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ON THE GEOMETRY INDUCED BY LORENTZ TRANSFORMATIONS IN PSEUDO-EUCLIDEAN SPACES

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Abstract. The Lorentz transformations of order (m, n) in pseudo-Euclidean spaces with indefinite inner product of signature (m, n) are extended in this work from $m = 1$ and $n \geq 1$ to all $m, n \geq 1$. A parametric realization of the Lorentz transformation group of any order (m, n) is presented, giving rise to generalized gyrogroups and gyrovector spaces called bi-gyrogroups and bi-gyrovector spaces. The latter, in turn, form the setting for generalized analytic hyperbolic geometry that underlies generalized balls called eigenballs.

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1. Introduction

The Lorentz transformations $\Lambda \in \mathrm{SO}(1, \mathrm{n})$ of special relativity are transformations of time-space points $(t, \mathbf{x})$, $t \in \mathbb{R}$, $\mathbf{x} \in \mathbb{R}^n$, of a pseudo-Euclidean space $\mathbb{R}^{1,n}$ with inner product of signature $(1, n)$. In physical applications $n = 3$, but in applications in geometry we allow $n \in \mathbb{N}$. The Lorentz transformation group $\mathrm{SO}(1, \mathrm{n})$ is a group of special linear transformations in $\mathbb{R}^{1,n}$ that leave the inner product invariant. They are special in the sense that the determinant of the $(1 + n) \times (1 + n)$ matrix representation of each $\Lambda \in \mathrm{SO}(1, \mathrm{n})$ is 1 and its $(1, 1)$ entry is positive.

A parametric realization of the Lorentz transformation group $\mathrm{SO}(1, \mathrm{n})$ in terms of the two parameters $V \in \mathbb{R}^n_c = \{V \in \mathbb{R}^n \,;\, \|V\| < c\}$ and $O_n \in \mathrm{SO}(\mathrm{n})$ is presented in [9], where $\mathbb{R}^n_c$ is the ball of all relativistically admissible velocities and $\mathrm{SO(n)}$ is the space of all space rotations of time-space coordinates $(t, \mathbf{x}) \in \mathbb{R}^{1,n}$. It is shown in [9] that the Lorentz transformation composition law, when expressed in terms of parameter composition, induces a group-like and a vector space-like structure that underlie the ball $\mathbb{R}^n_c$. These structures became known as a gyrogroup and a gyrovector space. Gyrogroups and gyrovector spaces, in turn, play a universal computational role, which extends far beyond the domain of special relativity as evidenced, for instance, from [11, 12, 14–18] and [1, 7, 20]. In particular, it became clear that gyrovector spaces form the setting for analytic hyperbolic geometry, just as vector spaces form the setting for analytic Euclidean geometry.

In this work we generalize the Lorentz transformation group $\mathrm{SO(m,n)}$ in a pseudo-Euclidean space $\mathbb{R}^{m,n}$ from $m = 1$ and $n \in \mathbb{N}$ to $m, n \in \mathbb{N}$, indicating the emergence of generalized analytic non-Euclidean geometry that the generalized Lorentz groups of order (m, n) induce in generalized balls, called eigenballs, of generalized relativistically admissible velocities. The group $\mathrm{SO}(m, n)$ of all Lorentz transformations of order (m, n), $m, n \in \mathbb{N}$, is also known as the special pseudo-orthogonal group [3, p. 478], or the group of pseudo-rotations [2]. A Lorentz transformation without rotations is called a boost when $m = 1$ and a bi-boost when $m > 1$. Bi-boosts are studied in [19].

2. How the Gyrovector Space was Discovered

By describing how the gyrovector space was discovered we pave in this section the road that leads in the next section to the discovery by inspection of the bi-gyrovector spaces that generalized Lorentz transformations induce. The generalization from gyrovector spaces to bi-gyrovector spaces is important and interesting since it results from the generalization of Lorentz transformations in pseudo-Euclidean spaces $\mathbb{R}^{m,n}$ from $m = 1$ and $n \in \mathbb{N}$ to all $m, n \in \mathbb{N}$. This generalization, in turn, leads to a generalization of the analytic non-Euclidean geometry that generalized Lorentz transformations induce.

René Descartes and Pierre de Fermat revolutionized the study of Euclidean geometry with their introduction of Cartesian coordinate systems. Significant outcomes were the rise of analytic Euclidean geometry and the introduction of Euclidean vector spaces as an appropriate framework for Euclidean geometry [5]. Hyperbolic gyrovector spaces form a natural generalization of Euclidean vector spaces. The gyrovector space structure forms the setting for analytic hyperbolic geometry, just as the vector space structure forms the setting for analytic Euclidean geometry, as demonstrated in many articles and in the seven books [11, 12, 14–18] that followed the pioneering discovery in 1988 [9]. Indeed, it was discovered in [9] that Einstein addition of relativistically admissible velocities plays in hyperbolic geometry the role that vector addition plays in Euclidean geometry, thus enabling the adaptation of Cartesian coordinates for use in hyperbolic geometry as well.

In order to extend the notion of the gyrovector space, we extend below the concept of the c-ball $\mathbb{R}_c^n$ in (2) to the concept of the c-eigenball $\mathbb{R}_c^{n\times m}$ in (1). Accordingly, let $\mathbb{R}^{n\times m}$ be the space of all rectangular real matrices of order $n\times m$, $m,n\in\mathbb{N}$, so that $\mathbb{R}^{n\times 1}=\mathbb{R}^n$ is the Euclidean n-space of all n-dimensional column vectors. The subset $\mathbb{R}_c^{n\times m}\subset\mathbb{R}^{n\times m}$

$$\begin{aligned}\mathbb{R}_c^{n\times m} &= \{V\in\mathbb{R}^{n\times m}\ ;\ \text{Each eigenvalue } \lambda \text{ of } VV^t \text{ satisfies } 0\le\lambda<c^2\}\\ &= \{V\in\mathbb{R}^{n\times m}\ ;\ \text{Each eigenvalue } \lambda \text{ of } V^tV \text{ satisfies } 0\le\lambda<c^2\}\end{aligned} \tag{1}$$

where exponent t denotes transposition, is called the eigenball of the ambient space $\mathbb{R}^{n\times m}$ with eigenradius c, or the c-eigenball of $\mathbb{R}^{n\times m}$ in short. It can readily be shown that in the special case when $m=1$, the c-eigenball $\mathbb{R}_c^{n\times 1}$ of $\mathbb{R}^{n\times 1}$ and the c-ball $\mathbb{R}_c^n$ of $\mathbb{R}^{n\times 1}=\mathbb{R}^n$

$$\mathbb{R}_c^n=\{V\in\mathbb{R}^n\ ;\ \|V\|<c\} \tag{2}$$

coincide [8], that is

$$\mathbb{R}_c^{n\times 1}=\mathbb{R}_c^n\,. \tag{3}$$

Hence, the c-eigenball $\mathbb{R}_c^{n\times m}$ of the ambient space $\mathbb{R}^{n\times m}$, $m,n\in\mathbb{N}$, forms a natural generalization of the c-ball $\mathbb{R}_c^n=\mathbb{R}_c^{n\times 1}$ of the ambient space $\mathbb{R}^n=\mathbb{R}^{n\times 1}$.

Following Einstein's special relativity theory, we say that $\mathbb{R}_c^n$ is the space of all relativistically admissible velocities (of material objects. Obviously, $n=3$ in physical applications). Hence, we may say that $\mathbb{R}_c^{n\times m}$ is the space of all *generalized relativistically admissible velocities*. We now wish to extend the gyrovector space structure of the ball $\mathbb{R}_c^n$ of all relativistically admissible velocities to a structure of the eigenball $\mathbb{R}_c^{n\times m}$ of all generalized relativistically admissible velocities, $m,n\in\mathbb{N}$.

In order to set the stage for the extension of

1. the gyrovector space structure of the ball $\mathbb{R}_c^n=\mathbb{R}_c^{n\times 1}$, $n\in\mathbb{N}$, of Euclidean spaces $\mathbb{R}^n=\mathbb{R}^{n\times 1}$ to
2. the bi-gyrovector space structure of the eigenball $\mathbb{R}_c^{n\times m}$ of spaces $\mathbb{R}^{n\times m}$, $m,n\in\mathbb{N}$

we present results from [8] that briefly unfold the history of the discovery of the gyrovector space structure. The brief history of the emergence of the gyrovector space concept, in turn, paves the road that leads from gyrogroups and gyrovector spaces to bi-gyrogroups and bi-gyrovector spaces, as we demonstrate in this article.

The Lorentz transformation Λ of order $(1,n)$ is the common Lorentz transformation of special relativity theory in one time dimension and n space dimensions ($n=3$ in physical applications). A Lorentz transformation without rotation is called a *boost*. The boost of order $(1,n)$ has the well known $(1+n)\times(1+n)$ matrix representation

$$B_c(V) = \begin{pmatrix} \gamma_V & \frac{1}{c^2}\gamma_V V^t \\ \gamma_V V & I_n + \frac{1}{c^2}\frac{\gamma_V^2}{1+\gamma_V} VV^t \end{pmatrix} \in \mathrm{SO(1,n)}\,. \tag{4}$$

Here, I_n is the $n\times n$ identity matrix, and γ_V is the gamma factor given by

$$\gamma_V = \frac{1}{\sqrt{1-\frac{\|V\|^2}{c^2}}} = \sqrt{1-c^{-2}V^tV}^{\,-1} \tag{5}$$

where $V\in\mathbb{R}^n_c$ is a relativistically admissible velocity.

Introducing a Cartesian coordinate system Σ into the ambient space $\mathbb{R}^n$ of the c-ball $\mathbb{R}^3_c$, any relativistically admissible velocity $V\in\mathbb{R}^3_c$ is expressible in terms of its coordinates by the column vector $V=(v_1,v_2,v_3)^t$. Then V^tV is the scalar

$$V^tV = v_1^2+v_2^2+v_3^2 = \|V\|^2 < c^2 \tag{6}$$

and VV^t is the 3×3 symmetric matrix

$$\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}(v_1,v_2,v_3) = \begin{pmatrix} v_1^2 & v_1v_2 & v_1v_3 \\ v_1v_2 & v_2^2 & v_2v_3 \\ v_1v_3 & v_2v_3 & v_3^2 \end{pmatrix}\,. \tag{7}$$

Hence, with $n=3$ and with respect to the Cartesian coordinate system Σ, the matrix representation (4) of the Lorentz transformation $\Lambda = B_c(V)$ of order $(m,n)=(1,3)$ is the 4×4 matrix

$$B_c(V) = \begin{pmatrix} \gamma_{\mathbf{v}} & \frac{1}{c^2}\gamma_{\mathbf{v}}\,v_1 & \frac{1}{c^2}\gamma_{\mathbf{v}}\,v_2 & \frac{1}{c^2}\gamma_{\mathbf{v}}\,v_3 \\ \gamma_{\mathbf{v}}\,v_1 & 1+\frac{1}{c^2}\frac{\gamma_{\mathbf{v}}^2}{\gamma_{\mathbf{v}}+1}v_1^2 & \frac{1}{c^2}\frac{\gamma_{\mathbf{v}}^2}{\gamma_{\mathbf{v}}+1}v_1v_2 & \frac{1}{c^2}\frac{\gamma_{\mathbf{v}}^2}{\gamma_{\mathbf{v}}+1}v_1v_3 \\ \gamma_{\mathbf{v}}\,v_2 & \frac{1}{c^2}\frac{\gamma_{\mathbf{v}}^2}{\gamma_{\mathbf{v}}+1}v_1v_2 & 1+\frac{1}{c^2}\frac{\gamma_{\mathbf{v}}^2}{\gamma_{\mathbf{v}}+1}v_2^2 & \frac{1}{c^2}\frac{\gamma_{\mathbf{v}}^2}{\gamma_{\mathbf{v}}+1}v_2v_3 \\ \gamma_{\mathbf{v}}\,v_3 & \frac{1}{c^2}\frac{\gamma_{\mathbf{v}}^2}{\gamma_{\mathbf{v}}+1}v_1v_3 & \frac{1}{c^2}\frac{\gamma_{\mathbf{v}}^2}{\gamma_{\mathbf{v}}+1}v_2v_3 & 1+\frac{1}{c^2}\frac{\gamma_{\mathbf{v}}^2}{\gamma_{\mathbf{v}}+1}v_3^2 \end{pmatrix}\,. \tag{8}$$

The application of the boost $B_c(V)$ in (8) to a point $(t,\mathbf{x})^t = (t,x_1,x_2,x_3)^t$ of the pseudo-Euclidean space $\mathbb{R}^{1,3}$ of signature $(1,3)$ leaves the inner product of signature $(1,3)$

$$\begin{pmatrix} t_1 \\ \mathbf{x}_1 \end{pmatrix}\cdot\begin{pmatrix} t_2 \\ \mathbf{x}_2 \end{pmatrix} = t_1t_2 - \frac{1}{c^2}\mathbf{x}_1\cdot\mathbf{x}_2 \tag{9}$$

invariant, that is

$$B_c(V)\begin{pmatrix} t_1 \\ \mathbf{x}_1 \end{pmatrix}\cdot B_c(V)\begin{pmatrix} t_2 \\ \mathbf{x}_2 \end{pmatrix} = \begin{pmatrix} t_1 \\ \mathbf{x}_1 \end{pmatrix}\cdot\begin{pmatrix} t_2 \\ \mathbf{x}_2 \end{pmatrix} = t_1t_2 - \frac{1}{c^2}\mathbf{x}_1\cdot\mathbf{x}_2 \tag{10}$$

for all $V \in \mathbb{R}^3_c$, $t_1, t_2 \in \mathbb{R}$ and $\mathbf{x}_1, \mathbf{x}_2 \in \mathbb{R}^3$.

The boost matrix representation (8) is well known in various equivalent forms as, for instance [6, p. 118] and [4, p. 541]. We prefer the form in (8) since this form reveals that in the limit of large c, $c \to \infty$, the Lorentz boost $B_c(V)$ tends to a corresponding Galilean boost [11, p. 254].

A Lorentz transformation Λ of order $(1, n)$ is the boost $B_c(V)$ preceded (or, followed) by a space rotation $\rho(O_n)$ $O_n \in \mathrm{SO(n)}$, that is

$$\Lambda = \Lambda(V, O_n) = B_c(V)\rho(O_n) \tag{11}$$

where $\rho(O_n)$ is the $(1+n) \times (1+n)$ matrix parametrized by O_n,

$$\rho(O_n) = \begin{pmatrix} 1 & 0_{1,n} \\ 0_{n,1} & O_n \end{pmatrix} \in \mathrm{SO(1,n)} \tag{12}$$

where $0_{m,n}$ is the zero matrix of order $m \times n$, so that

$$\rho(O_n) \begin{pmatrix} t \\ \mathbf{x} \end{pmatrix} = \begin{pmatrix} t \\ O_n\mathbf{x} \end{pmatrix} \in \mathbb{R}^{1,n}. \tag{13}$$

Here $(t, \mathbf{x})^t \in \mathbb{R}^{1,n}$ is a time-space point of the pseudo-Euclidean space $\mathbb{R}^{1,n}$ of order $(1, n)$, so that (13) represents a space rotation of time-space coordinates.

The parametric realization of the Lorentz transformation Λ of order $(1, n)$ in (11) enables the Lorentz transformation composition law to be expressed in terms of parameter composition law, as shown in [9]. The resulting parameter composition, in turn, induces a binary operation, $\oplus$, in the parameter space $\mathbb{R}^n_c$. The resulting groupoid $(\mathbb{R}^n_c, \oplus)$ turns out to form a group-like object that became known as a gyrogroup [10, 11, 13]. Furthermore, the gyrogroup $(\mathbb{R}^n_c, \oplus)$ admits scalar multiplication $\otimes$, turning itself into a gyrovector space $(\mathbb{R}^n_c, \oplus, \otimes)$, $n \in \mathbb{N}$. These gyrovector spaces, in turn, form the setting for analytic hyperbolic geometry of the c-ball $\mathbb{R}^n_c$, just as vector spaces form the setting for analytic Euclidean geometry of the ambient space $\mathbb{R}^n$. The resulting analytic hyperbolic geometry is studied, for instance, in [12, 14, 18].

Thus, the gyrovector space structure was discovered in two major steps

1. by a parametric realization of the Lorentz transformation of order $(1, n)$, and
2. by expressing the composition law of the Lorentz transformation of order $(1, n)$ in terms of parameter composition law, for all $n \in \mathbb{N}$.

The resulting parameter composition law gives rise to a binary operation, $\oplus$, in the space $\mathbb{R}^{n\times 1}_c = \mathbb{R}^n_c$ of the parameter V, and the binary operation, in turn, gives rise to a scalar multiplication $\otimes$, thus constructing the gyrovector space $(\mathbb{R}^{n\times 1}_c, \oplus, \otimes)$=$(\mathbb{R}^n_c, \oplus, \otimes)$.

Accordingly

1. the first major step in our mission to extend the gyrovector space notion from order $(1, n)$ to order (m, n), $m, n \in \mathbb{N}$, is to extend the parametric realization of the Lorentz group of order (m, n) from $m = 1$ and $n \in \mathbb{N}$ to all $m, n \in \mathbb{N}$, and
2. the second major step in our mission to extend the gyrovector space notion from order $(1, n)$ to order (m, n), $m, n \in \mathbb{N}$, is to express the composition law of the Lorentz transformation of order (m, n) in terms of parameter composition law for all $m, n \in \mathbb{N}$.

3. The Lorentz Boost of Any Order and its Composition Law

In this section we present a novel natural way to extend by inspection the classical Lorentz transformation group of order $(1, n)$, $n \in \mathbb{N}$, in a pseudo-Euclidean space $\mathbb{R}^{1,n}$ to the generalized Lorentz transformation group of order (m, n), in a pseudo-Euclidean space $\mathbb{R}^{m,n}$ for any $m, n \in \mathbb{N}$.

The important, elegant, novel identity

$$I_n + \frac{1}{c^2}\frac{\gamma_V^2}{1+\gamma_V} VV^t = \sqrt{I_n - c^{-2}VV^t}^{\,-1} \tag{14}$$

is verified in [8, equation (128)], where $V \in \mathbb{R}_c^n = \mathbb{R}_c^{n\times 1}$, $n \in \mathbb{N}$. It allows us to rewrite (4) in a form that reveals a hidden symmetry

$$B_c(V) = \begin{pmatrix} \sqrt{1 - c^{-2}V^tV}^{\,-1} & \frac{1}{c^2}\sqrt{1 - c^{-2}V^tV}^{\,-1}V^t \\ V\sqrt{1 - c^{-2}V^tV}^{\,-1} & \sqrt{I_n - c^{-2}VV^t}^{\,-1} \end{pmatrix} \in \mathrm{SO(1,n)} \tag{15}$$

for all $V \in \mathbb{R}_c^n = \mathbb{R}_c^{n\times 1}$, $n \in \mathbb{N}$.

Equation (15) suggests extending the notation in (5) for the gamma factor γ associated with $V \in \mathbb{R}_c^n = \mathbb{R}^{n\times 1}$ to left and right gamma factors, $\Gamma_{n,V}^L$ and $\Gamma_{m,V}^R$, in (16) below, associated with $V \in \mathbb{R}_c^{n\times m}$ for any $m, n \in \mathbb{N}$

$$\Gamma_{n,V}^L := \sqrt{I_n - c^{-2}VV^t}^{\,-1} \tag{16}$$

and

$$\Gamma_{m,V}^R := \sqrt{I_m - c^{-2}V^tV}^{\,-1} . \tag{17}$$

The left and right gamma factors are called collectively the *bi-gamma factor*. Clearly, in the special case when $m = 1$, (17) specializes to (5), that is

$$\Gamma_{m=1,V}^R = \gamma_V \,. \tag{18}$$

Interestingly, the bi-gamma factor and $V \in \mathbb{R}^{n\times m}$ possess the commuting relations [8, equation (102)]

$$\Gamma^L_{n,V} V = V\Gamma^R_{m,V}, \qquad \Gamma^R_{m,V} V^t = V^t \Gamma^L_{n,V} \tag{19}$$

for all $V \in \mathbb{R}^{n\times m}$.

The commuting relations (19) reveal hidden symmetries in the boost representation (15). Indeed, by (18), (16) and (19), the boost in (15) takes, in terms of the bi-gamma factor, a form that exhibits previously unnoticed symmetries that suggest the natural extension of the boost of order $(m = 1, n)$ to the *bi-boost* of order (m, n), $m, n \in \mathbb{N}$. Thus, the boost representation (15) can be written as

$$B_c(V) = \begin{pmatrix} \Gamma^R_{m=1,V} & \frac{1}{c^2}\Gamma^R_{m=1,V} V^t = \frac{1}{c^2} V^t \Gamma^L_{n,V} \\ V\Gamma^R_{m=1,V} = \Gamma^L_{n,V} V & \Gamma^L_{n,V} \end{pmatrix} \in \mathrm{SO}(1,\mathrm{n}) \tag{20}$$

for all $V \in \mathbb{R}^{n\times m}$, where $m = 1$ and $n \in \mathbb{N}$.

The boost $B_c(V)$ in (20) is a Lorentz transformation of order $(m = 1, n)$, which leaves invariant the inner product of signature $(m = 1, n)$, as shown in (10). However, the boost in (20) possesses symmetries that suggest that $m = 1$ is not privileged. Indeed, it is shown straightforwardly in [8] that the boost $B_c(V)$ in (20) is valid for all $m, n \in \mathbb{N}$. Thus, the boost (20) without the restriction $m = 1$ takes the elegant, novel form

$$B_c(V) = \begin{pmatrix} \Gamma^R_{m,V} & \frac{1}{c^2}\Gamma^R_{m,V} V^t = \frac{1}{c^2} V^t \Gamma^L_{n,V} \\ V\Gamma^R_{m,V} = \Gamma^L_{n,V} V & \Gamma^L_{n,V} \end{pmatrix} \in \mathrm{SO}(1,\mathrm{n})\,. \tag{21}$$

The boost $B_c(V)$ in (21) is a Lorentz transformation of order (m, n), which leaves invariant the inner product of signature (m, n) in a pseudo-Euclidean space $\mathbb{R}^{m,n}$ for all $V \in \mathbb{R}^{n\times m}_c$, $m, n \in \mathbb{N}$.

Thus through equation (21) we have generalized by inspection the boost $B_c(V) \in \mathrm{SO}(\mathrm{m},\mathrm{n})$ of order (m, n) from $m = 1$ and $n \in \mathbb{N}$ to all $m, n \in \mathbb{N}$.

A boost of order (m, n) is the common boost of special relativity when $m = 1$. A boost of order (m, n) with $m > 1$ is called a *bi-boost* since it involves a left and a right rotation that are called collectively a *bi-rotation*.

Indeed, the most general Lorentz transformation of order (m, n), $m > 1$, involves the boost $B_c(V)$ in equation (21) along with a time rotation (or, a right rotation) and a space rotation (or, a left rotation) according to the equation

$$\Lambda = \begin{pmatrix} O_m & 0_{m,n} \\ 0_{n,m} & I_n \end{pmatrix} \begin{pmatrix} \Gamma^R_{m,V} & \frac{1}{c^2}\Gamma^R_{m,V} V^t \\ \Gamma^L_{n,V} V & \Gamma^L_{n,V} \end{pmatrix} \begin{pmatrix} I_m & 0_{m,n} \\ 0_{n,m} & O_n \end{pmatrix} \in \mathrm{SO}(\mathrm{m},\mathrm{n})\,. \tag{22}$$

Parametrically, we may write (22) as

$$\Lambda = \Lambda(O_m, V, O_n) = \rho(O_m)B(V)\lambda(O_n) = \begin{pmatrix} V \\ O_n \\ O_m \end{pmatrix} \tag{23}$$

where the Lorentz transformation $\Lambda \in \mathrm{SO(m,n)}$ is parametrized by the parameters $V \in \mathbb{R}_c^{n\times m}$, $O_m \in \mathrm{SO(m)}$ and $O_n \in \mathrm{SO(n)}$.

The product of two bi-boosts

$$B_c(V_1)B_c(V_2) = \begin{pmatrix} \Gamma^R_{m,V_1} & \frac{1}{c^2}\Gamma^R_{m,V_1}V_1^t \\ \Gamma^L_{n,V_1}V_1 & \Gamma^L_{n,V_1} \end{pmatrix} \begin{pmatrix} \Gamma^R_{m,V_2} & \frac{1}{c^2}\Gamma^R_{m,V_2}V_2^t \\ \Gamma^L_{n,V_2}V_2 & \Gamma^L_{n,V_2} \end{pmatrix} \tag{24}$$

need not be a bi-boost, since it may involve time and space rotations. However, it must be a Lorentz transformation. Hence, the uniqueness of the representation of any Lorentz transformation of order (m,n) in (22) implies the bi-boost composition law

$$\begin{aligned} &B_c(V_1)B_c(V_2) \\ &= \begin{pmatrix} \mathrm{rgyr}[V_1,V_2] & 0_{m,n} \\ 0_{n,m} & I_n \end{pmatrix} \begin{pmatrix} \Gamma^R_{m,V_{12}} & \frac{1}{c^2}\Gamma^R_{m,V_{12}}V_{12}^t \\ \Gamma^L_{n,V_{12}}V_{12} & \Gamma^L_{n,V_{12}} \end{pmatrix} \begin{pmatrix} I_m & 0_{m,n} \\ 0_{n,m} & \mathrm{lgyr}[V_1,V_2] \end{pmatrix} \\ &= \begin{pmatrix} \mathrm{rgyr}[V_1,V_2]\Gamma^R_{m,V_{12}} & \frac{1}{c^2}\mathrm{rgyr}[V_1,V_2]\Gamma^R_{m,V_{12}}V_{12}^t\mathrm{lgyr}[V_1,V_2] \\ \Gamma^L_{n,V_{12}}V_{12} & \Gamma^L_{n,V_{12}}\mathrm{lgyr}[V_1,V_2] \end{pmatrix}. \end{aligned} \tag{25}$$

The expressions

$$V_{12} =: V_1 \oplus V_2 \in \mathbb{R}_c^{n\times m}, \ \ \mathrm{lgyr}[V_1,V_2] \in \mathrm{SO(n)}, \ \ \mathrm{rgyr[V_1,V_2] \in SO(m)} \tag{26}$$

that appear in (25) are determined uniquely in terms of V_1 and V_2, by comparing (24) and (25) as shown in detail in [8].

The resulting bi-gyrogroup and bi-gyrovector space structure of the eigenball $\mathbb{R}_c^{n\times m}$ involve the binary operation $\oplus$ in $\mathbb{R}_c^{n\times m}$ and its resulting scalar multiplication $\otimes$ as well as the *bi-gyration* $\mathrm{gyr} = (\mathrm{lgyr}[V_1,V_2], \mathrm{rgyr}[V_1,V_2])$. The details of these structures are presented in the work in [8] which is, in turn, a continuation of the work in [19].

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