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KAUFFMAN BRACKET ON RATIONAL TANGLES AND RATIONAL KNOTS

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Abstract. Computing Kauffman bracket grows exponentially with the number of crossings in the knot diagram. In this paper we illustrate how Kauffman bracket for rational tangles and rational knots can be computed so that it involves a low number of terms. Kauffman bracket and Jones polynomial are known to have connections with statistical mechanics, quantum theory and quantum field theory.

MSC: 57M27
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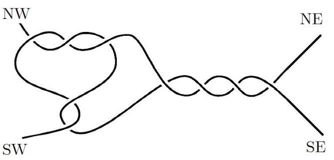
1. Introduction

A 2-tangle diagram is a region in a knot or link diagram surrounded by a rectangle such that the knot or link diagram crosses the rectangle in four points. These four points are usually thought of as fixed points occurring in the four corners NW, NE, SW, SE. See the following figure.



A *rational tangle* is a special 2-tangle that results from a finite sequence of twists of pairs of the four endpoints of two unknotted arcs. For the formal definition of a rational tangle see Definition 2. *Rational knots and links* are obtained by numerator closures of rational tangles; that is by connecting the two upper end points and the two lower endpoints of a given rational tangle by two unknotted and unlinked arcs. Rational knots and rational tangles are important in the study of DNA recombination (see [10]). Rational knots and links were considered early in [2] and [11]. The notion of a rational knot was introduced by Conway [5] in his work on classifying knots and links. Some other important references in the study of rational tangles and rational knots and links are [1], [3], [4], [6–9], [12]. A rational tangle is associated with a unique reduced rational number or ∞, called the fraction of the tangle. The fraction of the tangle classifies rational tangles, and it is also used in the classification of rational knots and links.

Kauffman bracket is a regular isotopy invariant of knots and links, which was used in redefining the well-known ambient isotopy invariant; the Jones polynomial. Computing Jones polynomial for a knot or link through Kauffman bracket grows exponentially with the number of crossings in the diagram. In this paper, we illustrate how Kauffman bracket for rational tangles and rational knots can be computed so that it involves a low number of terms.. For example the rational link $\mathrm{N}[5\,6\,7]$, in Example 3 at the end of this paper, has 18 crossings. Computing Kauffman bracket for this link might involve 2^{18} terms, which correspond to the 2^{18} states resulting from smoothing each crossing in two possible ways. While, in Example 3, it involves 8 terms, and each term consists of a product of three reduced factors. While we just work on rational tangles, knots and links, the ideas can be generalized to algebraic knots and links in general. In fact most of the ideas in this paper might be known, but we illustrate them from our perspective.

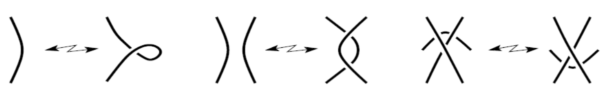
This paper is organized as follows. In Section two we give the needed background on rational tangles and Kauffman bracket. In section three we introduce operations on the module consisting of the values of the bracket on rational tangles. We show the existence of the structure of an algebra over a ring in this module. We also use this structure to deduce some important formulas and results on the values of the bracket of rational tangles. In Section four we give a main theorem that introduces a sum formula for calculating the bracket of a rational tangle and hence the bracket and the Jones polynomial of a rational knot.

2. Basic Concepts and Terminology

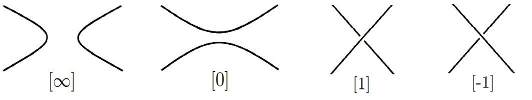
2.1. Rational Tangles

Most of the background in this section is based on Adams book [1]. Also we use many graphics from this book in the paper.

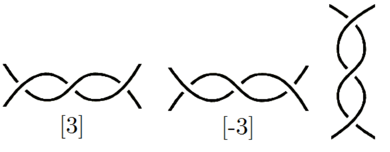
Definition 1. *Two 2-tangle diagrams are said to be isotopy equivalent if we can get from one of them to the other by a sequence of the well-known three Reidemeister moves inside the surrounding rectangle while the four points remain fixed.*



Let the $[\infty]$ tangle, the $[0]$ tangle, the $[1]$ tangle and the $[-1]$ tangle be as in the following figure

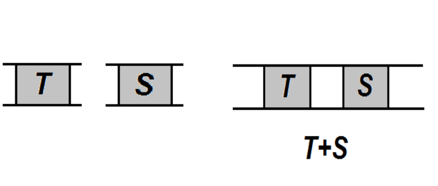


An integer tangle $[n]$ is the result of winding two horizontal strands around each other to get a tangle that involves $|n|$ crossings. If the upper strand at each crossing has positive slope, then n is positive. Otherwise, n would be negative. Note that $[-n]$ is the mirror image of $[n]$. The tangle denoted by $\frac{1}{[n]}$ is the result of winding two vertical strands around each other to get a tangle that involves $|n|$ crossings See the following figure for the tangle $[3]$, the tangle $[-3]$, and the tangle $\frac{1}{[3]}$.



Note that $\frac{1}{[3]}$ is a vertical tangle with crossings having an upper strand of positive slope again.

Consider the following two operations on general 2-tangles



where $-T$ is the mirror image of T, which can be obtained by switching all the crossings of T. In fact, $\frac{1}{T}$ can be considered as the mirror image of T in the diagonal line connecting NW and SE of T.

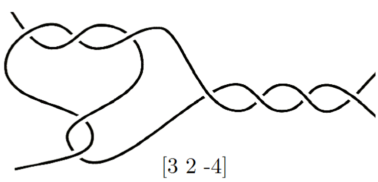
The definition of rational tangles can now be given as follows.

Definition 2. *A rational tangle is the isotopy class represented by the tangle diagram denoted by $[a_1\ a_2 \cdots a_m]$ and constructed as follows*

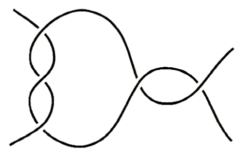
$$T = [a_1\ a_2 \cdots a_m] = [a_m] + \cfrac{1}{[a_{m-1}] + \cfrac{1}{\ddots \cfrac{1}{[a_2] + \cfrac{1}{[a_1]}}}}$$

where $a_2, \ldots, a_{m-1} \in \mathbb{Z} - \{0\}$, $a_1 \in \mathbb{Z}$ and $a_m \in \mathbb{Z} \cup \{\infty\}$. Such form of a tangle will be called standard form.

See the tangles $[3\ 2\ \ -4] = [-4] + \cfrac{1}{[2] + \cfrac{1}{[3]}}$, and $[3\ 2] = [2] + \cfrac{1}{[3]}$ below.



[3 2 -4]



[3 2]

Theorem 3. *Every rational tangle is the isotopy class represented by the tangle diagram $[a_1\ a_2 \cdots a_m]$ with all the $a_i^{;s}$ positive or all negative.*

2.2. Kauffman Bracket of Tangles

Consider the well-known Kauffman bracket in the following definition.

Definition 4. *The Kauffman bracket polynomial is a function from unoriented link diagrams in the oriented plane to Laurent polynomials with integer coefficients in an indeterminate A. It maps a diagram D of a link L to $\langle D \rangle \in \mathbb{Z}[A, A^{-1}]$ and it is characterized by the three rules*

i) $\left\langle \times \right\rangle = A \left\langle)\,(\right\rangle + A^{-1} \left\langle \asymp \right\rangle$

ii) $\left\langle D \cup \bigcirc \right\rangle = (-A^2 - A^{-2}) \langle D \rangle$

iii) $\left\langle \bigcirc \right\rangle = 1.$

The bracket polynomial is known to be a regular isotopy invariant, that is an invariant under the second and the third R-moves.

In the case when we have an oriented link D and we want to calculate the bracket polynomial for this link, we will denote by $|D|$ the unoriented diagram that is obtained from D by forgetting the orientations of all components. A crossing in an oriented diagram D has a sign of $+1$ or -1 according to the right-hand rule. The sum of signs of the crossings for a given diagram D is called *the writhe of D* and is denoted by $w(D)$.

Theorem 5. *Let D be a an oriented link diagram. Then the X polynomial defined by*

$$X(D) = (-A^3)^{-w(D)} \langle |D| \rangle$$

is an invariant of the links.

Now given a diagram D with n crossings. A *state S of D* is a choice of how to smooth all of the n crossings in the link diagram D. Each state is a set of nonoverlapping circles, and we have 2^n states. The following theorem provides us with a formula for calculating $\langle D \rangle$ called *a state sum formula.*

Theorem 6. *The bracket polynomial of the link diagram D is given by*

$$\langle D \rangle = \sum_S A^{a(S)-b(S)} (-A^2 - A^{-2})^{|S|-1}$$

where the sum runs over all possible states S, such that $|S|$ is the total number of circles in the state S, $a(S)$ is the number of splits of type  *of the crossing*  *in the state S.*

Let T be any 2-tangle and let $\langle T \rangle$ be the formal expansion of the bracket on this tangle. Then we will have a similar state sum formula except that a state S consists of circles and the tangle $\langle [0] \rangle$ or the tangle $\langle [\infty] \rangle$. Moreover, if we denote the number of circles in this S by $|S|$, the contribution of S in $\langle T \rangle$ will be $A^{a(S)-b(S)}(-A^2 - A^{-2})^{|S|} \langle [0] \rangle$, or $A^{a(S)-b(S)}(-A^2 - A^{-2})^{|S|} \langle [\infty] \rangle$. Therefore, we get

$$\langle T \rangle = \sum_S A^{a(S)-b(S)} (-A^2 - A^{-2})^{|S|} e(S)$$

where $e(S) = \langle [0] \rangle$, or $\langle [\infty] \rangle$. This implies the following lemma in [9].

Lemma 7. *Let T be any 2-tangle and let $\langle T \rangle$ be the formal expansion of the bracket on this tangle. Then there exist elements a(A) and b(A) in $\mathbb{Z}[A, A^{-1}]$, such that*

$$\langle T \rangle = a(A) \langle [\infty] \rangle + b(A) \langle [0] \rangle$$

and a(A) and b(A) are regular isotopy invariants of the tangle T.

For simplicity, from now on, we write the last formula as

$$\langle T\rangle = a\,\langle[\infty]\rangle + b\,\langle[0]\rangle\,.$$

Note that $\langle T\rangle$ is a linear combination of $\langle[\infty]\rangle\,,\langle[0]\rangle$, with coefficients of Laurent polynomials in A. In other words, $\langle T\rangle$ takes values in the module

$$M = \text{the free module over}\mathbb{Z}[A, A^{-1}] \text{ with basis } \{\langle[\infty]\rangle\,,\langle[0]\rangle\}\,.$$

3. Operations in the Bracketology of Rational Tangles

We define the following type of multiplication on the module M defined above as follows.

Definition 8. *Let* $\langle T\rangle = a\,\langle[\infty]\rangle + b\,\langle[0]\rangle$, *and* $\langle S\rangle = c\,\langle[\infty]\rangle + d\,\langle[0]\rangle$. *Let*

$$\begin{aligned}\langle T\rangle \oplus \langle S\rangle &= \left(a\,\langle[\infty]\rangle + b\,\langle[0]\rangle\right)\left(c\,\langle[\infty]\rangle + d\,\langle[0]\rangle\right)\\ &= ac\,\langle[\infty]\rangle \oplus \langle[\infty]\rangle + ad\,\langle[\infty]\rangle \oplus \langle[0]\rangle\\ &\quad + bc\,\langle[0]\rangle \oplus \langle[\infty]\rangle + bd\,\langle[0]\rangle \oplus \langle[0]\rangle\end{aligned}$$

subject to the relations

$$\langle[\infty]\rangle \oplus \langle[\infty]\rangle = \delta\,\langle[\infty]\rangle\,, \qquad where \qquad \delta = -A^2 - A^{-2}$$
$$\langle[\infty]\rangle \oplus \langle[0]\rangle = \langle[\infty]\rangle\,, \qquad \langle[0]\rangle \oplus \langle[\infty]\rangle = \langle[\infty]\rangle\,, \qquad \langle[0]\rangle \oplus \langle[0]\rangle = \langle[0]\rangle\,.$$

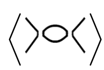
Note that, by definition, we have

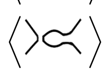
$$\langle T\rangle \oplus \langle S\rangle = (ac\delta + ad + bc)\,\langle[\infty]\rangle + bd\,\langle[0]\rangle\,.$$

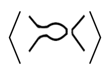
Lemma 9. *The following four linearity identities hold*

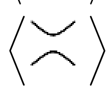
$$\langle[\infty]+[\infty]\rangle = \langle[\infty]\rangle \oplus \langle[\infty]\rangle\,, \qquad \langle[\infty]+[0]\rangle = \langle[\infty]\rangle \oplus \langle[0]\rangle$$
$$\langle[0]+[\infty]\rangle = \langle[0]\rangle \oplus \langle[\infty]\rangle\,, \qquad \langle[0]+[0]\rangle = \langle[0]\rangle \oplus \langle[0]\rangle\,.$$

Proof: Note that

$\langle[\infty]+[\infty]\rangle$ =  $= \delta\,\langle[\infty]\rangle = \langle[\infty]\rangle \oplus \langle[\infty]\rangle$

$\langle[\infty]+[0]\rangle$ =  $= \langle[\infty]\rangle = \langle[\infty]\rangle \oplus \langle[0]\rangle$

$\langle[0]+[\infty]\rangle$ =  $= \langle[\infty]\rangle = \langle[0]\rangle \oplus \langle[\infty]\rangle$

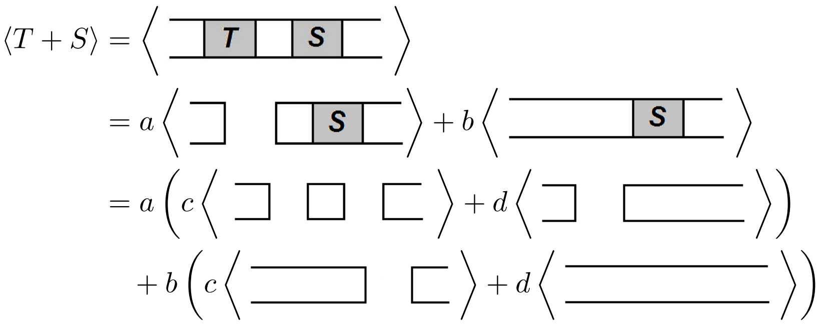
$\langle[0]+[0]\rangle$ =  $= \langle[0]\rangle = \langle[0]\rangle \oplus \langle[0]\rangle\,.$

■

Theorem 10. *Let T, and S be two rational tangles, then*

$$\langle T+S\rangle = \langle T\rangle \oplus \langle S\rangle\,.$$

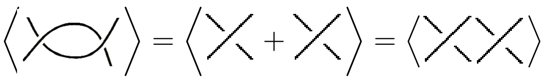
Proof: Let $\langle T\rangle = a\,\langle[\infty]\rangle + b\,\langle[0]\rangle$, and $\langle S\rangle = c\,\langle[\infty]\rangle + d\,\langle[0]\rangle$, then



$$\begin{aligned}
&= ac\delta\,\langle[\infty]\rangle + ad\,\langle[\infty]\rangle + bc\,\langle[\infty]\rangle + bd\,\langle[0]\rangle\\
&= ac\,\langle[\infty]\rangle \oplus \langle[\infty]\rangle + ad\,\langle[\infty]\rangle \oplus \langle[0]\rangle\\
&\quad + bc\,\langle[0]\rangle \oplus \langle[\infty]\rangle + bd\,\langle[0]\rangle \oplus \langle[0]\rangle\\
&= (a\,\langle[\infty]\rangle + b\,\langle[0]\rangle) \oplus (c\,\langle[\infty]\rangle + d\,\langle[0]\rangle) = \langle T\rangle \oplus \langle S\rangle\,.
\end{aligned}$$

■

Example 11.



$$\begin{aligned}
&= \left(A\,\langle[\infty]\rangle + A^{-1}\,\langle[0]\rangle\right) \oplus \left(A\,\langle[\infty]\rangle + A^{-1}\,\langle[0]\rangle\right)\\
&= A^2\delta\,\langle[\infty]\rangle + \langle[\infty]\rangle + \langle[\infty]\rangle + A^{-2}\,\langle[0]\rangle\\
&= \left(A^2\delta + 2\right)\langle[\infty]\rangle + A^{-2}\,\langle[0]\rangle\,.
\end{aligned}$$

Next we describe the algebraic structure of $(M,\oplus)$.

Lemma 12. $(M,\oplus)$ *is a commutative monoid, that is if* $\langle T\rangle$, $\langle S\rangle$, $\langle R\rangle \in M$, *then*

1) $\langle[0]\rangle \oplus \langle T\rangle = \langle T\rangle \oplus \langle[0]\rangle = \langle T\rangle$
2) $(\langle T\rangle \oplus \langle S\rangle) \oplus \langle R\rangle = \langle T\rangle \oplus (\langle S\rangle \oplus \langle R\rangle)$
3) $\langle T\rangle \oplus \langle S\rangle = \langle S\rangle \oplus \langle T\rangle$.

Proof: Parts (1), and (3) follow directly from the definition of the operation $\oplus$. For associativity, one needs to check the following

$$\begin{aligned}
(\langle[0]\rangle \oplus \langle[0]\rangle) \oplus \langle[0]\rangle &= \langle[0]\rangle = \langle[0]\rangle \oplus (\langle[0]\rangle \oplus \langle[0]\rangle)\\
(\langle[\infty]\rangle \oplus \langle[0]\rangle) \oplus \langle[0]\rangle &= \langle[\infty]\rangle = \langle[\infty]\rangle \oplus (\langle[0]\rangle \oplus \langle[0]\rangle)
\end{aligned}$$

$$\begin{aligned}
(\langle[0]\rangle \oplus \langle[\infty]\rangle) \oplus \langle[0]\rangle &= \langle[\infty]\rangle = \langle[0]\rangle \oplus (\langle[\infty]\rangle \oplus \langle[0]\rangle) \\
(\langle[0]\rangle \oplus \langle[0]\rangle) \oplus \langle[\infty]\rangle &= \langle[\infty]\rangle = \langle[0]\rangle \oplus (\langle[0]\rangle \oplus \langle[\infty]\rangle) \\
(\langle[0]\rangle \oplus \langle[\infty]\rangle) \oplus \langle[\infty]\rangle &= \delta \langle[\infty]\rangle = \langle[0]\rangle \oplus (\langle[\infty]\rangle \oplus \langle[\infty]\rangle) \\
(\langle[\infty]\rangle \oplus \langle[0]\rangle) \oplus \langle[\infty]\rangle &= \delta \langle[\infty]\rangle = \langle[\infty]\rangle \oplus (\langle[0]\rangle \oplus \langle[\infty]\rangle) \\
(\langle[\infty]\rangle \oplus \langle[\infty]\rangle) \oplus \langle[0]\rangle &= \delta \langle[\infty]\rangle = \langle[\infty]\rangle \oplus (\langle[\infty]\rangle \oplus \langle[0]\rangle) \\
(\langle[\infty]\rangle \oplus \langle[\infty]\rangle) \oplus \langle[\infty]\rangle &= \delta^2 \langle[\infty]\rangle = \langle[\infty]\rangle \oplus (\langle[\infty]\rangle \oplus \langle[\infty]\rangle) .
\end{aligned}$$

■

An example of a non-invertible element in this monoid is $\langle[\infty]\rangle$. Therefore $(M, \oplus)$ is not a group. However we have the following result.

Theorem 13. *$(M, \oplus)$ is an associative commutative algebra over the ring $\mathbb{Z}[A,A^{-1}]$, that is if $\langle T\rangle$, $\langle S\rangle$, $\langle R\rangle \in M$ and $p, q \in \mathbb{Z}[A, A^{-1}]$, then*

1) $\langle T\rangle \oplus (\langle S\rangle + \langle R\rangle) = (\langle T\rangle \oplus \langle S\rangle) + (\langle T\rangle \oplus \langle R\rangle)$
2) $(p \langle T\rangle) \oplus (q \langle S\rangle) = pq (\langle T\rangle \oplus \langle S\rangle)$.

Proof: Let $\langle T\rangle = a \langle[\infty]\rangle + b \langle[0]\rangle$, $\langle S\rangle = c \langle[\infty]\rangle + d \langle[0]\rangle$, $\langle R\rangle = e \langle[\infty]\rangle + f \langle[0]\rangle$, then

$$\begin{aligned}
\langle T\rangle \oplus (\langle S\rangle + \langle R\rangle) &= [a (c + e) \delta + a (d+f)+b (c + e)] \langle[\infty]\rangle + [b (d+f)] \langle[0]\rangle \\
&= [ac\delta + ad + bc] \langle[\infty]\rangle + bd \langle[0]\rangle \\
&\quad + [ae\delta + af + be] \langle[\infty]\rangle + bf \langle[0]\rangle \\
&= (\langle T\rangle \oplus \langle S\rangle) + (\langle T\rangle \oplus \langle R\rangle) .
\end{aligned}$$

On the other hand

$$\begin{aligned}
(p \langle T\rangle) \oplus (q \langle S\rangle) &= (pa \langle[\infty]\rangle + pb \langle[0]\rangle) \oplus (qc \langle[\infty]\rangle + qd \langle[0]\rangle) \\
&= (paqc\delta + paqd + pbqc) \langle[\infty]\rangle + (pbqd) \langle[0]\rangle \\
&= pq (ac\delta + ad + bc) \langle[\infty]\rangle + pq (bd) \langle[0]\rangle \\
&= pq (\langle T\rangle \oplus \langle S\rangle) .
\end{aligned}$$

■

After proving the properties of our operation $\oplus$, we can now get use of these properties in computation. We start with the following lemma.

Lemma 14. *For $n \in \mathbb{N} \cup \{0\}$, we have*

$$\begin{aligned}
\langle[n]\rangle &= \delta^{-1} \left[\left(-A^3\right)^n - \left(A^{-n}\right)\right] \langle[\infty]\rangle + \left[A^{-n}\right] \langle[0]\rangle , \quad \textit{and} \\
\langle[-n]\rangle &= \delta^{-1} \left[\left(-A^{-3}\right)^n - (A^n)\right] \langle[\infty]\rangle + [A^n] \langle[0]\rangle .
\end{aligned}$$

Proof: The equation clearly holds for $n = 0$. Let $n \in \mathbb{N}$. Note that

$$\begin{aligned}
\langle [n] \rangle &= \left\langle \quad + \quad + \cdots + \quad \right\rangle \\
&= \left\langle \quad \right\rangle \oplus \left\langle \quad \right\rangle \oplus \cdots \oplus \left\langle \quad \right\rangle \\
&= \left(A \langle [\infty] \rangle + A^{-1} \langle [0] \rangle\right) \oplus \cdots \oplus \left(A \langle [\infty] \rangle + A^{-1} \langle [0] \rangle\right)
\end{aligned}$$

which, by the nice properties of the operation $\oplus$, can be written as

$$\sum_{i=0}^{n} \binom{n}{i} \left(A \langle [\infty] \rangle\right)^i \oplus \left(A^{-1} \langle [0] \rangle\right)^{n-i} = \sum_{i=0}^{n} \binom{n}{i} A^{2i-n} \langle [\infty] \rangle^i \oplus \langle [0] \rangle^{n-i}$$

and by splitting the term when $i = 0$, reduces to

$$\begin{aligned}
&A^{-n} \langle [0] \rangle + \sum_{i=1}^{n} \binom{n}{i} A^{2i-n} \delta^{i-1} \langle [\infty] \rangle \oplus \langle [0] \rangle \\
&\qquad = \left[\sum_{i=1}^{n} \binom{n}{i} A^{2i-n} \delta^{i-1}\right] \langle [\infty] \rangle + \left[A^{-n}\right] \langle [0] \rangle .
\end{aligned}$$

Now we simplify the term involving the sum as follows

$$\begin{aligned}
\langle [n] \rangle &= \delta^{-1} \left[\sum_{i=1}^{n} \binom{n}{i} A^{i-n} A^{i} \delta^{i}\right] \langle [\infty] \rangle + \left[A^{-n}\right] \langle [0] \rangle \\
&= \delta^{-1} \left[\sum_{i=1}^{n} \binom{n}{i} \left(A^{-1}\right)^{n-i} (A\delta)^{i}\right] \langle [\infty] \rangle + \left[A^{-n}\right] \langle [0] \rangle \\
&= \delta^{-1} \left[\left(\sum_{i=0}^{n} \binom{n}{i} \left(A^{-1}\right)^{n-i} (A\delta)^{i}\right) - \left(A^{-n}\right)\right] \langle [\infty] \rangle + \left[A^{-n}\right] \langle [0] \rangle .
\end{aligned}$$

Therefore

$$\begin{aligned}
\langle [n] \rangle &= \delta^{-1} \left[\left(A^{-1} + A\delta\right)^{n} - \left(A^{-n}\right)\right] \langle [\infty] \rangle + \left[A^{-n}\right] \langle [0] \rangle \\
&= \delta^{-1} \left[\left(-A^{3}\right)^{n} - \left(A^{-n}\right)\right] \langle [\infty] \rangle + \left[A^{-n}\right] \langle [0] \rangle .
\end{aligned}$$

On the other hand

$$\begin{aligned}
\langle [-n] \rangle &= \left\langle \quad + \quad + \cdots + \quad \right\rangle \\
&= \left\langle \quad \right\rangle \oplus \left\langle \quad \right\rangle \oplus \cdots \oplus \left\langle \quad \right\rangle \\
&= \left(A^{-1} \langle [\infty] \rangle + A \langle [0] \rangle\right) \oplus \cdots \oplus \left(A^{-1} \langle [\infty] \rangle + A \langle [0] \rangle\right) .
\end{aligned}$$

Note that this last product is the same as the one above in this proof with A replacing A^{-1}. Note also that δ stays the same when A replaces A^{-1}. Therefore

$$\langle[-n]\rangle = \delta^{-1}\left[\left(-A^{-3}\right)^{n} - (A^{n})\right]\langle[\infty]\rangle + [A^{n}]\langle[0]\rangle .$$

$\blacksquare$

Definition 15. *Let $\alpha \in \mathbb{Z}[A, A^{-1}]$ then α^{*} is the result of replacing each A in α by A^{-1}. Also if $\langle T\rangle = \alpha\langle[\infty]\rangle + \beta\langle[0]\rangle$, then we define $\langle T\rangle^{*} = \alpha^{*}\langle[\infty]\rangle + \beta^{*}\langle[0]\rangle$.*

The following corollary follows from the last lemma.

Corollary 16. *For $n \in \mathbb{N}$, we have*

$$\langle[-n]\rangle = \langle[n]\rangle^{*} .$$

In fact, this can be generalized to any rational tangle as in the following lemma.

Lemma 17. *For any rational tangle T, we have*

$$\langle -T\rangle = \langle T\rangle^{*} .$$

Proof: Recall that

$$\langle T\rangle = \sum_{S} A^{a(S)-b(S)}(-A^{2} - A^{-2})^{|S|}e(S)$$

where $e(S) = \langle[0]\rangle$, or $\langle[\infty]\rangle$. Note that every state S contributing in $\langle -T\rangle$ involves a similar structure of the corresponding state in $\langle T\rangle$, except that each split of a crossing in $\langle -T\rangle$ replaces A by A^{-1} and viceversa. This means that the factor $A^{a(S)-b(S)}$ in $\langle T\rangle$ will be replaced by $A^{b(S)-a(S)}$ in $\langle -T\rangle$. The factor $(-A^{2} - A^{-2})^{|S|}$ in $\langle T\rangle$ stays the same in $\langle -T\rangle$ as it only depends on the number of circles in S. Also $e(S)$ in $\langle T\rangle$ stays the same in $\langle -T\rangle$ as $[-0] = [0]$, and $[-\infty] = [\infty]$. Therefore

$$\langle -T\rangle = \sum_{S} A^{b(S)-a(S)}(-A^{2} - A^{-2})^{|S|}e(S) = \langle T\rangle^{*} .$$

$\blacksquare$

Definition 18. *If $\langle T\rangle = \alpha\langle[\infty]\rangle + \beta\langle[0]\rangle$, then we define the reciprocal of $\langle T\rangle$ denoted by $\dfrac{1}{\langle T\rangle}$ or $\overline{\langle T\rangle}$ to be*

$$\overline{\langle T\rangle} = \alpha^{*}\langle[0]\rangle + \beta^{*}\langle[\infty]\rangle .$$

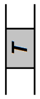
In particular $\overline{\langle[\infty]\rangle} = \langle[0]\rangle$, and $\overline{\langle[0]\rangle} = \langle[\infty]\rangle$.

Lemma 19. *If T is a rational tangle with $\langle T\rangle = \alpha\langle[\infty]\rangle + \beta\langle[0]\rangle$, then we have the following reciprocal formula*

$$\left\langle \frac{1}{T} \right\rangle = \frac{1}{\langle T\rangle} \equiv \overline{\langle T\rangle}.$$

Proof: Note that

$\left\langle \frac{1}{T} \right\rangle = \Big\langle$ 

$\Big\rangle = \Big\langle$ 

$\Big\rangle^{*} = (\alpha\langle[0]\rangle + \beta\langle[\infty]\rangle)^{*} = \alpha^{*}\langle[0]\rangle + \beta^{*}\langle[\infty]\rangle = \overline{\langle T\rangle}.$

■

As a corollary, we have the following.

Corollary 20. *For $n \in \mathbb{Z}$, we have*

$$\left\langle \frac{1}{[n]} \right\rangle = [A^{n}]\langle[\infty]\rangle + \delta^{-1}\left[\left(-A^{-3}\right)^{n} - (A^{n})\right]\langle[0]\rangle \equiv \overline{\langle[n]\rangle}.$$

Most importantly, now the bracket of a rational tangle $T = [a_1 \dots a_m]$ can be written and computed as a continued fraction as in the following proposition.

Proposition 21. *For a rational tangle $T = [a_1 \dots a_m]$, we have*

$$\langle T\rangle = \langle[a_m]\rangle \oplus \cfrac{1}{\langle[a_{m-1}]\rangle \oplus \cfrac{1}{\ddots \cfrac{1}{\langle[a_2]\rangle \oplus \cfrac{1}{\langle[a_1]\rangle}}}}.$$

Proof: This is clearly true by induction on m with the additivity of $\oplus$, and the reciprocal formula. ■

4. A Sum Formula for the Bracket of a Rational Tangle

Let $a_1, \dots, a_{m-1} \in \mathbb{Z} - \{0\}$, and $a_m \in \mathbb{Z}$. Let $[a_i] = \alpha_i\langle[\infty]\rangle + \beta_i\langle[0]\rangle$. Let x_i be a variable that takes the values α_i or β_i. Let P be the product given by

$$P = \begin{cases} x_m x_{m-1}^{*} \cdots x_2 x_1^{*}, & \text{if } m \text{ is even} \\ x_m x_{m-1}^{*} \cdots x_2^{*} x_1, & \text{if } m \text{ is odd.} \end{cases}$$

Note that for a fixed m, we will have 2^m products as each x_i varies over α_i and β_i.

Moreover, let $h(\alpha_i) = \langle[\infty]\rangle$, and $h(\beta_i) = \langle[0]\rangle$ for $i = 1, \ldots, m$. For each P, define $\overleftarrow{P}$, also denoted by $\overleftarrow{x_m x_{m-1} \cdots x_2 x_1}$ (without any $*$), by

$$\overleftarrow{P} = \overline{h(x_m) \oplus \overline{h(x_{m-1}) \cdots \overline{h(x_3) \oplus \overline{h(x_2) \oplus \overline{h(x_1)}}}}}.$$

Example 22. *Let $T = [a_1\ a_2\ a_3]$, and let*

$$\begin{aligned} \langle[a_1]\rangle &= \alpha_1 \langle[\infty]\rangle + \beta_1 \langle[0]\rangle, \qquad \langle[a_2]\rangle = \alpha_2 \langle[\infty]\rangle + \beta_2 \langle[0]\rangle \\ \langle[a_3]\rangle &= \alpha_3 \langle[\infty]\rangle + \beta_3 \langle[0]\rangle \end{aligned}$$

then we will have the following 8 products as values of P

$$\begin{gathered} \alpha_3\alpha_2^*\alpha_1, \alpha_3\alpha_2^*\beta_1, \alpha_3\beta_2^*\alpha_1, \alpha_3\beta_2^*\beta_1 \\ \beta_3\beta_2^*\beta_1, \beta_3\alpha_2^*\beta_1, \beta_3\beta_2^*\alpha_1, \beta_3\alpha_2^*\alpha_1 \end{gathered}$$

and we will have the following 8 values of $\overleftarrow{P}$, respectively. For simplicity, from now on, we will write these products without the operation sign $\oplus$

$$\begin{gathered} \langle[\infty]\rangle\, \overline{\langle[\infty]\rangle\, \overline{\langle[\infty]\rangle}},\ \langle[\infty]\rangle\, \overline{\langle[\infty]\rangle\, \overline{\langle[0]\rangle}},\ \langle[\infty]\rangle\, \overline{\langle[0]\rangle\, \overline{\langle[\infty]\rangle}},\ \langle[\infty]\rangle\, \overline{\langle[0]\rangle\, \overline{\langle[0]\rangle}} \\ \langle[0]\rangle\, \overline{\langle[0]\rangle\, \overline{\langle[0]\rangle}},\ \langle[0]\rangle\, \overline{\langle[\infty]\rangle\, \overline{\langle[0]\rangle}},\ \langle[0]\rangle\, \overline{\langle[0]\rangle\, \overline{\langle[\infty]\rangle}},\ \langle[0]\rangle\, \overline{\langle[\infty]\rangle\, \overline{\langle[\infty]\rangle}}. \end{gathered}$$

Theorem 23. *Let $T = [a_1 \ldots a_m]$, where $a_1, \ldots, a_{m-1} \in \mathbb{Z} - \{0\}$, and $a_m \in \mathbb{Z}$, then*

$$\langle T\rangle = \sum_P P\overleftarrow{P}.$$

Proof: Let $T = [a_1 \ldots a_m]$, where $a_1, \ldots, a_{m-1} \in \mathbb{Z} - \{0\}$, and $a_m \in \mathbb{Z}$ be written in the continued fraction form, then

$$\langle T\rangle = \left\langle [a_m] + \cfrac{1}{[a_{m-1}] + \cfrac{1}{\ddots + \cfrac{1}{[a_2] + \cfrac{1}{[a_1]}}}} \right\rangle$$

$$= \langle[a_m]\rangle \oplus \left\langle \cfrac{1}{[a_{m-1}] + \cfrac{1}{\ddots + \cfrac{1}{[a_2] + \cfrac{1}{[a_1]}}}} \right\rangle$$

$$= \langle [a_m] \rangle \oplus \overline{\left\langle [a_{m-1}] + \cfrac{1}{\ddots + \cfrac{1}{[a_2] + \cfrac{1}{[a_1]}}} \right\rangle}$$

$$= \langle [a_m] \rangle \oplus \overline{[a_{m-1}] \oplus \left\langle \cfrac{1}{\ddots + \cfrac{1}{[a_2] + \cfrac{1}{[a_1]}}} \right\rangle}$$

$$= \langle [a_m] \rangle \oplus \overline{[a_{m-1}] \oplus \overline{\overline{\left\langle \ddots + \cfrac{1}{[a_2] + \cfrac{1}{[a_1]}} \right\rangle}}}$$

$$\vdots$$

$$= \langle [a_m] \rangle \oplus \overline{\langle [a_{m-1}] \rangle \cdots \overset{\cdots}{\overline{\overline{\langle [a_2] \rangle \oplus \overline{\langle [a_1] \rangle}}}}}.$$

Continue in the same way with the following assumptions $h(\alpha_i) = \langle [\infty] \rangle$, and $h(\beta_i) = \langle [0] \rangle$ for $i = 1, \ldots, m$, and x_i a variable that takes the values α_i or β_i, and P the product given by

$$P = \left\{ \begin{array}{ll} x_m x_{m-1}^* \cdots x_2 x_1^*, & \text{if } m \text{ is even} \\ x_m x_{m-1}^* x_{m-2} \cdots x_2^* x_1, & \text{if } m \text{ is odd} \end{array} \right\}, \quad \text{and}$$

$\overleftarrow{P}$ the product given by

$$\overleftarrow{P} = h(x_m) \oplus \overline{h(x_{m-1}) \cdots \overset{\cdots}{\overline{\overline{h(x_3) \oplus \overline{h(x_2) \oplus \overline{h(x_1)}}}}}}.$$

Then we will finally have 2^m terms each consists of a possible product $P\overleftarrow{P}$. Therefore

$$\langle T \rangle = \sum_P P\overleftarrow{P}.$$

■

Example 24. *We compute the bracket for the tangle $T = [5\ 6\ 7]$. At first, using the formula*

$$\langle [n] \rangle = \delta^{-1} \left[\left(-A^3\right)^n - \left(A^{-n}\right) \right] \langle [\infty] \rangle + \left[A^{-n} \right] \langle [0] \rangle$$

we get

$$\begin{aligned}
\alpha_1 &= \delta^{-1} \left[\left(-A^3\right)^5 - \left(A^{-5}\right) \right], & \beta_1 &= \left[A^{-5} \right] \\
\alpha_2^* &= \delta^{-1} \left[\left(-A^{-3}\right)^6 - \left(A^{6}\right) \right], & \beta_2^* &= \left[A^{6} \right] \\
\alpha_3 &= \delta^{-1} \left[\left(-A^3\right)^7 - \left(A^{-7}\right) \right], & \beta_3 &= \left[A^{-7} \right]
\end{aligned}$$

then

$$\begin{aligned}
\langle T \rangle &= \sum_P P \overleftarrow{P} \\
&= \alpha_3 \alpha_2^* \alpha_1 \left(\langle [\infty] \rangle \overline{\langle [\infty] \rangle \overline{\langle [\infty] \rangle}} \right) + \alpha_3 \alpha_2^* \beta_1 \left(\langle [\infty] \rangle \overline{\langle [\infty] \rangle \overline{\langle [0] \rangle}} \right) \\
&\quad + \alpha_3 \beta_2^* \alpha_1 \left(\langle [\infty] \rangle \overline{\langle [0] \rangle \overline{\langle [\infty] \rangle}} \right) + \alpha_3 \beta_2^* \beta_1 \left(\langle [\infty] \rangle \overline{\langle [0] \rangle \overline{\langle [0] \rangle}} \right) \\
&\quad + \beta_3 \beta_2^* \beta_1 \left(\langle [0] \rangle \overline{\langle [0] \rangle \overline{\langle [0] \rangle}} \right) + \beta_3 \alpha_2^* \beta_1 \left(\langle [0] \rangle \overline{\langle [\infty] \rangle \overline{\langle [0] \rangle}} \right) \\
&\quad + \beta_3 \beta_2^* \alpha_1 \left(\langle [0] \rangle \overline{\langle [0] \rangle \overline{\langle [\infty] \rangle}} \right) + \beta_3 \alpha_2^* \alpha_1 \left(\langle [0] \rangle \overline{\langle [\infty] \rangle \overline{\langle [\infty] \rangle}} \right).
\end{aligned}$$

Therefore

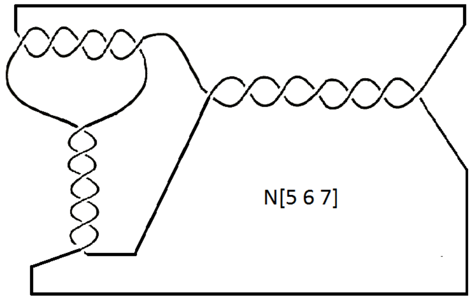
$$\begin{aligned}
\langle T \rangle = {}& \alpha_3 \alpha_2^* \alpha_1 \langle [\infty] \rangle + \alpha_3 \alpha_2^* \beta_1 \delta \langle [\infty] \rangle + \alpha_3 \beta_2^* \alpha_1 \delta \langle [\infty] \rangle + \alpha_3 \beta_2^* \beta_1 \langle [\infty] \rangle \\
& + \beta_3 \beta_2^* \beta_1 \langle [0] \rangle + \beta_3 \alpha_2^* \beta_1 \delta \langle [0] \rangle + \beta_3 \beta_2^* \alpha_1 \langle [\infty] \rangle + \beta_3 \alpha_2^* \alpha_1 \langle [0] \rangle .
\end{aligned}$$

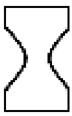
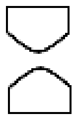
Finally

$$\begin{aligned}
\langle T \rangle = {}& \left[\alpha_3 \alpha_2^* \alpha_1 + \alpha_3 \alpha_2^* \beta_1 \delta + \alpha_3 \beta_2^* \alpha_1 \delta + \alpha_3 \beta_2^* \beta_1 + \beta_3 \beta_2^* \alpha_1 \right] \langle [\infty] \rangle \\
& + \left[\beta_3 \beta_2^* \beta_1 + \beta_3 \alpha_2^* \beta_1 \delta + \beta_3 \alpha_2^* \alpha_1 \right] \langle [0] \rangle
\end{aligned}$$

where $\alpha_1, \alpha_2^*, \alpha_3, \beta_1, \beta_2^*$, and β_3 are given above, and $\delta = -A^2 - A^{-2}$.

Moreover, if we want the bracket of the numerator of T*, that is the knot* $\mathrm{N}(T) = \mathrm{N}\,[5\ 6\ 7]$ *given below*



we just replace $\langle [\infty] \rangle$ *by* $\Big\langle$  $\Big\rangle = 1$, *and* $\langle [0] \rangle$ *by* $\Big\langle$  $\Big\rangle = \delta$. *Therefore*

$$\begin{aligned}\langle \mathrm{N}(T) \rangle = {} & [\alpha_3\alpha_2^*\alpha_1 + \alpha_3\alpha_2^*\beta_1\delta + \alpha_3\beta_2^*\alpha_1\delta + \alpha_3\beta_2^*\beta_1 + \beta_3\beta_2^*\alpha_1] \\ & + [\beta_3\beta_2^*\beta_1 + \beta_3\alpha_2^*\beta_1\delta + \beta_3\alpha_2^*\alpha_1]\,\delta.\end{aligned}$$

The methodology used in this paper can be generalized for other knots and links. Namely those which can be constructed using addition and reciprocals of rational tangles. For example, Pretzel links, or more generally, Algebraic links.

References

[1] Adams C., *The Knot Book*, Freeman, San Francisco 1994.

[2] Bankwitz C. and Schumann H., *Uber Viergeflechte*, Abh. Math. Sem. Univ. Hamburg **10** (1934) 263-284.

[3] Bleiler S. and Moriah J., *Heegaard Splittings and Branched Coverings of B3*, Math. Ann. **281** (1988) 531-543.

[4] Burde G. and Zieschang H., *Knots*, De Gruyter, Berlin 1985.

[5] Conway J., *An Enumeration of Knots and Links and some of Their Algebraic Properties*, In: Proceedings on Computational problems in Abstract Algebra held at Oxford in 1967, J. Leech (Ed), Pergamon, Oxford 1970, pp 329-358.

[6] Goldman J. and Kauffman L., *Rational Tangles*, Adv. Appl. Math. **18** (1997) 300-332.

[7] Kawauchi A., *A Survey of Knot Theory*, Birkhauser, Basel 1996.

[8] Kauffman L. and Lambropoulou S., *On the Classification of Rational Knots*, L' Enseignement Mathématique **49** (2003) 357–410.

[9] Kauffman L. and Lambropoulou S., *On the Classification of Rational Tangles*, Adv. Appl. Math. **33** (2004) 199-237.

[10] Kauffman L. and Lambropoulou S., *From Tangle Fractions to DNA*, In: Topology in Molecular Biology, M. Monastyrsky (Ed), Springer, Berlin 2007, pp 69-110.

[11] Reidemeister K., *Knoten und Verkettungen*, Math. Zeitschrift **29** (1929) 713-729.

[12] Rolfsen D., *Knots and Links*, Publish or Perish Press, Berkeley 1976.