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GEODESIC MAPPINGS ONTO RIEMANNIAN MANIFOLDS AND DIFFERENTIABILITY

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Abstract. In this paper we study fundamental equations of geodesic mappings of manifolds with affine connection onto (pseudo-) Riemannian manifolds. We proved that if a manifold with affine (or projective) connection of differentiability class C^r $(r \geq 2)$ admits a geodesic mapping onto a (pseudo-) Riemannian manifold of class C^1, then this manifold belongs to the differentiability class C^{r+1}. From this result follows if an Einstein spaces admits non-trivial geodesic mappings onto (pseudo-) Riemannian manifolds of class C^1 then this manifold is an Einstein space, and there exists a common coordinate system in which the components of the metric of these Einstein manifolds are real analytic functions.

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1. Introduction

To the theory of geodesic mappings and transformations were devoted many papers, these results are formulated in a large number of research papers and monographs [3, 5–12, 14–28], etc.

First we studied the general properties of geodesic mappings of manifolds with affine and projective connection onto (pseudo-) Riemannian manifolds in dependence on the smoothness class of these geometric objects. Here we present some well known facts, which were proved by Weyl [28], Thomas [26], Mikeš and Berezovski [17], see [5, 16, 20–22, 24].

In these results no details about the smoothness class of the metric, respectively connection, were stressed. They were formulated "for sufficiently smooth" geometric objects.

In the papers [10–12] we proved that these mappings preserve the smoothness class of the metrics of geodesically equivalent (pseudo-) Riemannian manifolds. We prove that this property generalizes in a natural way for a more general case.

2. Main Theorems

Let $A_n = (M, \nabla)$ and $P_n = (M, \blacktriangledown)$ be manifolds with affine and projective connection, respectively; and $\bar{V}_n = (M, \bar{g})$ be a (pseudo-) Riemannian manifold. The functions $\Gamma^h_{ij}(x)$, $\Pi^h_{ij}(x)$ and $\bar{g}_{ij}(x)$ are components of ∇, $\blacktriangledown$ and $\bar{g}$ in the coordinate system (U, x), $U \subset M$, and A_n, P_n and $\bar{V}_n$ belong to the differentiability class C^r if these functions are C^r.

Hinterleitner and Mikeš [12] proved the following theorems.

Theorem 1. *If $P_n \in C^r$ $(r \geq 2)$ admits geodesic mappings onto a* (pseudo-) *Riemannian manifold $\bar{V}_n \in C^2$, then $\bar{V}_n \in C^{r+1}$.*

Theorem 2. *If $A_n \in C^r$ $(r \geq 2)$ admits geodesic mappings onto a* (pseudo-) *Riemannian manifold $\bar{V}_n \in C^2$, then $\bar{V}_n \in C^{r+1}$.*

In this paper we proved a generalization of these theorems.

Theorem 3. *If $P_n \in C^r$ $(r \geq 2)$ admits geodesic mappings onto a* (pseudo-) *Riemannian manifold $\bar{V}_n \in C^1$, then $\bar{V}_n \in C^{r+1}$.*

Theorem 4. *If $A_n \in C^r$ $(r \geq 2)$ admits geodesic mappings onto a* (pseudo-) *Riemannian manifold $\bar{V}_n \in C^1$, then $\bar{V}_n \in C^{r+1}$.*

From the last Theorem and our results [10] for geodesic mappings of Einstein spaces we have the following theorem.

Theorem 5. *If the Einstein space V_n admits a non-trivial geodesic mapping onto a* (*pseudo-*) *Riemannian manifold $\bar{V}_n \in C^1$, then $\bar{V}_n$ is an Einstein space. Moreover, there exists a common coordinate system in which the components of the metric V_n and $\bar{V}_n$ are real analytic functions.*

Theorem 5 generalize results by Mikeš [15], see [6, 16, 20], which were proved in the case when V_n and $\bar{V}_n \in C^3$.

The above results about geodesic mappings of Einstein spaces are valid globally, this follows from the paper [4] by DeTurk and Kazhdan, see [1, p. 196], in which it is shown that in an Einstein manifold exists a real analytic coordinate system, i.e., in which the components of the metric tensor are real analytic functions.

3. Geodesic Mapping Theory for Manifolds With Affine and Projective Connections

Let $A_n = (M, \nabla)$ and $\bar{A}_n = (\bar{M}, \bar{\nabla})$ be manifolds with affine connections ∇ and $\bar{\nabla}$, respectively.

Definition 6. A diffeomorphism $f\colon A_n \to \bar{A}_n$ is called a *geodesic mapping* of A_n onto $\bar{A}_n$ if f maps any geodesic in A_n onto a geodesic in $\bar{A}_n$.

Because geodesics are independent of the antisymmetric parts of connections, we suppose that ∇ and $\bar{\nabla}$ are connections without torsion. A manifold A_n admits a geodesic mapping onto $\bar{A}_n$ if and only if the *Levi-Civita equations* (Weyl [28], see [5, p. 56], [20, p. 130], [21, p. 260], [22, p. 166])

$$\bar{\nabla}_X Y = \nabla_X Y + \psi(X)Y + \psi(Y)X \tag{1}$$

hold for any tangent fields X, Y and where ψ is a differential form on M $(= \bar{M})$. If $\psi \equiv 0$ then f is *affine* or *trivially geodesic*.

Eliminating ψ from the formula (1) Thomas [27], see [5, p. 98], [21, p. 263], obtained that equation (1) is equivalent to

$$\bar{\Pi}(X, Y) = \Pi(X, Y) \ \text{ for all tangent vectors } X, Y \tag{2}$$

where

$$\Pi(X, Y) = \nabla(X, Y) - \frac{1}{n+1}\left(\operatorname{trace}(V \to \nabla_V X) \cdot Y + \operatorname{trace}(V \to \nabla_V Y) \cdot X\right)$$

is the *Thomas' projective parameter* or *Thomas' object of projective connection*.

A geometric object Π that transforms according to a similar transformation law as Thomas' projective parameters is called a *projective connection*, and manifolds on which an object of projective connection is defined is called a *manifold with projective connection*, denoted by P_n. Such manifolds represent an obvious generalization of affine connection manifolds.

A projective connection on P_n will be denoted by $\blacktriangledown$. Obviously, $\blacktriangledown$ is a mapping $TP_n \times TP_n \to TP_n$, i.e., $(X, Y) \mapsto \blacktriangledown_X Y$. Thus, we denote a manifold M with projective connection $\blacktriangledown$ by $P_n = (M, \blacktriangledown)$. See [5, p. 99], [21, p. 264].

We restricted ourselves to the study of coordinate neighborhoods (U, x) of the points $p \in A_n$ (P_n) and $f(p) \in \bar{A}_n$ $(\bar{P}_n)$. The points p and $f(p)$ have the same coordinates $x = (x^1, \ldots, x^n)$.

We assume that A_n, $\bar{A}_n$, P_n, $\bar{P}_n \in C^r$ $(\nabla, \bar{\nabla}, \blacktriangledown, \bar{\blacktriangledown} \in C^r)$ if their components $\Gamma^h_{ij}(x), \bar{\Gamma}^h_{ij}(x), \Pi^h_{ij}(x), \bar{\Pi}^h_{ij}(x) \in C^r$ on (U, x), $U \subset M$, respectively. Here C^r is the smoothness class.

Formulae (1) and (2) in the common system (U, x) have the local form

$$\bar{\Gamma}^h_{ij}(x) = \Gamma^h_{ij}(x) + \psi_i(x)\delta^h_j + \psi_j(x)\delta^h_i \qquad \text{and} \qquad \bar{\Pi}^h_{ij}(x) = \Pi^h_{ij}(x)$$

respectively, where ψ_i are components of ψ and δ^h_i is the Kronecker delta.
It is seen that in a manifold $A_n = (M, \nabla)$ with affine connections ∇ there exists a projective connection $\blacktriangledown$ (i.e., Thomas projective parameter) with the same smoothness. The opposite statement is not valid, for example if $\nabla \in C^r$ ($\Rightarrow \blacktriangledown \in C^r$ and also $\bar{\blacktriangledown} \in C^r$) and $\psi(x) \in C^0$, then $\bar{\nabla} \in C^0$.
Let ∇, $\tilde{\nabla}$ and $\blacktriangledown$ be connections on M and their components Γ^h_{ij}, $\tilde{\Gamma}^h_{ij}$ and Π^h_{ij} in a certain common coordinate system (U, x) have the following form

$$\tilde{\Gamma}^h_{ij}(x) = \Pi^h_{ij}(x) = \Gamma^h_{ij}(x) - \frac{1}{n+1}\left(\delta^h_i \Gamma^\alpha_{\alpha j}(x) + \delta^h_j \Gamma^\alpha_{\alpha i}(x)\right). \tag{3}$$

These connections have common geodesics. The connection $\tilde{\nabla}$ is a *normal connection*, see Cartan [2] and Thomas [26], see (37.4) in [5, p. 105], [21, p. 282]. Because $\tilde{\Gamma}^\alpha_{\alpha j}(x) = 0$ the connection $\tilde{\nabla}$ is equaffine (if $\tilde{\Gamma}^h_{ij}(x) \in C^1$ the Ricci tensor in $\tilde{A}_n$ is symmetric). A global construction we obtained in the papers [10, 12]. So instead of the connection $\Gamma^h_{ij}(x)$ we can use $\tilde{\Gamma}^h_{ij}(x)$, which has the same differentiability (or greater), and Γ and $\bar{\Gamma}$ have common geodetics.

4. Geodesic Mappings From Equiaffine Manifolds Onto (Pseudo-) Riemannian Manifolds

Let a manifold $A_n = (M, \nabla) \in C^0$ admit a geodesic mapping onto a (pseudo-) Riemannian manifold $\bar{V}_n = (M, \bar{g}) \in C^1$, i.e., the components $\bar{g}_{ij}(x) \in C^1(U)$. It is known [17], see [20, p. 145], that equations (1) are equivalent to the following Levi-Civita equations

$$\nabla_k \bar{g}_{ij} = 2\psi_k \bar{g}_{ij} + \psi_i \bar{g}_{jk} + \psi \bar{g}_{ik}. \tag{4}$$

If A_n is an equiaffine manifold then ψ has the following form

$$\psi_i = \partial_i \Psi, \quad \Psi = \frac{1}{n+1} \ln \sqrt{|\det \bar{g}|} - \rho, \quad \partial_i \rho = \frac{1}{n+1}\, \Gamma^\alpha_{\alpha i}, \quad \partial_i = \partial/\partial x^i.$$

Mikeš and Berezovski [24], see [20, p. 150], proved that the Levi-Civita equations (1) and (4) are equivalent to

$$\nabla_k a^{ij} = \lambda^i \delta^j_k + \lambda^j \delta^i_k \tag{5}$$

where

$$\text{a) } a^{ij} = \mathrm{e}^{2\Psi}\, \bar{g}^{ij} \qquad \text{and} \qquad \text{b) } \lambda^i = -\mathrm{e}^{2\Psi}\, \bar{g}^{i\alpha}\psi_\alpha. \tag{6}$$

Here $\|\bar{g}^{ij}\| = \|\bar{g}_{ij}\|^{-1}$. On the other hand

$$\bar{g}_{ij} = \mathrm{e}^{2\Psi}\hat{g}_{ij}, \qquad \Psi = \ln \sqrt{|\det \hat{g}|} - \rho, \qquad \|\hat{g}_{ij}\| = \|a^{ij}\|^{-1}. \tag{7}$$

Equation (5) can be written in the following explicit form

$$\partial_k a^{ij} = \lambda^i \delta^j_k + \lambda^j \delta^i_k - a^{\alpha i}\Gamma^j_{\alpha k} - a^{\alpha j}\Gamma^i_{\alpha k}. \tag{8}$$

If we have $A_n = (M, \nabla)$ with general connection ∇, just replace this connection in formula (8) by a *normal* affine connection, which is equiaffine, from the discussion about formulas (3) follows

$$\partial_k a^{ij} = \lambda^i \delta^j_k + \lambda^j \delta^i_k - a^{\alpha i}\Pi^j_{\alpha k} - a^{\alpha j}\Pi^i_{\alpha k}. \tag{9}$$

5. Proof of Theorem 3

Evidently, from the discussion about formula (3) we obtain that from Theorem 3 follows Theorem 4. Below we can prove this Theorem.

The following lemma is true.

Lemma 7. *Let $P_n \in C^1$ admit a geodesic mapping onto the Riemannian space $\bar{V}_n \in C^1$, then for the tensor components $a^{ij}(x)$ exist partial derivatives of second order with the possible exception $\partial_{ii}a^{ii}$ and $\partial_{ij}a^{ij}$ ($i \neq j$ and no summation over indices*).

Proof: We will analyze formulas (9) under the conditions that $\Pi^h_{ij}(x) \in C^1$. In the following the Einstein summation convention will be used only for greek indices.

Formula (9) for $k \neq i$ and $k \neq j$ has the following form

$$\partial_k a^{ij} = -a^{\alpha i}\Pi^j_{\alpha k} - a^{\alpha j}\Pi^i_{\alpha k}. \tag{10}$$

Evidently, from (10) directly follows the existence of the partial derivatives $\partial_{kl}a^{ii}$ and $\partial_{kl}a^{ij}$ for any l and any different indices i, j, k.

After integrating (10) with $i = j$ we obtain

$$a^{ii} = \tilde{a}^{ii} - 2\int_{x^k_0}^{x^k} a^{\alpha i}\Pi^i_{\alpha k}\,\mathrm{d}\tau^k \tag{11}$$

where the function $\tilde{a}^{ii}$ does not depend on the variable x^k.

Because $a^{ii}(x)|_{x^k = x^k_0} = \tilde{a}^{ii}$, the function $\tilde{a}^{ii}$ is differentiable, and

$$\partial_i a^{ii} = \partial_i \tilde{a}^{ii} - 2\int_{x^k_0}^{x^k} \partial_i(a^{\alpha i}\Pi^i_{\alpha k})\,\mathrm{d}\tau^k. \tag{12}$$

Here we used properties of the integrals with parameters, see [13, p. 665].

Differentiating (12) with respect to x^k we obtain the derivative $\partial_{ik}a^{ii}$

$$\partial_{ik}a^{ii} = -2\,\partial_i(a^{\alpha i}\Pi^i_{\alpha k})\,.$$

From (9) with $i = j = k$ we get

$$\partial_i a^{ii} = 2\,\lambda^i - 2\,a^{\alpha i}\Pi^i_{\alpha i}. \tag{13}$$

Differentiating (13) with respect to x^k we show the existence of $\partial_k \lambda^i$.
Finally, after substituting $j = k$ to (9) we get

$$\partial_k a^{ik} = \lambda^i - a^{\alpha i}\Pi^k_{\alpha k} - a^{\alpha k}\Pi^i_{\alpha k} \tag{14}$$

and from this we obtain the existence of the partial derivative $\partial_{kl} a^{ik}$ for any $l \neq i$ and $i \neq k$.
Evidently, the lemma is proved. ■

Proof: *Finally we will prove* Theorem 3.
We analyze equations (9). We suppose $\Pi^k_{ij}(x) \in C^r$, $r \geq 2$. Based on Lemma 7 we obtain all second partial derivatives of $a^{ij}(x)$, except $\partial_{ii} a^{ii}$ and $\partial_{ij} a^{ij}$. Analogically all partial derivatives of $\lambda^i(x)$ exist, excluding $\partial_i \lambda^i(x)$.
Formula (9) with $i = j = 1$ and $k = 2$ has the following form

$$\partial_2 a^{11} = -2a^{11}\Pi^1_{12} + G \tag{15}$$

where $G = -\,2 \sum\limits_{\alpha=2}^{n} a^{1\alpha}\Pi^1_{\alpha 2}$. Evidently, $G \in C^1$, and from Lemma 7 follows the existence $\partial_{11} G$.
Further we solve equation (15) with respect to the unknown function a^{11}, we find

$$a^{11} = CA + B \tag{16}$$

where C is a function, that does not depend on the coordinate x^2

$$A = \exp\left(-2\int_{x^2_0}^{x^2} \Pi^1_{12}(x^1, \tau^2, x^3, \dots)\,\mathrm{d}\tau^2\right) \qquad \text{and} \qquad B = A\int_{x^2_0}^{x^2} G/A\,\mathrm{d}\tau^2.$$

The functions A and B are twice differentiable in x^1. This assertion follows from the differentiability of the functions G, Π^h_{ij} and from properties of integrals with parameters, see [13, p. 665]. Because $a^{11}(x^1, x^2_0, x^3, \dots, x^n) = C$, there exists the partial derivative $\partial_1 C$.

On the other side using equations (9) we get

$$\partial_1 a^{11} = 2\lambda^1 - 2a^{1\alpha}\Pi^1_{\alpha 1} \qquad \text{and} \qquad \partial_2 a^{12} = \lambda^1 - a^{1\alpha}\Pi^2_{\alpha 2} - a^{2\alpha}\Pi^1_{\alpha 2}. \tag{17}$$

After excluding λ^1 from (17) using (16) obtain the following condition

$$\partial_2 a^{12} = 1/2\,\partial_1 C A + H \tag{18}$$

where

$$H = 1/2\,(C\partial_1 A + \partial_1 B) + a^{1\alpha}\Pi^1_{\alpha 1} - a^{1\alpha}\Pi^2_{\alpha 2} - a^{2\alpha}\Pi^1_{\alpha 2}.$$

With the subsequent integration we get

$$a^{12} = \tilde{C} + 1/2\, \partial_1 C \int_{x_0^2}^{x^2} A(x^1, \tau^2, x^3, \dots)\, \mathrm{d}\tau^2 + \int_{x_0^2}^{x^2} H(x^1, \tau^2, x^3, \dots)\, \mathrm{d}\tau^2$$

where $\tilde{C}$ is a function that does not depend on the coordinate x^2.

Because $a^{12}(x^1, x_0^2, x^3, \dots, x^n) = \tilde{C}$, there exists $\partial_1 \tilde{C}$, and from the existence of $\partial_1 a^{12}$, $\partial_1 A$ and $\partial_1 H$ follows the existence of $\partial_{11} C$. Then, from (16), (17) and (18) the existence of $\partial_{11} a^{11}$, $\partial_1 \lambda^1$ and $\partial_{21} a^{12}$ follow. Elementary, $a^{ij} \in C^2$. From this and (7) follows that also $\bar{g}_{ij} \in C^2$ and $\bar{V}_n \in C^2$.

Finally, from Theorem 1 follows Theorem 3. ■

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