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ROTARY DIFFEOMORPHISM ONTO MANIFOLDS WITH AFFINE CONNECTION

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Abstract. In this paper we will introduce a newly found knowledge above the existence and the uniqueness of isoperimetric extremals of rotation on two-dimensional (pseudo-) Riemannian manifolds and on surfaces on Euclidean space. We will obtain the fundamental equations of rotary diffeomorphisms from (pseudo-) Riemannian manifolds for twice-differentiable metric tensors onto manifolds with affine connections.

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1. Introduction

A special diffeomorphism between (pseudo-) Riemannian manifolds and manifolds with affine and projective connections, for which maps any special curve onto a special curve, were studied in many works. For example geodesic mappings, for which any geodesic maps onto geodesic [1,3–5,13–16,18,19,21,22,25]. Analogically holomorphically-projective and F-planar mappings for which any analytic and F-planar curve maps onto analytic and F-planar curve, respectively [4, 13, 15, 16, 18, 20, 21]. An almost geodesic mapping is defined as, that one for which geodesic is mapped onto almost geodesic curve [13, 15, 16, 21].

In this sense was introduced the following definition.

Definition 1. A diffeomorphism between two-dimensional (pseudo-) Riemannian manifolds is called *rotary* if any geodesic is mapped onto isoperimetric extremal of rotation.

The above definition was introduced by Leiko [6, 7, 9–12] for surfaces S_2 on Euclidean space and two-dimensional (pseudo-) Riemannian manifold V_2.

The isoperimetric extremals of rotation have a physical meaning as can be interpreted as trajectories of particles with a spin, see [6, 8]. These results are local and are based on the known fact that a two-dimensional Riemannian manifold V_2 is implemented locally as a surface S_2 on Euclidean space. Therefore, we will deal more with the study of V_2, i.e., the inner geometry of S_2 and assuming that metrics of these manifolds have a differentiability class C^4. Further Mikeš, Sochor and Stepanova [17] improved above results for differentiability classes C^3.

In this paper we generalize the above notion of rotary diffeomorphism.

Let $V_2 = (M, g)$ be a two-dimensional (pseudo-) Riemannian manifold M with a metric g and $\bar{A}_2 = (\bar{M}, \bar{\nabla})$ be a two-dimensional manifold $\bar{M}$ with an affine connection $\bar{\nabla}$.

Definition 2. A diffeomorphism $f\colon V_2 \to \bar{A}_2$ is called *rotary* if any isoperimetric extremal of rotation on V_2 is mapped onto geodesic from $\bar{A}_2$.

We founded the fundamental equations for which V_2 admit rotary diffeomorphisms onto $\bar{A}_2$. These results are generalized results obtained in papers [7, 17].

2. Isoperimetric Extremals of Rotation

A (pseudo-) Riemannian manifold $V_2 = (M, g)$ belongs to the smoothness class C^r if its metric $g \in C^r$, i.e., its components $g_{ij}(x) \in C^r(U)$ in some local map (U, x), $U \subset M$. We suppose that the differentiability class r is equal to $0, 1, 2, \dots, \infty, \omega$, where $0, \infty$ and ω denote continuous, infinitely differentiable and real analytic functions, respectively.

Let ℓ: $(s_0, s_1) \to M$ be a parametric curve with the equation $x = x(s)$, $\lambda = \mathrm{d}x/\mathrm{d}s$ be a tangent vector and s is the arc length. The following formulas are developed by analogy with the Frenet formulas for manifold V_2 (cf. [2, 17])

$$\nabla_s \lambda = k \cdot \nu \qquad \text{and} \qquad \nabla_s \nu = -\varepsilon\, \varepsilon_\nu\, k \cdot \lambda \tag{1}$$

where k is the *Frenet curvature* (k is *geodesic curvature* if $\ell \subset S_2 \subset E_3$), ν represents a unit *normal* vector field along ℓ orthogonal to the unit tangent vector λ, i.e., $\langle \lambda, \lambda \rangle = g_{ij}\lambda^i\lambda^j = \varepsilon = \pm 1$ and $\langle \nu, \nu \rangle = g_{ij}\nu^i\nu^j = \varepsilon_\nu = \pm 1$, where λ^h and ν^h are components of λ and ν.

The operator ∇_s is covariant derivative along ℓ with respect to the Levi-Civita connection ∇ of metric g

$$\nabla_s \lambda^h \equiv \frac{d\lambda^h}{ds} + \lambda^\alpha \Gamma^h_{\alpha\beta}\left(x(s)\right)\,\lambda^\beta \quad \text{and} \quad \nabla_s \nu^h \equiv \frac{d\nu^h}{ds} + \nu^\alpha \Gamma^h_{\alpha\beta}\left(x(s)\right)\,\lambda^\beta$$

where Γ^h_{ij} are the Christoffel symbols of V_2, i.e., components of Levi-Civita connection ∇.

Recall the scalar product of the vectors λ, ξ which is defined by $\langle\lambda, \xi\rangle = g_{ij}\lambda^i\xi^j$ and $|\lambda| = \sqrt{|g_{\alpha\beta}\lambda^\alpha\lambda^\beta|}$ is the length of a vector λ.

Hence, we may conclude that formulas (1) hold if tangent vector λ and $\nabla_s\lambda$ are not isotropic, i.e., $|\lambda| \neq 0$ and $|\nabla_s\lambda| \neq 0$. Further, we present functionals of *length* and *rotation* of the curve $\ell\colon x = x(t)$

$$s[\ell] = \int_{t_0}^{t_1} \sqrt{|\lambda|}\,\mathrm{d}t \qquad \text{and} \qquad \theta[\ell] = \int_{t_0}^{t_1} k(t)\,\mathrm{d}t.$$

Using these functionals [7] introduce the following

Definition 3. A curve ℓ is called the *isoperimetric extremal of rotation* if ℓ is extremal of $\theta[\ell]$ and $s[\ell] = \mathrm{const}$ with fixed ends.

It is possible to prove (cf. [7, 10])

Theorem 1. *A curve ℓ is an isoperimetric extremal of rotation if and only if, its Frenet curvature k and Gaussian curvature K are proportional*

$$k = c \cdot K$$

where c is constant.

Mikeš, Sochor and Stepanova [17] proved the following

Theorem 2. *The equation of isoperimetric extremal of rotation can be written in the form*

$$\nabla_s\lambda = c \cdot K \cdot F\lambda \tag{2}$$

where c is constant.

The Theorem 2 follows from assertion, that holds for unit normal $\nu = \pm F\lambda$, where structure F is tensor $\binom{1}{1}$ which satisfies the conditions

$$F^2 = -e \cdot \mathrm{Id}, \qquad g(X, FX) = 0, \qquad \nabla F = 0.$$

For Riemannian manifold V_2 is $e = +1$ and F is a *complex structure* and for (pseudo-) Riemannian manifold is $e = -1$ and F is a *product structure*. This tensor F is uniquely defined (with the respect to the sign) with using skew-symmetric and covariantly constant discriminant tensor ε_{ij}, which is defined

$$F_j^h = g^{hi}\varepsilon_{ij}, \qquad \varepsilon_{ij} = \sqrt{|g_{11}g_{22} - g_{12}^2|} \cdot \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{3}$$

Above Theorem 2 for $V_2 \in C^2$ holds. In this case from equation (2) follows that in tangent direction λ_0 at the point x_0 passes through a isoperimetric extremal of rotation curve.

On (pseudo-) Riemannian manifold $V_2 \in C^3$ in tangent direction λ_0 at the point x_0 passes through just only one isoperimetric extremal of rotation curve [17]. Moreover, with simple analysis of equation (2) we find that sufficient condition of uniquely isoperimetric extremal of rotation curve is $V_2 \in C^2$ and Gaussian curvature K is differentiable [13, pp. 127–128]. This property proved Leiko [6,7] for $V_2 \in C^4$.

3. Necessary Conditions of Rotary Diffeomorphisms

Let V_2 be a two-dimensional (pseudo-) Riemannian manifold with the metric g, and $\bar{A}_2$ be a two-dimensional manifold $\bar{M}$ with affine connection $\bar{\nabla}$. On (pseudo-) Riemannian manifold V_2 is ∇ a Levi-Civita connection and F is above structure, for which the equation (2) is satisfied.

Assume a rotary diffeomorphism $f\colon V_2 \to \bar{A}_2$, i.e., any isoperimetric extremal of rotation of manifold V_2 maps onto a geodesic of $\bar{A}_2$. Since f is a diffeomorphism, we can impose local coordinate system on M and $\bar{M}$, respectively, such that locally $f\colon V_2 \to \bar{A}_2$ maps points onto points with the same coordinates x, and $M = \bar{M}$. Remark that $V_2 \in C^r$ if $g_{ij}(x) \in C^r$, and $\bar{A}_2 \in C^r$ if $\bar{\Gamma}_{ij}^h(x) \in C^r$. In next we consider that $K \neq 0$, otherwise the mapping is geodesic.

We obtain

Theorem 3. *Let V_2 admits rotary mapping f onto $\bar{A}_2$. If V_2 and $\bar{A}_2$ in common coordinate system belong differentiability class C^2 and C^1, respectively, then Gaussian curvature K on V_2 is differentiable.*

Proof: Let assumptions of Theorem 3 hold. Let $\gamma : x = x(s)$ be an isoperimetric extremal of rotation on V_2 for which the following equation is valid

$$\frac{\mathrm{d}\lambda^h}{\mathrm{d}s} + \Gamma_{ij}^h(x(s))\,\lambda^i\lambda^j = c\cdot K(x(s))\cdot F_i^h(x(s))\cdot\lambda^i \tag{4}$$

and $\bar{\gamma} = f(\gamma)\colon x = x(\bar{s})$ be a geodesic on $\bar{A}_2$ for which the following equation is valid

$$\frac{\mathrm{d}^2x^h}{\mathrm{d}\bar{s}^2} + \bar{\Gamma}_{ij}^h(x(\bar{s}))\,\frac{\mathrm{d}x^i}{\mathrm{d}\bar{s}}\,\frac{\mathrm{d}x^j}{\mathrm{d}\bar{s}} = 0,$$

where Γ^h_{ij} and $\bar{\Gamma}^h_{ij}$ are components of ∇ and $\bar{\nabla}$, parameters s is arc length on γ and $\bar{s}$ is canonical parameter of $\bar{\gamma}$, $\lambda^h = \mathrm{d}x^h(s)/\mathrm{d}s$ and $\bar{\lambda^h} = \mathrm{d}x^h(\bar{s})/\mathrm{d}\bar{s}$.
Evidently $\bar{s} = \bar{s}(s)$ holds. In this case, the equations of geodesic are modify:

$$\frac{\mathrm{d}\lambda^h}{\mathrm{d}s} + \bar{\Gamma}^h_{ij}(x(s))\,\lambda^i\lambda^j = \bar{\varrho}(s)\cdot\lambda^h \tag{5}$$

where $\bar{\varrho}(s)$ is a certain function of parameter s.
After subtraction equations (4) and (5) we obtain

$$P^h_{ij}(x)\lambda^i\lambda^j = \bar{\varrho}(s)\cdot\lambda^h - c\cdot K(x(s))\cdot F^h_i(x(s))\cdot\lambda^i, \tag{6}$$

where $P^h_{ij}(x) = \bar{\Gamma}^h_{ij}(x) - \Gamma^h_{ij}(x)$ is the *deformation tensor* of connections ∇ and $\bar{\nabla}$, see [13, pp. 181–183].
Contracting equations (6) with $g_{hi}\lambda^i$ we obtain

$$cK\,e\,\varepsilon = \lambda_\gamma F^\gamma_h\,P^h_{\alpha\beta}\,\lambda^\alpha\lambda^\beta$$

and we can rewrite this equation using (3) in the following form

$$cK\,e\,\varepsilon = \varepsilon_{\gamma h}\,P^h_{\alpha\beta}\lambda^\alpha\lambda^\beta\lambda^\gamma. \tag{7}$$

Through differentiation formulas (7) we make sure that $K(x(s)) \in C^1$. And because these properties apply in any direction, then K is differentiable. ■

Hence we may conclude from Theorem 3 following

Theorem 4. *If Gaussian curvature $K \notin C^1$, then rotary diffeomorphism $V_2 \to \bar{A}_2$ does not exist.*

4. Fundamental Equations of Rotary Diffeomorphisms

As it was mentioned in Introduction, we find fundamental equations of rotary diffeomorphism $V_2 \to \bar{A}_2$ from Definition 1, where $V_2 \in C^2$ and $\bar{A}_2 \in \bar{C}^1$. Moreover on the basis the Theorem 3, we can assume that necessary Gaussian curvature $K \in C^1$.
For rotary diffeomorphism $V_2 \to \bar{A}_2$ formulas (6) and (7) hold. After subsequent derivation formula (7) by parameter s we obtain

$$cK_\delta\,\lambda^\delta e\varepsilon\, = \varepsilon_{\gamma h}\,P^h_{\alpha\beta,\delta}\lambda^\alpha\lambda^\beta\lambda^\gamma\lambda^\delta + \varepsilon_{\gamma h}\,P^h_{\alpha\beta}(2\nabla_s\lambda^\alpha\lambda^\beta\lambda^\gamma + \lambda^\alpha\lambda^\beta\nabla_s\lambda^\gamma)$$

where and $K_\delta = \partial K/\partial x^\delta$ and " , " denotes the covariant derivative with respect to Levi-Civita connection. After substituting (2) we get

$$cK_\delta\,\lambda^\delta e\varepsilon\, = \varepsilon_{\gamma h}\,P^h_{\alpha\beta,\delta}\lambda^\alpha\lambda^\beta\lambda^\gamma\lambda^\delta + cK\,\varepsilon_{\gamma h}\,P^h_{\alpha\beta}(2F^\alpha_\delta\lambda^\beta\lambda^\gamma\lambda^\delta + \lambda^\alpha\lambda^\beta F^\gamma_\delta\lambda^\delta).$$

Using formula (7) we eliminate the constant c, and we obtain equation

$$\varepsilon_{\gamma h}\partial_\delta(\ln|K|)P^h_{\alpha\beta}\lambda^\alpha\lambda^\beta\lambda^\gamma\lambda^\delta - \varepsilon_{\gamma h}P^h_{\alpha\beta,\delta}\lambda^\alpha\lambda^\beta\lambda^\gamma\lambda^\delta = I_1\cdot I_2 \tag{8}$$

where

$$\begin{aligned} I_1 &= e\varepsilon\, \varepsilon_{\gamma h}\, P^h_{\alpha\beta}\lambda^\alpha\lambda^\beta\lambda^\gamma \\ I_2 &= \varepsilon_{\gamma h}\, P^h_{\alpha\beta}(2F^\alpha_\delta\lambda^\beta\lambda^\gamma\lambda^\delta + F^\gamma_\delta\lambda^\alpha\lambda^\beta\lambda^\delta). \end{aligned} \tag{9}$$

Evidently, on the right side of formula (8) is a polynomial of the sixth degree, respectively λ^1 and λ^2, but on the left side is a polynomial of the fourth degree. Further, we study formulas (8) at a point x_0 and we choose for it such a coordinate system, that at the point x_0 metric has form $\mathrm{d}s^2 = \mathrm{d}x^{1^2} + e\mathrm{d}x^{2^2}$, where $e = \pm 1$. At this point x_0 it holds

$$g_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & e \end{pmatrix}, \qquad \varepsilon_{ij} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad \text{and} \qquad F^h_i = \begin{pmatrix} 0 & 1 \\ -e & 0 \end{pmatrix}.$$

Because λ^h is in (pseudo-) Riemannian manifold V_2 a unit vector, then at the point x_0 holds $g_{ij}\lambda^i\lambda^j = \lambda^{1^2} + e\,\lambda^{2^2} = \varepsilon = \pm 1$, i.e.,

$$\lambda^{1^2} = \varepsilon - e\,\lambda^{2^2}.$$

Therefore we have to λ^1 consider as a function of variable λ^2 with domain of definition $\mathcal{D} = \langle -1; 1\rangle$ for $e = 1$ and $\mathcal{D} = \mathbb{R}$ for $e = -1$. With simple analysis of equation (8) we find members which contain maximum degree of λ^{2^6} and $\lambda^1 \cdot \lambda^{2^5}$ on the right side of equation

$$I = I_1 \cdot I_2. \tag{10}$$

We compute I_1 and I_2 in the special coordinate system at the point x_0

$$\begin{aligned} I_1 &= \lambda^{2^3} \cdot A + \lambda^{2^2} \cdot B + \dots \\ I_2 &= \lambda^{2^3} \cdot (-3B) + \lambda^{2^2}\lambda^1 \cdot (3eA) + \dots \end{aligned}$$

where " $\dots$ " means other members of polynomials I_1, I_2 and

$$A = P^1_{11} - 2P^2_{12} - eP^1_{22} \qquad \text{and} \qquad B = P^2_{22} - 2P^1_{12} - eP^2_{11}. \tag{11}$$

Finally, I has the following form

$$I = I_1 \cdot I_2 = \lambda^{2^6} \cdot 6eAB + \lambda^1\lambda^{2^5} \cdot (B^2 - eA^2) + \dots$$

Because $\lambda^2 \in \mathcal{D}$ is random, then coefficients by λ^{2^6} and $\lambda^1 \cdot \lambda^{2^5}$ have to be vanishing. It implies $AB = 0$ and $B^2 - eA^2 = 0$. From this follows $A = B = 0$. As a consequence of (11) the deformation tensor has the following form

$$P^h_{ij} = \delta^h_i\psi_j + \delta^h_j\psi_i + \theta^h g_{ij} \tag{12}$$

where ψ_i and θ^h are covector and vector fields.

Equation (6) is necessary and sufficient condition for existence of rotary diffeomorphism $f : V_2 \to \bar{A}_2$. Substitute from (12) into the equation (6). We obtain:

$$\varepsilon\theta^h = (\bar{\rho} - 2\psi_\alpha\lambda^\alpha)\lambda^h - cK \cdot F^h_\alpha\lambda^\alpha. \tag{13}$$

Contracting (13) with $g_{h\alpha}\lambda^\alpha$ we obtain $(\bar\rho - 2\psi_\alpha\lambda^\alpha) = \theta_\alpha\lambda^\alpha$ where $\theta_i = g_{i\alpha}\theta^\alpha$. Therefore formula (13) takes the form

$$\varepsilon\theta^h = \theta_\alpha\lambda^\alpha\lambda^h - cK \cdot F^h_\alpha\lambda^\alpha. \tag{14}$$

Differentiating (14) along the curve ℓ of parameter s, we obtain

$$\varepsilon \cdot \theta^h_{,\alpha}\lambda^\alpha = \theta^h_{\alpha,\beta}\lambda^\alpha\lambda^\beta \cdot \lambda^h - e\, F^\alpha_i\theta_\alpha\lambda^j \cdot (\theta_j - \partial_j \ln|K|)\, \lambda^j \cdot F^h_k\lambda^k. \tag{15}$$

After a detailed analysis of degrees of λ^h in the equation (15), we get

$$\theta^h_j = \theta^h(\theta_j + \partial_j \ln|K|) + \nu\, \delta^h_j \tag{16}$$

where ν is a function on V_2.

Theorem 5. (*Pseudo-*) *Riemannian manifold V_2 admits rotary mapping onto $\bar A_2$ if and only if equation* (16) *in V_2 holds.*

Proof: The statement of Theorem 5 follows from previous analysis of the equation (6). If in (pseudo-) Riemannian manifold V_2 equation (16) holds for any vector field θ^h, then the affine connection of $\bar A_2$ is constructed according to (12). ■

The vector field θ^h is a special case of torse-forming field, see [13, 18, 21, 24]. In general case this field satisfies

$$\theta^h_i = \nu\delta^h_j + \theta^h\, a_i$$

where a_i is a covector. If a function a_i is gradient-like, then a vector field θ^h is *concircular* [13, 18, 21, 23, 25]. In our sense, vector field θ^h is concircular, if covector $(\theta_j + \partial_j \ln|K|)$ is gradient-like.

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