = \Lambda^{(2)}\Lambda^{(1)}, \qquad \vec{Q}^{(1)} = \Lambda^{(1)}\vec{Q}, \qquad \vec{Q}' = \Lambda^{(2)}\vec{Q}^{(1)}. \tag{3.7}$$

The matrices $\Lambda^{(1)}$ and $\Lambda^{(2)}$ are seen to be solutions of the first order linear equations,

$$\frac{\mathrm{d}}{\mathrm{d}t}\Lambda^{(1)} = \Lambda^{(1)}F^{(1)}(t), \qquad \frac{\mathrm{d}}{\mathrm{d}t}\Lambda^{(2)} = F^{(2)}(t)\Lambda^{(2)}, \tag{3.8}$$

where

$$F^{(1)}(t) = -2J\mathcal{H}(t), \quad F^{(2)}(t) = 2J\mathcal{H}'(t), \qquad J = \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}. \tag{3.9}$$

If H' is diagonal as for the oscillator system (2.8) then the second Eq. (3.8) is easily solved: $\Lambda^{(2)}(t) = \exp(2J\mathcal{H}_{\rm Ho}t)\Lambda_0^{(2)}$. To perform the diagonalization of a quadratic H one has also to solve the first matrix equation in (3.8) and obtain $\Lambda^{(1)}(t)$, which in principle is always possible. In the case of stationary initial H the $\tilde{T}$ exponent becomes ordinary one, so the explicit solution is given by the matrix exponent $\Lambda_0^{(1)}\exp(-2J\mathcal{H}t)$. So for stationary H the total Λ matrix takes the form

$$\Lambda(t) = \exp(2J\mathcal{H}_{\rm Ho}t)\Lambda_0^{(2)}\Lambda_0^{(1)}\exp(-2J\mathcal{H}t)\,, \tag{3.10}$$

where $\Lambda_0^{(i)}$ are arbitrary symplectic matrices. One can put $\Lambda_0^{(1,2)} = 1$, which corresponds to $U_0 = 1$ in Eq. (2.4). Having obtained explicitly $\Lambda(t)$ one can try next to solve Eq. (3.6) and obtain the generating operator $U(t)$ in the form of the quadratic exponent (3.4).

Note, the resulting H' is diagonal in the variables, which we choose for $H_{\rm Ho}$. Let these variables are $p_k^{(1)}, q_k^{(1)}$. Then the final variables p'_k, q'_k obey Eqs (2.9). Inverting the transformations (2.9) we obtain H' diagonal in terms of the final variables as well: $H' = H_{\rm Ho}(\vec{p'}, \vec{q'})$. In this way we perform explicitly the first kind diagonalization. If $H' = H_{\rm Ho}$ in terms of old variables p_k, q_k (second kind diagonalization), then H' is evidently not diagonal in terms of p'_k, q'_k.

For some time-dependent $H(t)$ explicit solutions of Eqs (3.8) can also be found. Thus, in the case of $N = 1$, following the scheme of references [2, 7], one can express matrix the elements of $\Lambda^{(1)}(t)$ in terms of a complex function $z(t)$, that obeys the equation of the classical oscillator $\ddot{z} + \Omega^2(t)z = 0$, where $\Omega^2(t)$ is simply determined by the parameters $\mathcal{A}, \mathcal{B}, \mathcal{C}$ of the Hamiltonian (3.1) (for $N = 1$ these are not matrices, therefore we put $\mathcal{A} = a$, $\mathcal{B} = b$, $\mathcal{C} = c$),

$$\Omega^2(t) = 4ac + 2b\dot{a}/a + \ddot{a}/2a - 3\dot{a}^2/4a^2 - 4b^2 - 2\dot{b}\,.$$

For harmonic oscillator with varying frequency $\omega(t)$ we have $\Omega^2(t) = \omega^2(t)$. It is seen that $\Omega(t)$ corresponds to a class of quadratic $H(t)$. For example the constant Ω corresponds to the stationary oscillator and to the oscillators with varying mass (damped oscillators) $m(t) = m_0\exp(-2bt)$ and $m(t) = m_0\cos^2 bt$, considered later by many authors (see the references in [3, 6, 9]). The respective analytical solutions $z(t)$ are known for a variety of "frequencies" $\Omega(t)$. In the case of an oscillator with varying frequency the diagonalizing CT generator $U(t)$ has been expressed in terms of $z(t)$ in [8].

Let us briefly discuss the algebraic properties of the matrix $\Lambda(t)$ and its submatrices $\lambda_{pp}, \lambda_{pq}, \lambda_{qp}$, and λ_{qq}. From the canonical commutation relations it follows that $\Lambda(t)$ obeys the relation (the symplectic conditions, J being defined in Eq. (3.9))

$$\Lambda J\Lambda^{\mathsf{T}} = J\,, \tag{3.11}$$

which for the $N\times N$ matrices $\lambda_{qq}, \lambda_{pp}, \lambda_{qp}$ and λ_{pq}, defined by Eq. (3.5)), read

$$\lambda_{pp}\lambda_{qq}^{\mathsf{T}} - \lambda_{pq}\lambda_{qp}^{\mathsf{T}} = 1, \quad \lambda_{qq}\lambda_{qp}^{\mathsf{T}} = \lambda_{qp}\lambda_{qq}^{\mathsf{T}}, \quad \lambda_{pq}\lambda_{pp}^{\mathsf{T}} = \lambda_{pp}\lambda_{pq}^{\mathsf{T}}. \quad (3.12)$$

The set of matrices that obey the relation (3.11) is defined as the symplectic matrix group $Sp(N,\mathbb{R})$ (the transformation $\vec{x'} = \Lambda^{\mathsf{T}}\vec{x}$ preserves the quadratic form $\vec{x}J\vec{x}$). It has $N(2N+1)$ real parameters. The rank of its Lie algebra is N (following [19] we use the notation $Sp(N,\mathbb{R})$ instead of $Sp(2N,\mathbb{R})$). It is known that in classical mechanics the set of linear homogeneous CT generates the symplectic group $Sp(N,\mathbb{R})$. In the quantum case the set of matrices Λ, that realize homogeneous linear transformations of the operators of coordinates and momenta close the same group. However the set of unitary operators U for which $U\vec{q}U^{\dagger}$ and $U\vec{p}U^{\dagger}$ are linear combinations of $\vec{p}$ and $\vec{q}$ contains one extra parameter, namely the phase factor. If one considers CT in greater detail as transformations of coordinates, momenta and *vectors* in Hilbert space one has to count the phase factors as well and then we get the larger group $Sp(N,\mathbb{R})\times U(1) \equiv \widetilde{Mp}(N,\mathbb{R})$. If we consider transformations of coordinates, momenta and *states* we have to factorize over $U(1)$: $\widetilde{Mp}(N,\mathbb{R})/U(1) = Mp(N,\mathbb{R})$. The resulting group $Mp(N,\mathbb{R})$ is called **metaplectic group**. It is a double cover of $Sp(N,\mathbb{R})$. The Lie algebras of $Mp(N,\mathbb{R})$ and $Sp(N,\mathbb{R})$ are isomorphic [19, 25]. They are of dimensions $N(2N+1)$ and this is the number of independent matrix elements of matrix $\tilde{\mathcal{H}}$ in (3.4). The generators $U(t)$ of linear CT (3.5) can be considered as operators of the unitary (but not faithful) representation $U(\Lambda)$ of the symplectic group $Sp(N,\mathbb{R})$. One can use the group representation technique [19] to represent $U(t) \in Sp(N,\mathbb{R})$ in several factorized forms. In the case of one dimensional nonstationary harmonic oscillator the diagonalizing CT generator $U(t)$, $U(t) \in SU(1,1)$, and its factorized forms have been considered in [8].

If one considers Hamiltonians (3.1) with linear terms $\vec{d}(t)\vec{p}+\vec{e}(t)\vec{q}$ added, then in the same way one would get that such inhomogeneous quadratic Hamiltonians can be diagonalized to the form (2.8) by means of the same $U(t)$, Eq. (2.4), this time $U(t)$ being an element of the semidirect product group $Mp(N,\mathbb{R}) \circledS H_w(N)$, where $H_w(N)$ is the N dimensional Heisenberg–Weyl group.

4. Diagonalization of Uncertainty Matrix and Minimization of Characteristic Inequalities

The established possibility of converting (by means of time-dependent CT) any N-dimensional Hamiltonian H to that of the system of uncoupled harmonic oscillators suggests to expect that the dispersion matrix $\sigma(\vec{Q},\rho)$ of the canonical

observables Q_ν, $\nu = 1, \ldots, 2N$, in any (generally mixed) quantum state ρ could be diagonalized by means of some state dependent CT. It turns out that this really holds [15, 16].

Let us recall the notion of dispersion matrix $\sigma(\vec{X}, \rho)$ (called also fluctuation matrix, or uncertainty matrix). This is an $n \times n$ matrix constructed by means of the second moment (the variances and covariances) of observables $X_1, \ldots, X_n$ in a state ρ. The matrix elements $\sigma_{\mu\nu}$ of σ are defined as covariances $\Delta X_\mu X_\nu$ of the observables X_μ and X_ν, $\mu, \nu = 1, \ldots, n$,

$$\sigma_{\mu\nu}(\vec{X}, \rho) = \frac{1}{2}\langle X_\mu X_\nu + X_\nu X_\mu \rangle - \langle X_\mu \rangle \langle X_\nu \rangle \equiv \Delta X_\mu X_\nu(\rho)\,.$$

The matrix $\sigma(\vec{X}, \rho)$ is symmetric by construction and satisfies the *characteristic uncertainty relations* [20] $C_r^{(n)}(\sigma(\vec{X}, \rho)) \geq C_r^{(n)}(C(\vec{X}, \rho))$. Here $C(\vec{X}, \rho)$ is the $n \times n$ antisymmetric matrix of the means of the commutators of X_μ and X_ν, $C_{\mu\nu} = -\mathrm{i}\langle [X_\mu, X_\nu] \rangle / 2$, and $C_r^{(n)}(M)$, $r = 1, \ldots, n$, are the characteristic coefficients of a $n \times n$ matrix M [24]. The characteristic coefficient of maximal order $r = n$ is the determinant of M. The characteristic uncertainty relation of maximal order $r = n$,

$$\det \sigma(\vec{X}, \rho) \geq \det C(\vec{X}, \rho)\,, \tag{4.1}$$

has been established by Robertson [21] and is called *Robertson uncertainty relation*. For $N = 2$ inequality (4.1) recovers the Schrödinger uncertainty relation [22], $(\Delta X)^2 (\Delta Y)^2 - (\Delta XY)^2 \geq |\langle [X, Y] \rangle|^2 / 4$, which for the canonical pair q, p, $[q, p] = \mathrm{i}$, takes the simpler form of (hereafter we put $\hbar = 1$)

$$(\Delta p)^2 (\Delta q)^2 - (\Delta pq)^2 \geq 1/4\,. \tag{4.2}$$

The proof of (4.1) is based on the nonnegativity of the matrix $R = \sigma + \mathrm{i}C$ [21]. Properties of R (to be called Robertson matrix) are reviewed in [23]. Here we need the nonnegativity property of $\sigma(\vec{X}, \rho)$.

Proposition 4.1. *The uncertainty matrix for any n observables $X_1, \ldots, X_n$ is nonnegative definite, $\sigma(\vec{X}, \rho) \geq 0$.*

Proof: The proof relies to the Robertson inequality (4.1) and on the observation that the principal submatrices $m(X_{i_1}, \ldots, X_{i_r}, \rho)$, $r \leq n$, of σ can be regarded as uncertainty matrices for r observables $X_{i_1}, \ldots, X_{i_r}$ in the same state ρ. Therefore the submatrices $m(X_{i_1}, \ldots, X_{i_r}), \rho)$ also satisfy Robertson relation (4.1), i. e. their determinants (the principal minors of σ) are non-negative. And if all principal minors of a matrix M are nonnegative, then $M \geq 0$ [24]. □

The uncertainty matrix $\sigma(\vec{Q},\rho)$ for canonical observables Q_μ, $\mu=1,\ldots,2N$: $Q_k=p_k, Q_{N+k}=q_k$, $k=1,\ldots,N$, possess some further properties.

Proposition 4.2. *The uncertainty matrix for $2N$ canonical observables Q_μ is positive definite, $\sigma(\vec{Q},\rho)>0$.*

Proof: From the canonical commutation relations $[q_k,p_j]=\mathrm{i}\delta_{kj}$ it follows that $\det C(\vec{Q},\rho)=(1/4)^N$. Then (4.1) yields

$$\det\sigma(\vec{Q},\rho)\geq(1/4)^N\,. \tag{4.3}$$

As a symmetric matrix $\sigma(\vec{Q},\rho)$ can be diagonalized by an orthogonal transformation $Q_\mu\to Q'_\mu=\gamma_{\mu\nu}Q_\nu$. The uncertainty matrix for new observables Q'_μ is $\sigma'\equiv\sigma(\vec{Q}',\rho)=\gamma\sigma\gamma^{\mathsf{T}}$. This transformation preserves the determinant (and all the other characteristic coefficients) of σ. In view of $\det\sigma(\vec{Q},\rho)>0$ all diagonal elements of σ' are positive. Therefore $\sigma(\vec{Q},\rho)>0$. □

The desired diagonalization of $\sigma(\vec{Q},\rho)$ using *linear canonical transformations* [15, 16] now follows from the Proposition 4.2 and the known theorem [17, 12, 13] that any positive definite symmetric matrix M can be diagonalized by means of congruent transformation with a symplectic matrix Λ, $M\to M'=\Lambda M\Lambda^{\mathsf{T}}$. In [15] the diagonalization of $\sigma(\vec{Q},\rho)$ is performed explicitly by means of three consecutive linear canonical transformations. The diagonal elements s_μ of the diagonalized σ' are variances $(\Delta Q'_\mu)^2$. Additional scaling transformations $q'_i\to q''_i=\alpha_i q'_i$ with $\alpha_i=(\Delta p'_i/\Delta q'_i)^{1/2}$ equalize the variances of q''_i and $p''_i=p'_i/\alpha_i$. Note that: (a) the diagonalizing symplectic matrix Λ is not unique [12, 25]; (b) Λ is state-dependent. Therefore it may depends on time when the state is time-dependent.

Denoting the generator of the total diagonalizing canonical transformation by $U(\Lambda)$ [$Q'=U(\Lambda)QU^\dagger(\Lambda)$, $U(\Lambda)\subset Mp(N,\mathbb{R})$] we obtain the equality $\sigma(\vec{Q},\rho')=\sigma(\vec{Q}',\rho)$, where $\rho'=U\rho U^\dagger$. Thus every state ρ is *unitarily and metaplectically equivalent* to a state ρ', in which the uncertainty matrix $\sigma(\vec{Q},\rho')$ is diagonal with equal variances of coordinates q_i and momenta p_i: $\Delta q_i=\Delta p_i$. If the initial state $|\Psi\rangle$ is pure time-dependent state of system the $\mathcal{S}$ with the Hamiltonian H, then the CT is time-dependent and the new state $|\Psi'\rangle$ obey the Schrödinger equation with the new Hamiltonian (2.1).

Examples of pure states with diagonal uncertainty matrix with equal variances of coordinates and momenta are Glauber multimode coherent states $|\vec{\alpha}\rangle$ and multimode Fock states $|\vec{n}\rangle$. Therefore in the Klauder–Perelomov $Mp(2,\mathbb{R})$ CS $|g,\vec{\alpha}\rangle=U(g)|\vec{\alpha}\rangle$ and $|g,\vec{n}\rangle=U(g)|\vec{n}\rangle$ (g being the group element) the dispersion matrices $\sigma(\vec{Q}',g,\vec{\alpha})$ and $\sigma(\vec{Q}',g,\vec{n})$ are diagonal and with equal

variances of $q_i' = U^\dagger(g) q_i U(g)$ and $p_i' = U^\dagger(g) p_i U(g)$. In $|\vec{\alpha}\rangle$ *all variances are equal and minimal*, $\Delta q_i = \Delta p_i = 1/\sqrt{2}$, whereas in Fock states the variances are equal in pairs, $(\Delta q_i)^2 = (\Delta p_i)^2 = 1/2 + n_i$. Multimode CS $|\vec{\alpha}\rangle$ minimize Robertson inequality (4.3), whereas in $|\vec{n}\rangle$ one has $\det \sigma(\vec{Q}, \vec{n}) = \prod_i (1/2 + n_i)$. It is clear from the above consideration that the uncertainty matrix in any group-related CS $T(g)|\Psi_0\rangle$ with reference vector $|\Psi_0\rangle$ equal to $|\vec{\alpha}\rangle$ or $|\vec{n}\rangle$ is diagonalized by the CT $Q_\mu' = T^\dagger(g) Q_\mu T(g)$, that is linear for the group $Mp(N, \mathbb{R})$ only. In physical literature the $Mp(N, \mathbb{R})$ group-related CS $U(g)|\vec{\alpha}\rangle$ and $U(g)|\vec{n}\rangle$ are known as multimode *squeezed* CS and squeezed Fock states respectively. The operator $U(g) \in Mp(N, \mathbb{R})$ can be called multimode squeeze operator [26], its canonical form being $\exp[(\vec{a}^\dagger z \vec{a}^\dagger - \vec{a} z^* \vec{a})/2]$, where $\vec{a}^\dagger z \vec{a}^\dagger = a_i^\dagger z_{ij} a_j^\dagger$, $i, j = 1, \ldots, N$ [26]. It is more adequate to call it **squeeze** and **correlation operator** since, e.g., for pure imaginary z_{ii} it generates covariances of p_i and q_i and doesn't squeeze, while for real z_{ii} it generates squeezing and doesn't correlate. The wave function $\langle \vec{x}|U(g)|\vec{\alpha}\rangle$ of $Mp(N, \mathbb{R})$ CS is Gaussian (an exponent of N-dimensional quadratic form), thereby that states are also called Gaussian pure states [5, 18].

It is interesting to note that the multimode squeezed states $U(g)|\vec{\alpha}\rangle$ are the unique states to minimize the Robertson inequality (4.3).

Proposition 4.3. *The equality in the multimode Robertson uncertainty relation, Eq. (27), holds in the multimode squeezed states* $U(g)|\vec{\alpha}\rangle$ $(g \in Mp(N, \mathbb{R}))$ *only.*

Proof: Let $\Lambda(\rho)$ be a symplectic matrix that diagonalizes the dispersion matrix $\sigma(\vec{Q}, \rho)$, and $U = U(\Lambda)$ — the generator of the diagonalizing CT $\vec{Q}' = \Lambda(\rho)\vec{Q} = U(\Lambda)\vec{Q}U^\dagger(\Lambda)$. $U(\Lambda)$ belongs to $Mp(N, \mathbb{R})$. We have

$$\sigma(\vec{Q}', \rho) = \Lambda(\rho)\sigma(\vec{Q}, \rho)\Lambda^\top(\rho) = \sigma(\vec{Q}, \rho'), \quad \rho' = U(\Lambda)\rho U^\dagger(\Lambda)\,, \tag{4.4}$$

$$\sigma(\vec{Q}, \rho') = \mathrm{diag}\{s_1, s_2, \ldots, s_{2N}\}\,, \tag{4.5}$$

where the diagonal elements s_ν are the variances of q_k and p_k in the new state ρ': $s_k = (\Delta p_k(\rho'))^2$, $s_{N+k}(\rho') = (\Delta q_k(\rho'))^2$. The determinant of $\sigma(\vec{Q}, \rho')$ is a product of all diagonal elements s_ν, $\nu = 1, \ldots, 2N$,

$$\det \sigma(\vec{Q}, \rho) = \det \sigma(\vec{Q}, \rho') = [s_1 s_{N+1}][s_2 s_{N+2}] \ldots [s_N s_{2N}]. \tag{4.6}$$

From Heisenberg uncertainty relation we have for every factor in Eq. (4.6) the inequality

$$s_k s_{N+k} = (\Delta p_k)^2 (\Delta q_k)^2 \geq 1/4\,. \tag{4.7}$$

From Eqs (4.7) and (4.6) we derive that the equality in Robertson relation (4.3) holds iff the equality in Eq. (4.7) holds for all modes (for every $k = 1, \ldots, N$). The minimal value of $1/4$ of the product of variances of q and p cannot be reached in mixed state [15] — it is reached in the Stoler states [4] $|\alpha, r\rangle = \exp[r(a^{\dagger 2} - a^2)]\,|\alpha\rangle$ only (see the proof in the Appendix). Thus the equality in (4.3) holds in pure states $U(\Lambda)\prod_k S(r_k)|\vec{\alpha}\rangle$ only. The unitary operator $S(r) = \exp[r(a^{\dagger 2} - a^2)/2]$ (the squeeze operator) belongs to $U(1,1) \sim Mp(1,\mathbb{R})$. Therefore the unitary operator $U(\Lambda)\prod_k S(r_k) = U(g)$ belongs to $Mp(N,\mathbb{R})$, and the unique minimizing states are $Mp(N,\mathbb{R})$-group related CS with reference vector $|\vec{\alpha}\rangle$. □

Since Glauber CS $|\vec{\alpha}\rangle$ are eigenstates of every annihilation operator a_k (with eigenvalues α_k, $k = 1, \ldots, N$), the minimizing states $U(g)|\vec{\alpha}\rangle$ are eigenstates of the canonically transformed annihilation operators $a'_k = U(g)a_kU^\dagger(g)$, which are linear combinations of $a_1, \ldots, a_N$: $a'_k = u_{kj}a_j + v_{kj}a_j^\dagger$. Therefore the minimizing states $U(g)|\vec{\alpha}\rangle$ (the multimode squeezed states) can be denoted equivalently as $|\vec{\alpha}, u, v\rangle$. For $v = 0$ (and $u = 1$) they coincide with $|\vec{\alpha}\rangle$.

For quadratic Hamiltonians the time evolution operator $U_{quad}(t) \in Mp(N,\mathbb{R})$. Therefore the time evolution of $|\vec{\alpha}, u, v\rangle$ for quadratic Hamiltonians is stable, i. e., $U(t)|\vec{\alpha}, u, v\rangle = |\vec{\alpha}, u(t), v(t)\rangle$. The evolved states $|\vec{\alpha}, u(t), v(t)\rangle$ are eigenstates of the new annihilation operators $A_k(t) = U(t)a'_kU^\dagger(t)$, which are again linear in a_j and $a_j^\dagger$ and are integrals of motion of quadratic system. Overcomplete system of eigenstates $|\vec{\alpha}, t\rangle$ of integrals of motion $A_k(t)$ has been constructed in ref. [5] and used later in many papers [10].

A further property of the uncertainty matrix (the fourth one) we want to note here is referred to its **symplectic character**: the normalized uncertainty matrix $\tilde{\sigma} = \sigma/\left(\det\sigma\right)^{1/2N}$ is symplectic for a certain class of states. In order to find out that states we note the invariance of the symplectic property of a matrix M under the congruent transformation $\Lambda M\Lambda^\mathsf{T}$ with a symplectic Λ: if M is symplectic, that is $MJM^\mathsf{T} = J$, then $M' = \Lambda M\Lambda^\mathsf{T}$ is also symplectic. This symplectic invariance can be easily proved using the known property that if $\Lambda J\Lambda^\mathsf{T} = J$ then one also has $\Lambda^\mathsf{T}J\Lambda = J$: $M'JM'^\mathsf{T} = \Lambda M\Lambda^\mathsf{T}J\Lambda M^\mathsf{T}\Lambda^\mathsf{T} = \Lambda MJM^\mathsf{T}\Lambda^\mathsf{T} = \Lambda J\Lambda^\mathsf{T} = J$. This invariance enables us to study the symplectic properties of σ in its simpler diagonal form. For diagonal uncertainty matrix $\tilde{\sigma} = \mathrm{diag}\{s_1, \ldots, s_N\}$ the symplectic condition $\tilde{\sigma}J\tilde{\sigma}^\mathsf{T} = J$ reduces to

$$s_1s_{N+1} = s_2s_{N+2} = \cdots = s_Ns_{2N} = \left(\det\sigma\right)^{1/N}. \tag{4.8}$$

One solution to (4.8) can be immediately pointed out, recalling the meaning of s_μ as the variance of Q_μ: the uncertainty matrix in the multimode Glauber CS $|\vec{\alpha}\rangle$ is diagonal with $s_k = (\Delta p_k)^2 = 1/2$, $s_{N+k} = (\Delta p_k)^2 = 1/2$, $k =$

$1, \ldots, N$, which clearly satisfy (4.8). Therefore the normalized uncertainty matrix in pure states $U(g)|\vec{\alpha}\rangle$ that are unitary equivalent to Glauber CS with $U(g) \in Mp(N, \mathbb{R})$ *is symplectic*. These states, as we have already noted, are called Gaussian pure states or multimode squeezed CS. In fact the symplectic character of the normalized uncertainty matrix for Gaussian pure states was established in [18]: in that states our $\tilde{\sigma}$ is equal to 2σ and this quantity coincides with the matrix $\underline{G}(\underline{U}^{-1}, -\underline{V})$ of [18], which was shown to be symplectic [18].

A second solution to (4.8) is provided by the uncertainty matrix in (multimode) Fock states $|\vec{n}\rangle$ with equal numbers $n_k = n$ (equal numbers of photons in every mode). In $|\vec{n}\rangle$ we have $(\Delta p_k)^2 = 1/2 + n_k = (\Delta q_k)^2$. Therefore in states $U(g)|\vec{n}\rangle$ with $n_1 = \cdots = n_N$ and $U(g) \in Mp(N, \mathbb{R})$ the normalized uncertainty matrix is symplectic. The above two families of states do not exhaust the set states with symplectic (normalized) uncertainty matrix.

Let us write down the symplectic conditions and the Robertson relation for $\sigma(\vec{Q}, \rho)$ in terms of the four $N \times N$ blocks $\sigma_{pp}(\rho)$, $\sigma_{qq}(\rho)$, $\sigma_{pq}(\rho)$ and $\sigma_{qp}(\rho)$,

$$\sigma(\vec{Q}, \rho) = \begin{pmatrix} \sigma_{pp}(\rho) & \sigma_{pq}(\rho) \\ \sigma_{qp}(\rho) & \sigma_{qq}(\rho) \end{pmatrix} , \tag{4.9}$$

Inserting this into $\tilde{\sigma} J \tilde{\sigma}^{\mathsf{T}} = J$, and taking into account that σ_{pp} and σ_{qq} are symmetric, and $\sigma_{pq} = \sigma_{qp}^{\mathsf{T}}$ we obtain

$$\sigma_{pp}\sigma_{qq} - (\sigma_{pq})^2 = (\det \sigma)^{1/N} , \tag{4.10}$$

$$\sigma_{pp}\sigma_{qp} - \sigma_{pq}\sigma_{pp} = 0, \quad \sigma_{qp}\sigma_{qq} - \sigma_{qq}\sigma_{pq} = 0 . \tag{4.11}$$

Squeezed CS $U(g)|\vec{\alpha}\rangle$ minimize (4.3), i. e. $\det \sigma = (1/4)^N$. Therefore in $U(g)|\vec{\alpha}\rangle$ the symplectic condition (4.10) reads $\sigma_{pp}\sigma_{qq} - (\sigma_{pq})^2 = 1/4$. The latter formula was obtained in [26] for the squeezed CS of the form $\exp[(\vec{a}^{\dagger} z \vec{a}^{\dagger} - \vec{a} z^* \vec{a})/2]|\vec{\alpha}\rangle$ by direct calculations (but with no reference to Robertson inequality, neither to the symplecticity of the uncertainty matrix). In squeezed Fock states $U(g)|\vec{n}\rangle$ we have $\det \sigma(g, \vec{n}) = \prod_k (1/2 + n_k) \geq (1/4)^N$. For these states the symplectic condition (4.10) is valid iff $n_k = n$, and reads $\sigma_{pp}\sigma_{qq} - (\sigma_{pq})^2 = (1/2 + n)^2$.

In terms of the $N \times N$ matrices Robertson inequality (4.3) takes the form (using known formulas for the block matrices [24])

$$\det[\sigma_{pp}\sigma_{qq} - \sigma_{pp}\sigma_{qp}\sigma_{pp}^{-1}\sigma_{pq}] \geq (1/4)^N . \tag{4.12}$$

For $\tilde{\sigma}$ symplectic we have $\sigma_{pp}\sigma_{qp} = \sigma_{pq}\sigma_{pp}$, and the Robertson relation simplifies to $\det[\sigma_{pp}\sigma_{qq} - (\sigma_{pq})^2] \geq (1/4)^N$. This form is quite similar to that of Schrödinger inequality (4.2) for p and q: for $N = 1$ we have $\sigma_{pq} = \Delta pq$, $\sigma_{pp} = \Delta pp \equiv (\Delta p)^2$, and $\sigma_{qq} = \Delta qq \equiv (\Delta q)^2$.

It is curious to note that the Robertson matrix $\tilde{R}$ for normalized $\tilde{\sigma}(\vec{Q},\rho)$ and $\tilde{C}(\vec{Q},\rho) = C(\vec{Q},\rho)/\left(\det C(\vec{Q},\rho)\right)^{1/2N}$, $\tilde{R} = \tilde{\sigma} + \mathrm{i}\tilde{C}$, is also symplectic for squeezed CS and squeezed Fock states with $n_k = n$: $\tilde{R}J\tilde{R}^\dagger = J$, that is $\tilde{R} \in Sp(N,\mathbb{C})$.

Appendix A

Proposition A.1. *Heisenberg inequality* $(\Delta q)^2(\Delta p)^2 \geq 1/4$ *is minimized in the Stoler states* $|\alpha, r\rangle = \exp[r(a^{\dagger 2} - a^2)/2]|\alpha\rangle$ *only.*

Let ρ be a general mixed state. Any mixed state can be represented in the form $\rho = \sum_k \rho_k |\psi_k\rangle\langle\psi_k|$, where $\rho_k \geq 0$, and $\{|\psi_k\rangle\}$ is some complete orthonormal set of pure states. The mean value of an operator X in ρ is given by $\langle X\rangle = \mathrm{Tr}(X\rho)$. Consider the mean value of the operator $b^\dagger(\lambda)b(\lambda)$, where

$$b(\lambda) = \lambda q + \mathrm{i}p - (\lambda\langle q\rangle + \mathrm{i}\langle p\rangle, \quad \lambda \in \mathbb{R}\,. \tag{A.1}$$

For positive λ the operator $b(\lambda)$ is, up to a factor $1/\sqrt{2\lambda}$, a boson annihilation operator, $[b, b^\dagger] = 2\lambda$, and $b^\dagger b$ is, up to a factor $1/2\lambda$, the number operator, which is nonnegative definite. For negative λ the operators b and $b^\dagger$ are interchanged. The mean of $b^\dagger(\lambda)b(\lambda)$ in ρ reads

$$\langle b^\dagger(\lambda)b(\lambda)\rangle = \sum_k \rho_k \langle\psi_k|b^\dagger(\lambda)b(\lambda)|\psi_k\rangle\,, \tag{A.2}$$

where all means $\langle\psi_k|b^\dagger(\lambda)b(\lambda)|\psi_k\rangle$ are nonnegative. On the other hand, by the use of (A.1), this nonnegative mean $\langle b^\dagger(\lambda)b(\lambda)\rangle$ can be written as

$$\langle b^\dagger(\lambda)b(\lambda)\rangle = \lambda^2(\Delta q)^2 - \lambda + (\Delta p)^2 \geq 0\,. \tag{A.3}$$

The λ-roots of the equation $\lambda^2(\Delta q)^2 - \lambda + (\Delta p)^2 = 0$ must be real, wherefrom one deduces Heisenberg inequality. The equality in Heisenberg relation corresponds to the equality in (A.3), i. e. to the vanishing $\langle b^\dagger(\lambda)b(\lambda)\rangle$. From (A.2) it is seen that $\langle b^\dagger(\lambda)b(\lambda)\rangle = 0$ if and only if $\langle\psi_k|b^\dagger(\lambda)b(\lambda)|\psi_k\rangle = 0$ for every $k = 1, \ldots$ (in view of $\rho_k \geq 0$). From the *uniqueness of the vacuum state* it follows that all ρ_k but one (say ρ_1) must be zero. Therefore $\langle b^\dagger(\lambda)b(\lambda)\rangle = 0$ in pure state $|\psi\rangle$ only and iff it is an eigenstate of $\lambda q + \mathrm{i}p$. The final step is to identify the minimizing pure state with $|\alpha, r\rangle$. The minimizing state must be eigenstate of $\lambda q + \mathrm{i}p$ for some real λ. $|\alpha, r\rangle$ are eigenstates of $\lambda q + \mathrm{i}p$) with $\lambda = [\cosh(2r) - \sinh(2r)]/[\cosh(2r) + \sinh(2r)]$ and eigenvalue $\alpha\sqrt{2}/[\cosh(2r) + \sinh(2r)]$. Thus $(\Delta q)^2(\Delta p)^2 = 1/4$ holds in states $|\alpha, r\rangle$ only. In slightly different notations the proof of the statement that a state with absolute minimum of the product $(\Delta q)^2(\Delta p)^2$ is a pure state is given in [15].

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