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SOME ASPECTS OF THE SPECTRAL THEORY FOR $\mathfrak{sl}(3,\mathbb{C})$ SYSTEM WITH $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ REDUCTION OF MIKHAILOV TYPE WITH GENERAL POSITION BOUNDARY CONDITIONS

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Abstract. We consider some aspects of the spectral theory of a system that is a generalization to a pole gauge Zakharov-Shabat type system on the Lie algebra $\mathfrak{sl}(3,\mathbb{C})$ but involving rational dependence on the spectral parameter and subjected to $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ reduction of Mikhailov type. The question of the existence of analytic fundamental solutions under some special type of boundary conditions has been considered, recently we consider boundary conditions in general position.

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1. Introduction

In this article we shall consider the linear problem $L_{S_{\pm 1}}\psi = 0$ of the type

$$\mathrm{i}\partial_x\psi + \begin{pmatrix} 0 & (\lambda-\lambda^{-1})u & (\lambda+\lambda^{-1})v \\ (\lambda-\lambda^{-1})u^* & 0 & 0 \\ (\lambda+\lambda^{-1})v^* & 0 & 0 \end{pmatrix}\psi = 0 \tag{1}$$

with ‘boundary’ conditions

$$\lim_{x\to\pm\infty} u(x) = u_0, \qquad \lim_{x\to\pm\infty} v(x) = v_0.$$

In the above $u(x), v(x)$ (the potentials) are smooth complex valued functions on x where x belongs to the real line and by * is denoted the complex conjugation. In addition, the functions $u(x)$ and $v(x)$ satisfy the relation

$$|u(x)|^2 + |v(x)|^2 = 1. \tag{2}$$

We shall call the above system the rational GMV (RGMV) system. The history of the above problem is the following. In number of papers [5–8], there has been studied the auxiliary linear problem

$$L_{S_1}\psi = (\mathrm{i}\partial_x + \lambda S_1)\psi = 0, \qquad S_1 = \begin{pmatrix} 0 & u & v \\ u^* & 0 & 0 \\ v^* & 0 & 0 \end{pmatrix} \tag{3}$$

where $u(x)$, $v(x)$ have the same properties as above. We call this system GMV system.

The GMV system arises naturally when one looks for integrable system having a Lax representation $[L, A] = 0$ with L of the form $\mathrm{i}\partial_x + \lambda S(x)$ and A of the form $\mathrm{i}\partial_t + U(x,\lambda)$ where $S(x) \in \mathfrak{sl}(3,\mathbb{C})$ and L, A are subject to Mikhailov-type reduction requirements. The notion of Mikhailov-type reductions, see [10–12] has nice applications since it permits to reduce the number of the independent functions in the Lax representations in a way compatible with the evolution given by $[L, A] = 0$. The presence of reductions affects all the theory of the Nonlinear Evolution Equations solvable by an auxilliary linear problem L, see [9]. In the particular case of the system (3) the Mikhailov reduction group G_M is generated by two elements g_0 and g_1 acting in the following way on the fundamental solutions:

$$g_0(\psi)(x,\lambda) = \left[\psi(x,\lambda^*)^\dagger\right]^{-1}, \qquad g_1(\psi)(x,\lambda) = H_1\psi(x,-\lambda)H_1 \tag{4}$$

where $\dagger$ denotes Hermitian conjugation and $H_1 = \mathrm{diag}(-1,1,1)$. Since $g_0g_1 = g_1g_0$ and $g_0^2 = g_1^2 = \mathrm{id}$ the reduction group is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$. According to the definition of Mikhailov reduction group it leaves the set of the fundamental solutions invariant. One can prove that this forces the matrix S_1 in the GMV system to be of the form we see in (3). The condition (2) is of different nature, in fact one can prove that it ensures that S_1 has constant eigenvalues $+1, 0, -1$, so that for each x the matrix $S_1(x)$ belongs to the orbit $\mathcal{O}_{J_0}(\mathrm{SU}(3))$ of $J_0 = \mathrm{diag}(1,0,-1)$ with respect to the adjoint action of the group $\mathrm{SU}(3)$. Thus one is able to show that that the GMV system is gauge-equivalent to a Generalized Zakharov-Shabat (GZS) system on $\mathfrak{sl}(3,\mathbb{C})$ and as a consequence it has nice spectral properties, identical to that of the GZS. This could be used to develop the spectral theory of the so-called Recursion Operators in two different ways. Either develop the theory independently (but in analogy with GZS), as has been done in the works we cited in the above, or to develop it using the gauge covariant theory of the Recursion Operators, see [4]. The last approach has been adopted in [13–15].

The linear problem we shall consider has been introduced in [5, 6] simultaneously with the GMV and it is a sort of its generalization. Indeed, assume that one wants bigger Mikhailov reduction group, generated this time by the following three elements

$$
\begin{aligned}
&g_0(\psi)(x,\lambda) = \left[\psi(x,\lambda^*)^\dagger\right]^{-1} \\
&g_1(\psi)(x,\lambda) = H_1\psi(x,-\lambda)H_1, \qquad H_1 = \mathrm{diag}(-1,1,1) \\
&g_2(\psi)(x,\lambda) = H_2\psi(x,\frac{1}{\lambda})H_2, \qquad H_2 = \mathrm{diag}(1,-1,1).
\end{aligned} \tag{5}
$$

Since the elements g_i, $i = 0,1,2$ commute and $g_i^2 = \mathrm{id}$ the reduction group is $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. Naturally, L_{S_1} cannot admit such such reduction group for which rational dependence on λ is needed. Thus one comes to consider the linear problem

$$
L_{S_{\pm 1}}\psi = (\mathrm{i}\partial_x + \lambda S_1 + \lambda^{-1}S_{-1})\psi = 0 \tag{6}
$$

subject to reduction generated by g_0, g_1, g_2. If one sets $S_L(\lambda) = \lambda S_1 + \lambda^{-1}S_{-1}$ one obtains that the reduction group forces $S_L(\lambda)$ to obey

$$
(S_L(\lambda^*))^\dagger = S(\lambda), \quad H_1S_L(-\lambda)H_1 = S(\lambda), \quad H_2S_L(\lambda^{-1})H_2 = S(\lambda). \tag{7}
$$

In this way one sees that S_1 is as in (3) and $S_{-1} = H_2S_1H_2$, that is

$$
S_{-1} = \begin{pmatrix} 0 & -u & v \\ -u^* & 0 & 0 \\ v^* & 0 & 0 \end{pmatrix} \tag{8}
$$

so $L_{S_{\pm 1}}$ becomes exactly the linear problem (1) we started with. The question about the Recursion Operators for RGMV , has been considered in [8], and in [5] have been made the first steps into considering its spectral properties. In that work have been made also some important observations how to construct the fundamental solutions analytic in λ (FAS) for (1). However, in [8] the authors limited their scope to the degenerate cases when either u_0 or v_0 is equal to zero. We shall consider now boundary conditions for which both $u_0, v_0 \neq 0$ and will call them either non-degenerate boundary conditions or boundary conditions in general position.

2. FAS. Asymptotic Behavior for $x \to \pm\infty$

In order to write down the integral equations that will permit us to analyze the fundamental analytic solutions (FAS) we need to know their asymptotic behavior when $x \to \pm\infty$. We expect that the when $x \to \pm\infty$ the solutions of $L_{S_{\pm 1}}\psi = 0$ will behave as $(\exp \mathrm{i}J(\lambda)x)A$ where $A = A(\lambda)$ is a matrix that does not depend on x and

$$
J(\lambda) = (\lambda S_1 + \lambda^{-1}S_{-1})|_{u=u_0, v=v_0}. \tag{9}
$$

It is not hard to find that $J(\lambda)$ has eigenvalues

$$
\mu_0 = 0, \qquad \mu_\pm = \pm\sqrt{2(|v_0|^2 - |u_0|^2) + (\lambda^2 + \lambda^{-2})}. \tag{10}
$$

Since $|u_0|^2 + |v_0|^2 = 1$ one can also cast $\mu_\pm$ in the following equivalent forms

$$\mu_\pm = \pm\sqrt{4|v_0|^2 + (\lambda - \lambda^{-1})^2} = \pm\sqrt{-4|u_0|^2 + (\lambda + \lambda^{-1})^2}. \tag{11}$$

Hence $J(\lambda)$ is diagonalizable and there is a constant matrix C (depending of course on u_0 and v_0 and λ) such that

$$C^{-1}J(\lambda)C = \mu(\lambda)\operatorname{diag}(1,0,-1) = \mu(\lambda)J_0, \qquad J_0 = \operatorname{diag}(1,0,-1). \tag{12}$$

Denote

$$r(\lambda) = 2(|v_0|^2 - |u_0|^2) + (\lambda^2 + \lambda^{-2}). \tag{13}$$

Then μ is a square root of $r(\lambda)$ which of course has two branches. In the degenerate cases $r(\lambda)$ becomes a square of an analytic functions having simple poles at $\lambda = 0$ and $\lambda = \infty$ (In fact $r(\lambda) = (\lambda + \lambda^{-1})^2$ when $u_0 = 0$ and $r(\lambda) = (\lambda - \lambda^{-1})^2$ when $v_0 = 0$) so the two branches are $\pm(\lambda + \lambda^{-1})$ when $u_0 = 0$ and $\pm(\lambda - \lambda^{-1})$ when $v_0 = 0$.

In case both u_0 and v_0 are different from zero the situation is not so trivial. One of the ways to describe the branches will be to cut the plane into simply connected regions such that in each of them the function $r(\lambda)$ does not have zeros, then in each of them there will be exactly two branches of the square root of $r(\lambda)$. Since all the zeros of $r(\lambda)$ lie on the unit circle $\mathbb{S}^1$ centered at $\lambda = 0$, see below (15), it is natural to introduce the four regions $G_\pm$, $\Omega_\pm$

$$\begin{aligned} G_+ &= \{\lambda\ ;\ |\lambda| < 1,\ \operatorname{Im}\lambda > 0\}, \qquad G_- = \{\lambda\ ;\ |\lambda| < 1,\ \operatorname{Im}\lambda < 0\} \\ \Omega_+ &= \{\lambda\ ;\ |\lambda| > 1,\ \operatorname{Im}\lambda > 0\}, \qquad \Omega_- = \{\lambda\ ;\ |\lambda| > 1,\ \operatorname{Im}\lambda < 0\}. \end{aligned} \tag{14}$$

These regions are obtained cutting $\mathbb{C}$ using the circle $\mathbb{S}^1$ and the real line $\mathbb{R}$. On each them one can define branches of the logarithm of $r(\lambda)$ and hence the branches of the square root.

However, there is a better way to investigate $\mu(\lambda)$ and it is to make analytic continuation of the square root $\sqrt{r(\lambda)}$. To this end let us consider $r(\lambda)$ more closely. The function $r(\lambda)$ is meromorphic on the extended plane (Riemann sphere $\mathbb{P}^1$). At the points $\lambda = 0$ and $\lambda = \infty$ it has a poles of order two. Further, $r(\lambda)$ has simple zeros at the four points

$$\begin{aligned} z_1 &= |u_0| + \mathrm{i}|v_0|, \qquad z_2 = -|u_0| + \mathrm{i}|v_0| \\ z_3 &= -|u_0| - \mathrm{i}|v_0|, \qquad z_4 = |u_0| - \mathrm{i}|v_0|. \end{aligned} \tag{15}$$

They degenerate into two points in case either u_0 or v_0 equals zero. (At ± 1 in the case $v_0 = 0$ and at $\pm\mathrm{i}$ in the case $u_0 = 0$). All the zeros lie on the unit circle $\mathbb{S}^1 = \{\lambda : |\lambda| = 1\}$. Let us also note that since the function $r(\lambda)$ is invariant under the involutions mapping the Riemann sphere into itself

$$\lambda \mapsto \lambda^*, \qquad \lambda \mapsto -\lambda, \qquad \lambda \mapsto \lambda^{-1}. \tag{16}$$

Naturally, the set of zeros is invariant under these involutions which could be checked immediately.

For the analytic continuation of $\sqrt{r(\lambda)}$ we first remark that $r(\lambda)$ could be written into the form

$$r(\lambda)=\lambda^{-2}(\lambda-z_1)(\lambda-z_2)(\lambda-z_3)(\lambda-z_4).$$

Then one can apply to $r(\lambda)$ the standard technique for analytic continuation of a germ of the square root. For this consider the closed arcs $a_\pm$ where a_+ is the open arc with ends $z_{2,3}=-|u_0|\pm \mathrm{i}|v_0|$ containing $\lambda=-1$ and a_- is the arc with ends $z_{3,4}=|u_0|\mp|v_0|$ containing $\lambda=1$. Let G_0 be the region $\mathbb{C}\setminus(\{0\}\cup\bar{a}_+\cup\bar{a}_-)$ where by bar is denoted the closure. Thus G_0 consists of the punctured plane from which are eliminated the arcs $\bar{a}_+,\bar{a}_-$ which we shall call the cuts. A standard technique considering the increment $\Delta_\gamma\arg(r(\lambda))$ of the argument of $r(\lambda)$ along arbitrary partially smooth closed curve in G_0 then shows that $\sqrt{r(\lambda)}$ allows analytic continuation starting from any λ_0 and ζ_0 such that $\zeta_0^2=r(\lambda_0)$. Of course, there are only two square roots, so if one of them is denoted by $\mu(\lambda)$, the other will be $-\mu(\lambda)$. We shall assume that $\mu(\lambda)$ will be the function that for $\lambda=\mathrm{i}$ takes the value $2\mathrm{i}|u_0|$. In more general (in the case of non-degenerate boundary conditions case the requirement we use amounts to the same) we shall always take as μ the branch for which $\mathrm{Im}(\mu(\mathrm{i}))$ is positive for $a\mathrm{i}$, a-real $a>1$.

We need to investigate the sign of $\mathrm{Im}(\mu(\lambda))$ (we need that for the construction of the FAS). Using just the definition of the analytic continuation it is not so easy though for example for the points on the imaginary axis one can do it. For example, one calculates that $\mu(-\mathrm{i})=-2\mathrm{i}|u_0|^2$. However, there is a simpler way to investigate the sign of $\mathrm{Im}(\mu(\lambda))$. Indeed, if $\mathrm{Im}(\mu)=0$ then $\mathrm{Im}(\mu^2)=0$. So we can find the set of points at which $\mathrm{Im}(\mu^2)=\mathrm{Im}(r(\lambda))=0$ and then remove from it the points at which μ^2 is real and $\mu^2\le 0$. Thus we discover that $\mathrm{Im}(\mu)=0$ either if λ is real ($\lambda\neq 0$) or if λ belongs to the arcs $\bar{a}_\pm$. However, the arcs $\bar{a}_\pm$ are the cuts we eliminated in order to extend μ. Then we get that $\mathrm{Im}(\mu)(\lambda)>0$ for $\lambda\in G_0$ in the upper half-plane and $\mathrm{Im}(\mu)(\lambda)<0$ for $\lambda\in G_0$ in the lower half-plane. On the real line (except $\lambda=0$) we have that $\mathrm{Im}(\mu)=0$, $\mu\neq 0$. (The function $\mathrm{Im}(\mu)$ (but not μ since $\mathrm{Re}\,(\mu)$ has a jump) could be extended setting it equal to zero to all points of the arcs $a_\pm$ and μ could be extended setting it equal to zero at $\lambda=z_i$, $i=1,2,3,4$. The function μ is meromorphic in $G_0\cup\{\infty\}$ and has simple poles at $\lambda=0$, $\lambda=\infty$. Next, the symmetry properties of $r(\lambda)$ lead to symmetry properties for $\mu(\lambda)$. The first one follows from the fact that if we expand μ in Taylor or Laurent series the coefficients in these expansions will be real. The two other ones follow taking into account that the maps $\lambda\mapsto-\lambda$, $\lambda\mapsto\lambda^{-1}$, $\lambda\mapsto\lambda^*$ interchange the upper and lower half-planes, that in each connected open set $\sqrt{r(\lambda)}$ has exactly two branches μ and $-\mu$, and that we know the sign of $\mathrm{Im}(\mu(\lambda))$ is same as the sign of $\mathrm{Im}(\lambda)$. Thus we obtain

$$\mu^*(\lambda^*) = \mu(\lambda), \qquad \mu(\lambda^{-1}) = -\mu(\lambda), \qquad \mu(-\lambda) = -\mu(\lambda). \tag{17}$$

All this (and also what follows) has a nice interpretation in terms of the theory of compact Riemann surfaces, since the square root defines a two-fold covering of the Riemannian sphere, see [2] but we are not going to discuss it here since up to now our results for the FAS are local.

Now, as we shall see, for the construction of the fundamental analytic solutions (FAS) of $L_{S_{\pm 1}}$ it is important to know the regions in which $\mathrm{Im}(\mu(\lambda))$ is positive (negative). Naturally, we must also know what branch we shall use since on this choice depends the matrix $C = C(\lambda, \mu(\lambda))$ that diagonalizes $J(\lambda)$

$$C = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 1 \\ \mu^{-1}(\lambda - \lambda^{-1})u_0^* & -\sqrt{2}\mu^{-1}(\lambda + \lambda^{-1})v_0 & -\mu^{-1}(\lambda - \lambda^{-1})u_0^* \\ \mu^{-1}(\lambda + \lambda^{-1})v_0^* & \sqrt{2}\mu^{-1}(\lambda - \lambda^{-1})u_0 & -\mu^{-1}(\lambda + \lambda^{-1})v_0^* \end{pmatrix}.$$

The matrix $C(\lambda, \mu(\lambda))$ is not unique, and this is the first difficulty one must overcome. We have chosen it to be unitary for real λ since in this case $J(\lambda)$ is Hermitian. Of course, here $\mu(\lambda)$ is one of the branches of the square root. Changing μ to $-\mu$, that is passing from $C(\lambda, \mu(\lambda))$ to $C(\lambda, -\mu(\lambda))$ is equivalent to multiplying $C(\lambda, \mu(\lambda))$ to the left by $\mathrm{diag}(1, -1, -1) = -H_1$. We shall write $C_+(\lambda) = C(\lambda, \mu(\lambda))$ and $C_-(\lambda) = C(\lambda, -\mu(\lambda))$. Thus $C_- = -H_1 C_+$ and

$$C_+^{-1} J(\lambda) C_+ = \mu J_0, \qquad C_-^{-1} J(\lambda) C_- = -\mu J_0, \qquad J_0 = \mathrm{diag}(1, 0, -1).$$

Let us mention that for the degenerate cases we have that the matrices $C_\pm$ do not depend on λ, which makes the things much easier.

3. FAS. Integral Equations

Now, let us assume that ϕ is a solution to the equation

$$L_{S_{\pm 1}} \phi = (\mathrm{i}\partial_x + \lambda S_1(x) + \lambda^{-1} S_{-1}(x))\phi = 0. \tag{18}$$

As pointed out in [5] in order to investigate fundamental solutions of (18) it is useful to introduce the functions

$$\Phi_\pm(x, \lambda) = C_\pm^{-1}(\lambda)\phi(x, \lambda) \exp\left(\mp \mathrm{i}\mu(\lambda) J_0 x\right) \tag{19}$$

which satisfy the equation

$$\mathrm{i}\partial_x \Phi_\pm + [\lambda(C_\pm^{-1} S_1 C_\pm) + \lambda^{-1}(C_\pm^{-1} S_{-1} C_\pm)]\Phi_\pm \mp \mu(\lambda)\Phi_\pm J_0 = 0. \tag{20}$$

Conversely, if $\Phi_+(x, \lambda)$ and $\Phi_+(x, \lambda)$ satisfy the corresponding equation in (20) then both functions

$$\psi_\pm = C_\pm \Phi_\pm \exp\left(\pm \mathrm{i}\mu(\lambda) J_0\right)$$

satisfy the equation (18). For the sake of brevity let is put

$$S_{\pm}[\lambda,x]=\lambda(C_{\pm}^{-1}S_1C_{\pm})+\lambda^{-1}(C_{\pm}^{-1}S_{-1}C_{\pm}) \tag{21}$$

we shall even put $S_{\pm}[\lambda,x]=S_{\pm}[\lambda]$ sometimes, and then (20) is written as

$$\mathrm{i}\partial_x\Phi_{\pm}+S_{\pm}[\lambda]\Phi\mp\mu(\lambda)\Phi_{\pm}J_0=0. \tag{22}$$

Since the number of subscripts and superscripts starts to grow quickly we shall concentrate first on the functions Φ_+ and shall not write the subscript $+$. Thus such functions satisfy

$$\mathrm{i}\partial_x\Phi+S_{+}[\lambda]\Phi-\mu(\lambda)\Phi J_0=0. \tag{23}$$

Let us try to find functions $\Phi^{p,n}$ that satisfy the above equation and in addition satisfy $\lim_{x\to-\infty}\Phi^n(x)=\mathbf{1}$, $\lim_{x\to+\infty}\Phi^p(x)=\mathbf{1}$. Let us first assume that λ is such that $\mathrm{Im}(\mu(\lambda))=0$. Let the functions $\Phi^{n,p}$ are solutions of the following integral equations of Volterra type

$$\begin{aligned}
\Phi^n(x,\lambda)&=\mathbf{1}+\mathrm{i}\int_{-\infty}^{x}\mathrm{d}y\left[\mathrm{e}^{\mathrm{i}\mu(x-y)J_0}(S_+[\lambda]-\mu J_0)\Phi^n(y,\lambda)\mathrm{e}^{-\mathrm{i}\mu(x-y)J_0}\right]\\
\Phi^p(x,\lambda)&=\mathbf{1}+\mathrm{i}\int_{+\infty}^{x}\mathrm{d}y\left[\mathrm{e}^{\mathrm{i}\mu(x-y)J_0}(S_+[\lambda]-\mu J_0)\Phi^p(y,\lambda)\mathrm{e}^{-\mathrm{i}\mu(x-y)J_0}\right].
\end{aligned} \tag{24}$$

Then assuming that one can differentiate under the sign of the integral one checks that $\Phi^{n,p}$ will satisfy (23) with the required asymptotic. So in case $\mathrm{Im}(\mu(\lambda))=0$ the question is transformed to the question of the solutions to the integral equations (24). Let us remark that the function $S_+[\lambda]-\mu J_0$ goes to zero when $x\to\pm\infty$. If this convergence is fast enough the above integral equations have unique solutions. In particular, this will happen if $S_+[\lambda]-\mu(\lambda)J_0$ have compact support (with respect to x for fixed λ). The functions $\phi^{n,p}=C_+\Phi^{n,p}\exp\left(\mathrm{i}\mu(\lambda)xJ_0\right)$ for the problem (18) are the analogs of the so-called Jost solutions for the GZS system. We should not forget that there is another pair of equations that correspond to the choice $-\mu(\lambda)$, that is, the superscript gives information about the asymptotic when $x\to\pm\infty$ and not of the branch so in fact we have four functions, namely $\phi_{\pm}^{n,p}$. In case $\mathrm{Im}(\mu(\lambda))\neq 0$, some of the exponents that appear in the integrand when one writes the integral equations (24) components-wise are growing, so for considering the fundamental analytic solutions (FAS) of (22) and consequently of the system $L_{S_{\pm1}}\psi=0$ we must modify the integral equations. It is a fact that is well known for the GZS system and its generalization – the Caudrey-Beals-Coifman (CBC) system, see [1, 3]. One considers two separate cases: a) $\mathrm{Im}(\mu(\lambda))>0$ and b) $\mathrm{Im}(\mu(\lambda))<0$. The systems of integral equations will be written for the components of the 3×3 matrix functions $\zeta^n(x,\lambda)$ and $\zeta^p(x,\lambda)$.

Case a). The condition $\operatorname{Im}(\mu(\lambda)) > 0$, fixes the asymptotic at $-\infty$. (For boundary conditions in general position this means that λ belongs to the upper half-plane without the cuts). Solutions are denoted by $\zeta^n(x,\lambda)$ and the system of the integral equations runs as

$$\begin{aligned}\zeta^n_{jk}(x,\lambda) &= \delta_{jk}\\ &\quad +\mathrm{i}\sum_{s=1}^{3}\int_{-\infty}^{x}\mathrm{d}y(S_+[\lambda](y)-\mu(\lambda)J_0)_{js}\zeta^n_{sk}(y,\lambda)\exp\left[\mathrm{i}\mu_{jk}(x-y)\right]\\ &\text{when } j\leq k \text{ and when } j>k \qquad (25)\\ \zeta^n_{jk}(x,\lambda) &=\\ &\quad \mathrm{i}\sum_{s=1}^{3}\int_{+\infty}^{x}\mathrm{d}y(S_+[\lambda](y)-\mu(\lambda)J_0)_{js}\zeta^n_{sk}(y,\lambda)\exp\left[\mathrm{i}\mu_{jk}(x-y)\right].\end{aligned}$$

Case b). The condition $\operatorname{Im}(\mu(\lambda)) < 0$, fixes the asymptotic at $-\infty$. (For boundary conditions in general position this means that λ belongs to the lower half-plane without the cuts). Then we must consider the following system

$$\begin{aligned}\zeta^n_{jk}(x,\lambda) &= \delta_{jk}\\ &\quad +\mathrm{i}\sum_{s=1}^{3}\int_{-\infty}^{x}\mathrm{d}y(S_+[\lambda](y)-\mu(\lambda)J_0)_{js}\zeta^n_{sk}(y,\lambda)\exp\left[\mathrm{i}\mu_{jk}(x-y)\right]\\ &\text{when } j\geq k \text{ and when } j<k \qquad (26)\\ \zeta^n_{jk}(x,\lambda) &=\\ &\quad \mathrm{i}\sum_{s=1}^{3}\int_{+\infty}^{x}\mathrm{d}y(S_+[\lambda](y)-\mu(\lambda)J_0)_{js}\zeta^n_{sk}(y,\lambda)\exp\left[\mathrm{i}\mu_{jk}(x-y)\right].\end{aligned}$$

In the above formulas $\mu_{kk} = 0$, $\mu_{12} = -\mu_{21} = \mu$, $\mu_{13} = -\mu_{31} = 2\mu$, $\mu_{23} = -\mu_{32} = \mu$. One sees that for $i < j$ we have that $\operatorname{Im}(\mu_{ij}) > 0$ in the upper half-plane and $\operatorname{Im}(\mu_{ij}) < 0$ in the lower half-plane while for $i > j$ we have $\operatorname{Im}(\mu_{ij}) < 0$ in the upper half-plane and $\operatorname{Im}(\mu_{ij}) > 0$ in the lower half-plane. Thus for $j \neq k$ in the integrands we always have falling exponents ensuring that the kernels of the above integral operators fall exponentially when $x \to \pm\infty$. As we shall see this circumstance in case the function $(S_+[\lambda] - \mu(\lambda)J_0)$ has a small $L^1(\mathbb{R})$ norm ensures that the above equations have solutions $\zeta^{n,+}(x,\lambda)$ (for λ in the upper half-plane without the cuts) and $\zeta^{n,-}(x,\lambda)$ (for λ in the lower half-plane without the cuts). We shall omit the superscripts $+$ $(-)$ assuming that when λ is in the upper half-plane we have $\zeta^{n,+}(x,\lambda)$ and when it is in the lower half-plane we have $\zeta^{n,-}(x,\lambda)$. We shall preserve the superscripts only when it is necessary to avoid ambiguity, for example, when λ is real and we must know whether we

are having the extension of the corresponding functions from above or from below. Let us mention again that in fact we should write also subscripts $\pm$ to $\zeta^n(x,\lambda)$ depending what is the branch of the square root we use. In other words, in the upper half-plane we have the solutions e $\zeta_+^{n,+}(x,\lambda),\zeta_-^{n,+}(x,\lambda)$ and in the lower the solutions $\zeta_+^{n,+}(x,\lambda),\zeta_-^{n,+}(x,\lambda)$. They all tend to $\mathbf{1}$ when $x\to-\infty$.
In a similar way, we can consider the integral equations for the functions $\zeta^p(x,\lambda)$. Finally we just mention that for the degenerate cases the construction of the solutions $\zeta^{n,p}$ in the degenerate cases do not differ from the one we had in the above.

Our intention is to establish results, that are analogs of similar results for the CBC system, namely

- For potentials that go fast to their limit values (this words should be given precise meaning) the integral equations for $\zeta^n_\pm(x,\lambda),\zeta^p_\pm(x,\lambda)$ have unique solutions which satisfy
$$\lim_{x\to-\infty}\zeta^n_\pm(x,\lambda)=\mathbf{1},\qquad \lim_{x\to+\infty}\zeta^p_\pm(x,\lambda)=\mathbf{1}. \tag{27}$$
- These functions are analytic in the regions where $\mathrm{Im}(\mu(\lambda))\neq 0$ and could be extended by continuity to their boundaries.
- For a class of potentials that do not go fast to their limit values, and that should be further specified, the fundamental solutions $\zeta^n_\pm(x,\lambda),\zeta^p_\pm(x,\lambda)$ possibly do not exist in a finite number of points where they have pole type singularities – the discrete spectrum of the linear problem (18).
- The Mikhailov reduction symmetries g_1,g_2,g_3 should result in symmetries of the solutions $\zeta^n_\pm(x,\lambda),\zeta^p_\pm(x,\lambda)$.

The above is a program that we expect to follow and the present article is a first step. For the lack of space we shall prove only the theorem of uniqueness and will formulate a simple theorem of existence. The proofs follow the ideas in [1, 3].

Proposition 1. *Suppose for given potentials $u(x),v(x)$ and $\mathrm{Im}(\lambda)\neq 0$ the bounded fundamental solutions $\zeta^n_\pm(x,\lambda),\zeta^p_\pm(x,\lambda)$ exist. Then they are unique.*

Proof: Let us assume for example that $\mathrm{Im}(\lambda)>0$ and we have two bounded solutions of the type $\zeta_-^{n,+}(x,\lambda)$, let us denote them by $\zeta_1(x,\lambda)$ and $\zeta_2(x,\lambda)$. The solutions$\zeta_{1,2}$ satisfy (23) and $\lim_{x\to-\infty}\zeta_1(x,\lambda)=\lim_{x\to-\infty}\zeta_2(x,\lambda)=\mathbf{1}$. But then
$$\psi_1=C_+(\lambda)\zeta_1(x,\lambda)\mathrm{e}^{\mathrm{i}\mu(\lambda)xJ_0},\qquad \psi_2=C_+(\lambda)\zeta_2(x,\lambda)\mathrm{e}^{\mathrm{i}\mu(\lambda)xJ_0}$$
are fundamental solutions of (18). Therefore, there exists a non-degenerate matrix $A(\lambda)$ which does not depend on x such that $\psi_2(x,\lambda)=\psi_1(x,\lambda)A(\lambda)$. This of course means that
$$\zeta_2(x,\lambda)=\zeta_1(x,\lambda)\mathrm{e}^{-\mathrm{i}\mu(\lambda)xJ_0}A(\lambda)\mathrm{e}^{\mathrm{i}\mu(\lambda)xJ_0}.$$

Since $\zeta_1(x,\lambda)$ and $\zeta_2(x,\lambda)$ are bounded, then $\mathrm{e}^{-\mathrm{i}\mu(\lambda)xJ_0}A(\lambda)\mathrm{e}^{\mathrm{i}\mu(\lambda)xJ_0}$ should also be bounded for $x\in\mathbb{R}$. But as $\mathrm{Im}(\mu(\lambda))\neq 0$ one sees that this could happen only if $A(\lambda)$ is diagonal and then $\mathrm{e}^{-\mathrm{i}\mu(\lambda)xJ_0}A(\lambda)\mathrm{e}^{\mathrm{i}\mu(\lambda)xJ_0}=A(\lambda)$. This means that $\zeta_2(x,\lambda)=\zeta_1(x,\lambda)A(\lambda)$. Taking the limit when $x\to+\infty$ shows that we must have $A(\lambda)=\mathbf{1}$ so $\zeta_2(x,\lambda)=\zeta_1(x,\lambda)$. ■

Let us note that the reasoning in the proof could be applied also to investigate what will be the relation between the solutions $\zeta^{n,+}(x,\lambda)$ and $\zeta^{p,+}(x,\lambda)$. (We consider the general position case). As we have seen there should be a diagonal matrix $D^+(\lambda)$ (which must analytic in λ in the upper half-plane without the cuts) such that

$$\zeta^{n,+}(x,\lambda)D^+(\lambda)=\zeta^{p,+}(x,\lambda)$$

for λ in the upper half-plane (without the cuts) and

$$\zeta^{n,-}(x,\lambda)D^-(\lambda)=\zeta^{p,-}(x,\lambda)$$

for λ in the lower half-plane (without the cuts). This time $D^-(\lambda)$ is diagonal and analytic in the lower half-plane (without the cuts). Since

$$\lim_{x\to+\infty}\zeta^{n,+}(x,\lambda)=D^+(\lambda),\qquad \lim_{x\to+\infty}\zeta^{n,-}(x,\lambda)=D^+(\lambda) \tag{28}$$

one can recover for example $\zeta^p(x,\lambda$ from $\zeta^n(x,\lambda)$ and therefore only one of them is considered.

From now on we shall consider only the solutions $\zeta^n(x,\lambda)$ and usually shall not write the superscript 'n'.

It is more interesting however to consider the relation between the solutions $\zeta^+(x,\lambda)$ and $\zeta^-(x,\lambda)$ which leads to the definition of scattering data for our linear problem but we shall not discuss this issue in the present article.

4. FAS. The Effect of the Symmetries

The issue about the existence of solutions is quite involved and must be investigated thoroughly but as a first and simple step we shall consider the existence and analytical properties of the solution of the corresponding system of integral equations for $\mathrm{Im}(\lambda)>0$ (without the cuts) in case of 'small' potentials. Let us introduce the following classical norms. In the space of the functions $R(x)$ where $R(x)$ is a 3×3 matrix whose entries are complex functions on the line $\mathbb{R}$ and belong to $L^1(\mathbb{R})$ we set

$$||R||_1=\max_{1\leq i,j\leq 3}\int_{-\infty}^{+\infty}|R_{ij}(x)|\mathrm{d}x \tag{29}$$

and for $R(x)$ whose entries are bounded we define

$$||R||_\infty = \sup_{x\in\mathbb{R};1\le i,j\le 3} |R_{ij}(x)|. \tag{30}$$

The space with the norm (29) we denote by $L^1(\mathbb{R};9)$ and the space with the norm (30) by $L^\infty(\mathbb{R};9)$. Of course, both these spaces are Banach spaces.

Proposition 2. *Suppose Ω is an open subset in the upper half-plane (without the cuts) with compact closure that do not contain $\lambda = 0$. Suppose that for any $\lambda \in \bar{\Omega}$ the the function $Q(x,\lambda) = (S_+[\lambda,x] - \mu(\lambda)J_0)$ belongs to $L^1(\mathbb{R},9)$ and $||Q(x,\lambda)||_1 < 1$. Then for $\lambda \in \Omega$ there exists unique solution $\zeta^+(x,\lambda)$ of the integral equations* (25) *with the following properties*

1. *The solution $\zeta^+(x,\lambda)$ together with its λ-derivatives allows continuous extension to the closure $\bar{\Omega}$ of Ω.*
2. *The solution $\zeta^+(x,\lambda)$ and its inverse obey the estimates*

$$||\zeta^+||_\infty < (1-\alpha)^{-1}, \qquad ||(\zeta^+)^{-1}||_\infty < (1-\alpha)^{-1}$$

where $\alpha = \max_{\lambda\in\bar{\Omega}} ||Q(x,\lambda)||_1 < 1$.

We shall not give the proof of the above result here, we shall only mention that the assumptions we made permit to regard the integral equations for ζ^+ as a fixed point theorem for which the Banach fixed point theorem could be applied.

From now on, we shall assume that we have FAS defined everywhere exept for the cuts and the real line and shall find how the symmetries we had for our linear problem (6) affect the solutions we introduced. In this subsection we shall assume that the fundamental solutions $\zeta_\pm(x,\lambda)$ exist.

Lemma 3. *Suppose we have the general position boundary conditions. The matrces $C_+(\lambda)$, $C_-(\lambda) = -H_1C_+(\lambda)$ satiisfy the relations*

$$\begin{aligned} &C_-(\lambda) = -H_1C_+(\lambda), \qquad [(C_\pm(\lambda^*)^\dagger]^{-1} = C_\pm(\lambda,\mu(\lambda)) \\ &C_\mp(\lambda) = H_2C_\pm(\lambda^{-1})H_2, \qquad C_\pm(-\lambda) = C_\pm(\lambda). \end{aligned} \tag{31}$$

As a consequence we obtain that

Corollary 4. *In the case of general position boundary conditions the functions $S_\pm[\lambda,x]$ satisfy*

$$S_\pm^\dagger[\lambda^*] = S_\pm[\lambda], \qquad H_2S_\pm[\lambda^{-1}]H_2 = S_\mp[\lambda], \qquad S_\pm[-\lambda] = S_\mp[\lambda]. \tag{32}$$

Then using the uniqueness of the solutions ζ one gets

Proposition 5. *In the case of general position boundary conditions the solutions* $\zeta(x,\lambda)$ *have the following properties*

$$\begin{aligned} &[(\zeta_{\pm}(x,\lambda^*)^{\dagger}]^{-1} = \zeta_{\pm}(x,\lambda), \qquad \zeta_{\pm}(x,-\lambda) = \zeta_{\mp}(x,\lambda) \\ &H_2\zeta_{\pm}(x,\lambda^{-1})H_2 = \zeta_{\mp}(x,\lambda). \end{aligned} \tag{33}$$

And finally, in terms of the solutions

$$\chi_{\pm}(x,\lambda) = C_{\pm}(\lambda)\zeta_{\pm}(x,\lambda)\exp\left(\pm \mathrm{i}\mu(\lambda)xJ_0\right)$$

of the linear problem RGMV the above symmetries take the form

Theorem 6. *In the case of general position boundary conditions the solutions* $\chi(x,\lambda)$ *have the following properties*

$$\begin{aligned} &[(\chi_{\pm}(x,\lambda^*)^{\dagger}]^{-1} = \chi_{\pm}(x,\lambda) \\ &H_2\chi_{\pm}(x,\lambda^{-1})H_2 = \chi_{\mp}(x,\lambda), \qquad H_1\chi_{\pm}(x,-\lambda)H_1 = \chi_{\mp}(x,\lambda)(-H_1). \end{aligned} \tag{34}$$

The relations in the above propositions should be properly read since they are written in the form we see for shortness. For example, what they mean in case $\mathrm{Im}(\lambda) > 0$ is the following

$$\begin{aligned} &((\zeta_{\pm}^{-})^{\dagger}(x,\lambda^*))^{-1} = \zeta_{\mp}^{+}(x,\lambda), \qquad H_2(\zeta_{\pm}^{-}(x,\lambda^{-1})H_2 = \zeta_{\mp}^{+}(x,\lambda) \\ &H_1(\zeta_{\pm}^{-}(x,-\lambda)H_1 = \zeta_{\mp}^{+}(x,\lambda). \end{aligned}$$

5. Conclusions

We have investigated the problem of fundamental analytic solutions (FAS) for the operator $L_{S_{\pm 1}}$ in case of boundary conditions in general position. We established the uniqueness and the symmetry property for these solutions. Also, we have established some results about the existence of FAS which however should be considerably improved in order to prove that FAS exist for a reasonable class of potentials. This is a trend we are going to follow in the next future. Finally, the big goal is to establish completeness relations constructed through the FAS of $L_{S_{\pm 1}}$. This will permit to extend the known theory of expansions over the so-called adjoint solutions which is basic for the Recursion Operators approach to soliton equations to the equations solvable though the auxiliary linear problem $L_{S_{\pm 1}}\psi = 0$.

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