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ON CONFORMAL MAPPINGS ONTO COMPACT EINSTEIN MANIFOLDS

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Abstract. In the present paper we prove non-existence theorems for conformal mappings of compact (pseudo-)Riemannian manifolds onto Einstein manifolds without boundary. We obtained certain conditions for which these mappings are only trivial.

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This paper is devoted to conformal mappings of special (pseudo-)Riemannian spaces onto Einstein spaces.

On the basis of studying the fundamental linear equations which were obtained by Mikeš, Gavril'chenko and Gladyscheva [13] we found new results of conformal mappings of compact n-dimensional pseudo-Riemannian manifolds onto Einstein manifolds.

Conformal mappings of n-dimensional Riemannian spaces V_n were studied in many papers, see for example [2,3,5,6,10–12,15,16]. We assume that the metrics g of V_n under study are of arbitrary signature, i.e., V_n is either a proper Riemannian or a pseudo-Riemannian space. Conformal mappings have applications in the general theory of relativity (see, e.g., [1,4,5,15]).

In 1920 Brinkmann [1] started researching on conformal mapping of (pseudo-)Riemannian manifolds V_n onto Einstein spaces $\bar{V}_n$. He obtained the fundamental system of differential equations in covariant derivatives of Cauchy type with respect to $(n+1)$ unknown functions. This problem is stated in detail in Petrov's monograph [15].

As we said above, in [13] J. Mikeš, M. Gavril'chenko and E. Gladyscheva found the mentioned fundamental system of differential equations in linear form, see [9], [14, pp. 112–116], [12, pp. 242–246]. We studied this system precisely in [7].

The number p $(\leq\ n+2)$ of substantial parameters on which the general solutions depend is called a *degree of mobility of a Riemannian space with respect to conformal mappings onto Einstein spaces*.

Lacunas in the distribution of mobility degrees of Riemannian spaces with respect to conformal mappings onto Einstein spaces were found [7–9, 13].

In the above mentioned papers, it is assumed that the geometric objects in question are of rather high differentiability class. Our paper is devoted to the study of minimal conditions on the differentiability of the geometric objects under conformal mappings of V_n onto Einstein spaces. As it is known, by means of the choice of a coordinate system one can decrease the differentiability class to the "minimal level". We assume that the dimensions of the spaces under consideration is greater than three.

1. Basic Facts on Conformal Mappings

Consider a Riemannian space V_n with metric g of arbitrary signature. If in terms of a coordinate chart (U,x) the components $g_{ij}(x)$ of the metric belong to the class C^r, we will write $V_n \in C^r$. We assume that $r = 2,3,\ldots,\infty,\omega$, where C^∞ is the class of functions possessing continuous partial derivatives of any order, C^ω is the class of real analytic functions.

In $V_n \in C^1$ there are defined the Christoffel symbols of the first and the second types

$$\Gamma_{ijk} = 1/2\ (\partial_i g_{jk} + \partial_j g_{ik} - \partial_k g_{ij}) \qquad \text{and} \qquad \Gamma^h_{ij} = g^{h\alpha}\Gamma_{ij\alpha}$$

where g^{ij} are the components of the inverse of the matrix $\|g_{ij}\|$ and $\partial_i = \partial/\partial x^i$.

In $V_n \in C^2$, there are defined the Riemann and the Ricci tensors and the scalar curvature

$$R_{hijk} = g_{h\alpha}R^\alpha_{ijk}, \qquad R^h_{ijk} = \partial_j\Gamma^h_{ik} - \partial_k\Gamma^h_{ij} + \Gamma^\alpha_{ik}\Gamma^h_{\alpha j} - \Gamma^\alpha_{ij}\Gamma^h_{\alpha k}$$

$$R_{ij} = R^\alpha_{i\alpha j}, \qquad R = R_{\alpha\beta}g^{\alpha\beta}.$$

Note that in many works (e.g., [16,17]) the Ricci tensor is defined with the opposite sign. Further, in V_n, there are defined the Weyl tensor of conformal curvature C

$$C^h_{ijk} = g^{h\alpha}C_{\alpha ijk}, \qquad C_{hijk} = R_{hijk} - g_{hj}L_{ik} - g_{ik}L_{hj} + g_{hk}L_{ij} + g_{ij}L_{hk}$$

and the tensors

$$L_{ij}=\frac{1}{n-2}\,(R_{ij}-\frac{R}{2(n-1)}\,g_{ij}),\quad L_i^h=g^{h\alpha}L_{\alpha i}\quad\text{and}\quad L_{ijk}=L_{ij,k}-L_{ik,j}.$$

Here and in what follows the comma stands for the covariant derivative with respect to the Levi-Civita connection of V_n. If $V_n\in C^r$, then

$$\Gamma^h_{ij},\Gamma_{ijk}\in C^{r-1},\ R^h_{ijk},R_{hijk},R_{ij},R,C^h_{ijk},C_{hijk},L_{ij},L^h_i\in C^{r-2},\ L_{ijk}\in C^{r-3}.$$

Two Riemannian spaces V_n and $\bar V_n$ are in a *conformal correspondence* if, in a coordinate system (U,x) which is common with respect to a mapping between V_n and $\bar V_n$, their metric tensors g_{ij} and $\bar g_{ij}$ are related as follows [5, 6, 12, 14–16]

$$\bar g_{ij}(x)=\mathrm{e}^{2\sigma(x)}g_{ij}(x) \tag{1}$$

where σ is an invariant.

As it is known, for spaces V_n and $\bar V_n$ which are in a conformal correspondence, the following relations hold

$$\begin{aligned}
\bar\Gamma^h_{ij}\ &=\Gamma^h_{ij}+\delta^h_i\sigma_j+\delta^h_j\sigma_i-\sigma^h g_{ij}\\
\bar R^h_{ijk}&=R^h_{ijk}+\delta^h_k\sigma_{ij}-\delta^h_j\sigma_{ik}+\sigma^h_k g_{ij}-\sigma^h_j g_{ik}+\Delta_1\sigma\,(\delta^h_k g_{ij}-\delta^h_j g_{ik})\\
\bar R_{ij}\ &=R_{ij}-(n-2)\sigma_{ij}-(\Delta_2\sigma+(n-2)\Delta_1\sigma)g_{ij}\\
\bar C^h_{ijk}&=C^h_{ijk}.
\end{aligned}$$

Here Γ^h_{ij} $(\bar\Gamma^h_{ij})$ are the Christoffel symbols, R^h_{ijk} $(\bar R^h_{ijk})$ the Riemann tensors, R_{ij} $(\bar R_{ij})$ the Ricci tensors, and C^h_{ijk} $(\bar C^h_{ijk})$ the conformal curvature tensors of V_n and $\bar V_n$, respectively. In addition, δ^h_i is the Kronecker's delta, $\Delta_1\sigma=g^{\alpha\beta}\sigma_\alpha\sigma_\beta$ and $\Delta_2\sigma=g^{\alpha\beta}\sigma_{,\alpha\beta}$ are the first and the second Beltrami operators, respectively

$$\sigma_i=\sigma_{,i},\qquad \sigma_{ij}=\sigma_{i,j}-\sigma_i\sigma_j,\qquad \sigma^h=\sigma_\alpha g^{\alpha h},\qquad \sigma^h_i=\sigma_{\alpha i}g^{\alpha h}.$$

The above formulas hold when V_n and $\bar V_n\in C^2$ are considered in a common coordinate system with respect to the mapping between V_n and $\bar V_n$.

2. Conformal Mappings onto Einstein Spaces

In [13] it is proved that a Riemannian space V_n admits a conformal mapping onto an Einstein space $\bar V_n$ if and only if in V_n there exists a solution of the following system of linear homogeneous differential equations in covariant derivatives of Cauchy type with respect to the functions $u(x)$ and $s(x)$ (>0)

$$s_{,ij}=u\,g_{ij}-s\,L_{ij}. \tag{2}$$

In this case, $s=\mathrm{e}^{-\sigma}$, and (1) takes the form

$$\bar g_{ij}=s^{-2}g_{ij}.$$

These conditions are fulfilled for minimal requirements on the smoothness class of the functions under consideration, i.e., when $s \in C^2$ and u is a continuous function. Then obviously V_n and $\bar{V}_n \in C^2$.

In [7] we have obtained the following theorem.

Theorem 1. *A Riemannian space $V_n \in C^r$, $r > 2$, admits a conformal mapping onto an Einstein space $\bar{V}_n \in C^2$ if and only if in V_n there exists a solution to the closed system of linear homogeneous differential equations in covariant derivatives of Cauchy type*

$$\text{a)}\ s_{,i} = s_i, \qquad \text{b)}\ s_{i,j} = u\, g_{ij} - s\, L_{ij}, \qquad \text{c)}\ u_{,i} = -s_\alpha L_i^\alpha \tag{3}$$

with respect to the functions $u(x)$, $s(x)\,(>0)$ and a vector $s_i(x)$. In this case, space $\bar{V}_n \in C^r$.

In [13], as a matter of fact, under the condition that V_n and $\bar{V}_n \in C^3$ the above Theorem 1 is proved.

3. Modification of a Certain Theorem by Hopf

After modification of a well known Hopf Theorem about the existence of the solutions of differential equations in partial derivative on manifolds [17] (pp. 26) we prove the new results about conformal mappings onto Einstein spaces.

In advance we modify the well known Hopf Theorem.

Theorem 2. *Let $\varphi \in C^2$ be a function on a connected compact manifold M_n without boundary. Then if for every point $P_0 \in M_n$ there exists the coordinate neighbourhood $U(x^1, x^2, \ldots, x^n) \in M_n$ and in this neighbourhood there exist the continuous functions $A^{ij}(x)$ and $B^i(x)$ of $P(x) \in U$ such that on the whole U the following inequality holds*

$$A^{ab}(x)\, \frac{\partial^2 \varphi}{\partial x^a \partial x^b} + B^a(x)\, \frac{\partial \varphi}{\partial x^a} \geq 0 \quad \textit{respectively} \quad \leq 0 \tag{4}$$

where $A^{ab}(x) z_a z_b$ is a positive form then φ is constant on M_n, and "$\geq$" or "$\leq$" in the inequalities (4) *for all coordinate neighbourhoods are the same.*

Here and later we suppose that the studied manifolds are connected without boundary.

Obviously, in (pseudo-)Riemannian manifolds and manifolds with affine connection the partial derivative in formula (4) may be replaced by the covariant derivative.

We point out that in Theorem 2 we do not suppose that the functions $A^{ij}(x)$ and $B^i(x)$ define in all coordinate neighborhoods geometric objects which are determined "in the whole" of M_n, in such a way as required, as for example in a theorem by K. Yano and S. Bochner [17, p. 26].

Proof: For a compact manifold M_n we can choose a finite set of coordinate neighbourhoods U, such that the unification of these coordinate neighbourhoods covered M_n and in each of them the condition (4) holds (for definition take the sign "$\geq$").

Because φ is smooth and M_n is compact, the function φ on M_n reaches its maximum at a point $P_0 \in U_0$, where U_0 is one of the neighbourhoods chosen earlier. Then $\varphi(P) \leq \varphi(P_0)$ for each points $P \in U_0$ and on U_0 all the conditions of Hopf's Theorem hold, and we have $\varphi(P) = \varphi(P_0)$ for any point $P \in U_0$.

Further consider a coordinate neighbourhood U_1 which has an intersection with U_0. Evidently, $\varphi(P) \leq \varphi(P_0)$ for any point $P \in U_1$. From this $\varphi(P) = \varphi(P_0)$ for any point $P \in U_1$. Analogically we exhausted any coordinate neighbourhood U. Because the number of them is finite and M_n is connected, we verify that $\varphi(P) = \varphi(P_0)$ for any point $P \in M_n$. The theorem is proved. ■

4. Conformal Mappings of Pseudo-Riemannian Manifolds onto Compact Einstein Spaces

Mikeš, Gavrilchenko and Gladysheva [13] obtained certain results, which are devoted to the problems of conformal mappings of pseudo-Riemannian manifolds onto compact Einstein spaces

Theorem 3. *A compact space V_n in which the tensor S_{ijk} vanishes at not more than one point does not admit nontrivial conformal mappings onto Einstein spaces.*

Theorem 4. *Compact nonconformally flat symmetric Riemannian spaces do not admit nontrivial mappings onto Einstein spaces.*

We proved the following result

Theorem 5. *Let V_n be a compact pseudo-Riemannian manifold. If the Ricci tensor constitutes a positive (or negative) form, then a conformal mapping V_n onto Einstein spaces is only homothetic.*

From differential prolongations of integrability conditions of the fundamental equations (3) we prove the following theorems.

Firstly we obtain integrability conditions of the fundamental equations (3). From the Ricci identity $s_{i,jk} - s_{ik,j} = s_\alpha R^\alpha_{ijk}$ and after substitution of (3) we get (see [13])

$$s_\alpha C^\alpha_{ijk} + s\, S_{ijk} = 0. \tag{5}$$

After differentiating this integrability condition we obtain

$$u\, C_{hijk} + s\,(S_{ijk,h} - L_{h\alpha} C^\alpha_{ijk}) + s_\alpha\, C^\alpha_{ijk,h} + s_h\, S_{ijk} = 0. \tag{6}$$

If $C_{hijk} \neq 0$, then from the last formula we have obtained

$$u = s \cdot a + s_\alpha b^\alpha \tag{7}$$

where a is a function and b^α is a vector field on V_n. This formula is true on all of the manifold V_n if the condition $C_{hijk} \neq 0$ is true on all V_n with the exception of a set of measure zero.

After exclusion of u from the formula (6) we obtain

$$s\,(a\,C_{hijk} + S_{ijk,h} - L_{h\alpha}C^\alpha_{ijk}) + s_\alpha\,(b^\alpha C_{hijk} + C^\alpha_{ijk,h}) + s_h\,S_{ijk} = 0. \qquad (8)$$

From formula (8) the following theorem holds.

Theorem 6. *A compact space V_n in which the tensor*

$$a\,C_{hijk} + S_{ijk,h} - L_{h\alpha}C^\alpha_{ijk} \qquad (9)$$

vanishes at not more than one point does not admit nontrivial conformal mappings onto Einstein spaces.

Proof: The proof follows from the fact that, if a conformal mapping is nontrivial, there must exist a maximum and a minimum for the function s. But at these points $s_i = 0$. Since $s > 0$, from (8) we find that the tensor (9) vanishes at least at two points in the space V_n. ■

Finally, the following theorem holds.

Theorem 7. *Let V_n be a compact (pseudo-)Riemannian manifold for which C_{hijk} $\neq 0$ is true on all V_n with the exception of a set of measure zero.*

If for the tensor $L^*_{ij} \equiv L_{ij} + a \cdot g_{ij}$ in any point $x_0 \in V_n$ exists a vector v^i for which $L^*_{ij}v^iv^j > 0$ $(< 0$, respectively), then a conformal mapping V_n onto Einstein spaces is only homothetic.

Proof: Let V_n be a compact (pseudo-)Riemannian manifold for which C_{hijk} is non vanishing on all V_n with the exception of a set of measure zero, and V_n admits a conformal mapping onto the Einstein space $\bar{V}_n$. Then formula (7) will be true and we can rewrite equation (2) in the form

$$s_{,ij} - s_{,\alpha}b^\alpha\,g_{ij} = s \cdot L^*_{ij}. \qquad (10)$$

We note that the functions α and b^i determine on V_n.

We suppose that a vector v^i that $L^*_{ij}v^iv^j > 0$ $(< 0$, respectively) exists at all points $x_0 \in V_n$.

For any point x_0 of V_n there exists a coordinate neighbourhood $U^*(x)$ such that $v^i = \delta^i_1$ and also $L^*_{11}(x_0) > 0$ (< 0).

We assume a positive constant A for which

$$A > -(L^*_{22}(x_0) - L^*_{33}(x_0) - \cdots - L^*_{nn}(x_0))/L^*_{11}(x_0).$$

Therefore

$$A^{ij}(x) \overset{\text{def}}{=} \operatorname{diag}(\mathrm{A}, 1, 1, 1, \ldots, 1)$$

gives rise to a positive form $A^{ij}(x)z_\alpha z_\beta$ on the neighbourhood U^*.
In the point x_0 holds

$$L^*_{ij}(x_0)\cdot A^{ij}(x_0)>0, \qquad \text{respectively} < 0.$$

From (10) it follows that

$$(s_{,ij}-s_{,\alpha}b^\alpha\, g_{ij})(x_0)\cdot A^{ij}(x_0)>0, \qquad \text{respectively} < 0.$$

This implies the existence of the coordinate neighbourhood $U(x)\subset U^*$ in which for any point $x\in U$

$$(s_{,ij}-s_{,\alpha}b^\alpha\, g_{ij})(x)\cdot A^{ij}(x)>0, \qquad \text{respectively} < 0 \tag{11}$$

holds.
Finally, because $s_{,ij}=\partial_{ij}s-\partial_\alpha s\,\Gamma^\alpha_{ij}$ we have that from Theorem 2 follows that the function s is constant, i.e. the conformal mapping is homothetic. ■

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