SURFACES FROM DEFORMATION OF PARAMETERS

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Abstract. We construct surfaces from modified Korteweg-de Vries (mKdV)
and sine-Gordon (SG) soliton solutions by the use of parametric deforma-
tions. For each case there are two types of deformations. The first one gives
surfaces on spheres and the second one gives highly complicated surfaces
in three dimensional Euclidean space ($\mathbb{R}^3$). The SG surfaces that we ob-
tained are not the critical points of functional where the Lagrange function is
a polynomial function of the Gaussian (K) and mean (H) curvatures of the
surfaces. We also give the graph of interesting mKdV and SG surfaces arise
from parametric deformations.

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CONTENTS

1. Introduction

Soliton surfaces or Integrable surfaces are the surfaces in $\mathbb{R}^3$ obtained by a technique which uses the symmetries of both the integrable nonlinear partial differential equations and their Lax equations. In this technique, the main step is to associate a deformation corresponding to each symmetry. These are the scaling of the spectral parameter, gauge transformations and the classical and generalized symmetries of the integrable equations. By the use of corresponding deformations of different symmetries of different integrable equations such as mKdV, SG, nonlinear Schrödinger (NLS), and KdV equations, several surfaces have been obtained by several authors [3–7], [12–20], [24–29].

In this work, we present a new deformation which corresponds to the integration parameters of the solutions of the integrable equations. These deformations are given by the formulas

$$A_i = \delta_i U = \mu_2 \frac{\partial U}{\partial k_i}, \qquad B_i = \delta_i V = \mu_2 \frac{\partial V}{\partial k_i}, \qquad F_i = \mu_2 \Phi^{-1} \frac{\partial \Phi}{\partial k_i} \qquad (1)$$

where $i = 0, 1, \cdots n$ and k_i are parameters of the solution $u(x, t, k_0, \ldots, k_n)$ of the integrable equations, μ_2 is a constant. Here x and t are independent variables and n is the number of integration parameters [28]. Some of these deformations may generate exactly the same surface generated by other deformations such as symmetry and spectral parameter deformations but we have different surfaces corresponding to the deformation for different parameters. For a review of the integrable surfaces see [16].

2. General Theory

Let Φ be a group G-valued and U and V be the corresponding algebra $\mathfrak{g}$-valued functions satisfying the Lax equations

$$\Phi_x = U\, \Phi, \qquad \Phi_t = V\, \Phi. \qquad (2)$$

In the Lax equations all functions such as Φ, U and V are functions of x, t and the spectral parameter λ. Here and in what follows, subscripts x and t denote the derivatives of the objects with respect to x and t, respectively. The integrability condition of the Lax equations is the well known equation called the zero curvature condition [1], [2], [9] has the following form

$$U_t - V_x + [U, V] = 0. \qquad (3)$$

This equation is the source of all integrable nonlinear partial differential equations. There are symmetries which leave both the Lax equations and the integrable nonlinear partial differential equations invariant. They are i) scaling the spectral parameter λ, ii) gauge transformations, iii) transformations whose infinitesimals are the generalized symmetries, iv) scaling of the parameters of the solutions. For each transformations we can define infinitesimal symmetries. Let δ denotes such infinitesimal deformation and let $\delta U = A$ and $\delta V = B$. Deformation operator δ commutes with D_x and D_t. Both A and B are $\mathfrak{g}$-valued functions. Taking δ deformation of the Lax equations in equation (2), we obtain the following equations

$$(\delta\Phi)_x = A\Phi + U\delta\Phi, \qquad (\delta\Phi)_t = B\Phi + V\delta\Phi \tag{4}$$

From the integrability condition of the equations in equations (4) or from the zero curvature condition given in equation (3), we get the following equation

$$A_t - B_x + [U, B] + [A, V] = 0. \tag{5}$$

In addition, the equations in equations (4) imply the followings

$$F_x = \Phi^{-1} A \Phi, \qquad F_t = \Phi^{-1} B \Phi \tag{6}$$

where

$$F = \Phi^{-1} \delta\Phi + F_0. \tag{7}$$

Notice that $\mathrm{trace}(F) = \delta(\det(\Phi))$. It is clear that, from the Lax equations given in equation (2), $\det(F)$ does not depend on x and t but it may depend on the spectral parameter λ and also other variables. Hence the constant term F_0 in equation (7) is added to make $\mathrm{trace}(F) = 0$. In the theory of soliton surfaces, the $\mathfrak{g}$-valued function F is taken as the parametrization of the surfaces embedded in $\mathbb{R}^3$ or in M_3 (three dimensional Minkowski space).

We give four types of these deformations below. The first three were given by Sym [24–26], Fokas and Gel'fand [12], Fokas *et al* [5, 13] and Cieśliński [7] and others [27–29], [15, 16]. The last one is a new one and introduced in [28].

i) Spectral parameter λ invariance of the equation

$$A = \delta U = \mu_1 \frac{\partial U}{\partial \lambda}, \qquad B = \delta V = \mu_1 \frac{\partial V}{\partial \lambda}, \qquad F = \mu_1 \Phi^{-1} \frac{\partial \Phi}{\partial \lambda}$$

where μ_1 is an arbitrary function of λ. This kind of deformation was first used by Sym [24–26].

ii) Symmetries of the (integrable) differential equations

$$A = \delta U, \qquad B = \delta V, \qquad F = \Phi^{-1}\delta\Phi$$

where δ represents the classical Lie symmetries and (if integrable) the generalized symmetries of the nonlinear PDE's [5, 7, 12, 13].

iii) The Gauge symmetries of the Lax equation

$$A = \delta U = M_x + [M, U], \qquad B = \delta V = M_t + [M, V], \qquad F = \Phi^{-1} M \Phi$$

where M is any traceless 2×2 matrix [7, 12, 13].

iv) The deformation of parameters for solution of integrable equation

$$A = \delta U = \mu_2 \frac{\partial U}{\partial k_i}, \qquad B = \delta V = \mu_2 \frac{\partial V}{\partial k_i}, \qquad F = \mu_2 \Phi^{-1} \frac{\partial \Phi}{\partial k_i}$$

where $i = 0, 1, \ldots, n$ and k_i are parameters of the solution $u(x, t, k_0, \ldots, k_n)$ of the PDEs, μ_2 is constant. Here x and t are independent variables [28].

Surfaces corresponding to integrable equations are called integrable surfaces and a connection formula, relating integrable equations to surfaces, was first established by Sym [24–26]. His formula gives a relation between family of immersions and Lax pairs defined in a Lie algebra. In this work, we consider deformation of parameters to construct new surfaces. This means that we take $A_i = \delta_i U = \mu_2 (\partial U / \partial k_i)$, $B_i = \delta_i V = \mu_2 (\partial V / \partial k_i)$ where k_i $(i = 0, 1, 2, \cdots n)$ are the arbitrary parameters in the solutions the integrable equations.

For the sake of completeness, we now give a brief summary that gives the relation between solutions of the integrable equations and surfaces. For the differential geometry of surfaces in $\mathbb{R}^3$ see [10], [11] and [8]. Let $F : \mathcal{U} \to \mathbb{R}^3$ be an isometric immersion of a domain $\mathcal{U} \in R_2$ into $\mathbb{R}^3$, where R_2 is the 2-plane. Let $(x, t) \in \mathcal{U}$. The surface $F(x, t)$ is uniquely defined up to rigid motions by the first and second fundamental forms. Let $N(x, t)$ be the normal vector field defined at each point of the surface $F(x, t)$. Then the triple $\{F_x, F_t, N\}$ at a point $p \in S$ defines a basis of the tangent space at p, $T_p(S)$, where S is the surface parameterized by $F(x, t)$. The first and the second fundamental forms of S are

$$\begin{aligned}
\mathrm{d}s_I^2 &= g_{ij} \, \mathrm{d}x^i \, \mathrm{d}x^j = \langle F_x, F_x \rangle \mathrm{d}x^2 + 2\langle F_x, F_t \rangle \mathrm{d}x \, \mathrm{d}t + \langle F_t, F_t \rangle \mathrm{d}t^2 \\
&= \langle A, A \rangle \, \mathrm{d}x^2 + 2\langle A, B \rangle \, \mathrm{d}x \, \mathrm{d}t + \langle B, B \rangle \, \mathrm{d}t^2 \\
\mathrm{d}s_{II}^2 &= h_{ij} \, \mathrm{d}x^i \, \mathrm{d}x^j = \langle F_{xx}, N \rangle \mathrm{d}x^2 + 2\langle F_{xt}, N \rangle \mathrm{d}x \, \mathrm{d}t + \langle F_{tt}, N \rangle \mathrm{d}t^2 \\
&= -\langle A_x + [A, U], C \rangle \, \mathrm{d}x^2 - 2\langle A_t + [A, V], C \rangle \, \mathrm{d}x \, \mathrm{d}t \\
&\quad -\langle B_t + [B, V], C \rangle \, \mathrm{d}t^2
\end{aligned}$$

where $i, j = 1, 2$, $x^1 = x$ and $x^2 = t$, $\langle A, B \rangle = (1/2)\mathrm{trace}(AB)$, $[A, B] = AB - BA$, $\|A\| = \sqrt{|\langle A, A \rangle|}$, and $C = [A, B]/\|[A, B]\|$. A frame on this surface S is

$$\Phi^{-1} A \Phi, \qquad \Phi^{-1} B \Phi, \qquad \Phi^{-1} C \Phi.$$

The Gauss and the mean curvatures of S are given by

$$K = \det(g^{-1} h), \qquad H = \frac{1}{2} \mathrm{trace}(g^{-1} h).$$

The function Φ, which is defined by equations given in equation (2) exists if and only if U and V satisfy the equation given in equation (3) [12]. In other words, equation (3) is the compatibility condition of equation (2). The equations in equation (6) define a surface F if and only if A and B satisfy the equation given in equation (5) [12]. Namely, equation (5) is the condition to define a surface F in Lie algebra $\mathfrak{g}$ which is obtained from the equation given by equation (7). Furthermore, to have regular surfaces, F_x and F_t (or A and B) must be linearly independent at each point of the surface S. This is the regularity condition of the mapping $F : S \to \mathbb{R}^3$. Hence the commutator $[A, B]$ is nowhere zero on the surface. This ensures that the three vectors A, B and C form a triad at each point of the surface.

For the given U and V, finding A and B from the equation in equation (5) is in general a difficult task. However, there are some deformations which provide A and B directly. Some of these deformations are given by Sym [24–26], Fokas and Gel'fand [12], Fokas et al [13] and Cieśliński [7].

In order to obtain the surfaces using the given technique, we have to find position vector F which is given by [13]

$$F = \Phi^{-1} \left(\mu_1 \frac{\partial \Phi}{\partial \lambda} + \mu_2 \frac{\partial \Phi}{\partial k_i} + \delta \Phi + M \, \Phi \right).$$

In order to calculate F explicitly, the Lax equations provided by equation (2) need to be solved for a given solution of an integrable equation.

2.1. Surfaces From a Variational Principle

Let H and K be the mean and Gaussian curvatures of a surface S (either in M_3 or in $\mathbb{R}^3$) then we have the following definition.

Definition 1. *Let S be a surface with its Gaussian (K) and mean (H) curvatures. A functional $\mathcal{F}$ is defined by*

$$\mathcal{F} = \int_S \mathcal{E}(H, K) \mathrm{d}A + p \int_V \mathrm{d}V \tag{8}$$

where $\mathcal{E}$ is some function of H and K, p is a constant and V is the volume enclosed within the surface S.

The following proposition gives the first variation of the functional $\mathcal{F}$.

Proposition 2. *Let $\mathcal{E}$ be a twice differentiable function of H and K. Then the Euler-Lagrange equation for $\mathcal{F}$ is given by the following equation* [22, 23, 30–33]

$$(\nabla^2 + 4H^2 - 2K)\frac{\partial \mathcal{E}}{\partial H} + 2(\nabla \cdot \bar{\nabla} + 2KH)\frac{\partial \mathcal{E}}{\partial K} - 4H\mathcal{E} + 2p = 0 \tag{9}$$

where ∇^2 and $\nabla \cdot \bar{\nabla}$ are the differential operators defined in the following way

$$\nabla^2 = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^i} \left(\sqrt{g} g^{ij} \frac{\partial}{\partial x^j} \right), \qquad \nabla \cdot \bar{\nabla} = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^i} \left(\sqrt{g} K h^{ij} \frac{\partial}{\partial x^j} \right)$$

and $g = \det(g_{ij})$, g^{ij} and h^{ij} are components of the inverse of the first and second fundamental forms, $i, j = 1, 2$, and we assume Einstein's summation convention on repeated indices over their ranges.

Example 3. *The following are some examples of the surfaces in* Proposition 2 *for different choices of $\mathcal{E}$*

    i) *Minimal surfaces* [21]: *$\mathcal{E} = 1$, $p = 0$.*

    ii) *Constant mean curvature surfaces: $\mathcal{E} = 1$.*

    iii) *Linear Weingarten surfaces: $\mathcal{E} = aH + b$, where a and b are some constants.*

    iv) *Willmore surfaces: $\mathcal{E} = H^2$* [34, 35].

    v) *Surfaces solving the shape equation of lipid membrane: $\mathcal{E} = (H - c)^2$, where c is a constant* [22, 23, 30–33].

Definition 4. *The surfaces obtained from the solutions of the equation*

$$\nabla^2 H + aH^3 + bH\,K = 0 \tag{10}$$

are called W*illmore-like surfaces, where a and b are arbitrary constants.*

Remark 5. *The case $a = -b = 2$ in equation* (10) *corresponds to the Willmore surfaces.*

In this work, we assume $p = 0$. In addition, for surfaces derivable from a variational principal, we require asymptotic conditions such that H goes to a constant value and K goes to zero asymptotically. This is consistent with vanishing of boundary terms in obtaining Euler-Lagrange equation given by equation (9). This requires that the soliton equations such as KdV, mKdV, SG and NLS equations must have solutions decaying rapidly to zero at $|x| \to \pm \infty$. For this purpose, we shall calculate H and K for all surfaces obtained by mKdV and SG equation and look for possible solutions (surfaces) of the Euler-Lagrange equation [equation (9)].

3. Surfaces in $\mathbb{R}^3$

In this section, closely following [5, 12, 13] and [16] we give the immersion of a surface in $\mathbb{R}^3$ in terms of a group representation explicitly. For this purpose, we use Lie group SU(2) and its Lie algebra $\mathfrak{su}(2)$ with basis $e_j = -i\,\sigma_j$, $j = 1, 2, 3$, where σ_j denote the usual Pauli sigma matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{11}$$

Define an inner product on $\mathfrak{su}(2)$ in the following way

$$\langle X, Y \rangle = -\frac{1}{2}\ \mathrm{trace}(XY)$$

where $X, Y \in \mathfrak{su}(2)$ valued vectors. $\mathfrak{su}(2)$ valued representation F of a vector Y in $\mathbb{R}^3$ can be written as

$$F(x,t) = \mathrm{i} \sum_{k=1}^{3} Y^k \sigma_k \tag{12}$$

where Y^k, $k = 1, 2, 3$ are components of the vector F. We can write the vector F given in equation (12) more explicitly as

$$F = \mathrm{i} \begin{pmatrix} Y^3 & Y^1 - \mathrm{i}Y^2 \\ Y^1 + \mathrm{i}Y^2 & -Y^3 \end{pmatrix}.$$

In this representation, the inner product of two vectors is defined as

$$\langle F, G \rangle = -\frac{1}{2}\ \mathrm{trace}(FG)$$

where $F, G \in \mathfrak{su}(2)$. The length of the vector is defined as

$$\|F\| = \sqrt{|\langle F, F \rangle|}. \tag{13}$$

If F is the $\mathfrak{su}(2)$ representation of the position vector $\mathbf{Y}(x,t)$, then F_x and F_t are the $\mathfrak{su}(2)$ representation of the tangent vectors $\mathbf{Y}_{,x}$ and $\mathbf{Y}_{,t}$. If we let $\mathfrak{su}(2)$ representation of unit normal vector N as Z, then we find Z as

$$Z = \frac{[F_x, F_t]}{\|[F_x, F_t]\|}.$$

Here $[.\,,.]$ denotes the usual commutator. Hence we can give the $\mathfrak{su}(2)$ representation of a triad defined at every point of a surface as

$$\{F_x, F_t, Z\}.$$

In soliton theory, surfaces are developed in this way. Fundamental forms are given as

$$g = \begin{pmatrix} \langle F_x, F_x \rangle & \langle F_x, F_t \rangle \\ \langle F_x, F_t \rangle & \langle F_t, F_t \rangle \end{pmatrix}, \qquad h = -\begin{pmatrix} \langle F_x, Z_x \rangle & \langle F_x, Z_t \rangle \\ \langle F_t, Z_x \rangle & \langle F_t, Z_t \rangle \end{pmatrix}$$

and the Gaussian and mean curvatures take the following forms

$$K = \det\left(g^{-1}h\right), \qquad H = \frac{1}{2}\mathrm{trace}(g^{-1}h).$$

3.1. mKdV Surfaces From Deformation of Parameters

In [5], [14–16], [27–29], we considered spectral parameter deformation and combination of spectral and Gauge deformations to develop surfaces. In this section, we consider the mKdV surfaces arising from deformations of parameters of the integrable equations' solution.

Let $u(x,t)$ satisfy the mKdV equation

$$u_t = u_{xxx} + \frac{3}{2}u^2 u_x. \tag{14}$$

Substituting the travelling wave ansatz $u_t - \alpha\, u_x = 0$ in the equation given by equation (14), we get

$$u_{xx} = \alpha u - \frac{u^3}{2} \tag{15}$$

where α is an arbitrary real constant and integration constant is taken to be zero. mKdV equation in equation (15) can be obtained from the following Lax pairs U and V

$$U = \frac{\mathrm{i}}{2}\begin{pmatrix} \lambda & -u \\ -u & -\lambda \end{pmatrix}$$

$$V = -\frac{\mathrm{i}}{2}\begin{pmatrix} \frac{1}{2}u^2 - (\alpha + \alpha\lambda + \lambda^2) & (\alpha + \lambda)u - \mathrm{i}u_x \\ (\alpha + \lambda)u + \mathrm{i}u_x & -\frac{1}{2}u^2 + (\alpha + \alpha\lambda + \lambda^2) \end{pmatrix} \tag{16}$$

and λ is a spectral parameter.

Consider the one soliton solution of mKdV equation (15) as

$$u = k_1 \operatorname{sech} \xi_1 \tag{17}$$

where $\alpha = k_1^2/4$, $\xi_1 = k_1(k_1^2 t + 4x)/8 + k_0$, and k_0 and k_1 are arbitrary constants. The following Proposition gives the mKdV surfaces arising from deformation of parameter k_0.

Proposition 6. *Let u (which describes a travelling mKdV wave), given by equation (17), satisfy the equation (15). The corresponding $\mathfrak{su}(2)$ valued Lax pairs U and V of the mKdV equation are provided by equations (16). The $\mathfrak{su}(2)$ valued matrices A and B are obtained as*

$$A = -\frac{\mathrm{i}\mu}{2}\begin{pmatrix} 0 & \phi_0 \\ \phi_0 & 0 \end{pmatrix}$$

$$B = -\frac{\mathrm{i}\mu}{2}\begin{pmatrix} u\,\phi_0 & (k_1^2/4 + \lambda)\phi_0 - \mathrm{i}\,(\phi_0)_x \\ (k_1^2/4 + \lambda)\phi_0 + \mathrm{i}\,(\phi_0)_x & -u\,\phi_0 \end{pmatrix}$$

where $A = \mu\left(\partial U/\partial k_0\right)$, $B = \mu\left(\partial V/\partial k_0\right)$, $\phi_0 = \partial u/\partial k_0$, k_0 is a parameter of the one soliton solution u, and μ is a constant. Then the surface S, generated by U, V, A and B, has the following first and second fundamental forms $(j, k = 1, 2)$

$$\mathrm{d}s_I^2 \equiv g_{jk}\,\mathrm{d}x^j\,\mathrm{d}x^k, \qquad \mathrm{d}s_{II}^2 \equiv h_{jk}\,\mathrm{d}x^j\,\mathrm{d}x^k$$

where

$$g_{11} = \frac{1}{4}\mu^2\phi_0^2, \qquad g_{12} = g_{21} = \frac{1}{16}\mu^2\phi_0^2(k_1^2 + 4\lambda)$$

$$g_{22} = \frac{1}{64}\mu^2\left(16\,(\phi_0)_x^2 + \phi_0^2\left(16\,u^2 + (k_1^2 + 4\lambda)^2\right)\right)$$

$$h_{11} = -16\,\Delta_1\,\lambda\,u\,\phi_0^2$$

$$h_{12} = 4\,\Delta_1\phi_0\left(4\,(\phi_0)_x\,u_x + u\,\phi_0\left(2\,u^2 - k_1^2(\lambda+1) - 4\,\lambda^2\right)\right)$$

$$h_{22} = -\Delta_1\left(u\,\phi_0^2\,(k_1^2 + 4\lambda)\left(2u^2 + 4\lambda^2 + k_1^2(\lambda+1)\right)\right)$$

$$+\,4\phi_0\left(4u(\phi_0)_{xt} - (\phi_0)_x\left((k_1^2 + 4\lambda)u_x + 4u_t\right)\right)$$

$$+\,4u(\phi_0)_x\left((\phi_0)_x(k_1^2 + 4\lambda) - 4(\phi_0)_t\right)\right)$$

$$\Delta_1 = \frac{\mu}{32\left((\phi_0)_x^2 + u^2\phi_0^2\right)^{1/2}}$$

and the corresponding Gaussian and mean curvatures have the following form

$$K = \frac{16\lambda^2}{k_1^2\mu^2}, \qquad H = -\frac{4\lambda}{k_1\mu}$$

where $x^1 = x$, $x^2 = t$.

The mKdV surfaces obtained from k_0 deformation given in Proposition 6 are spheres.

Another parameter of the one soliton solution of mKdV equation is k_1. We use k_1 parameter deformation to construct new mKdV surfaces in the following proposition.

Proposition 7. *Let u, given by equation (17), satisfy the equation (15). The corresponding $\mathfrak{su}(2)$ valued Lax pairs U and V of the mKdV equation are given by equations (16). The $\mathfrak{su}(2)$ valued matrices A and B are obtained as*

$$A = -\frac{\mathrm{i}\mu}{2}\begin{pmatrix} 0 & \phi_1 \\ \phi_1 & 0 \end{pmatrix}$$

$$\tag{18}$$

$$B = -\frac{\mathrm{i}\mu}{8}\begin{pmatrix} 4\,u\,\phi_1 - 2k_1(\lambda+1) & \tau - 4\mathrm{i}(\phi_1)_x \\ \tau + 4\mathrm{i}(\phi_1)_x & -4\,u\,\phi_1 + 2k_1(\lambda+1) \end{pmatrix}$$

where $A = \mu \left(\partial U / \partial k_1 \right)$, $B = \mu \left(\partial V / \partial k_1 \right)$, $\tau = 2 k_1 u + (k_1^2 + 4\lambda) \phi_1$ and $\phi_1 = \partial u / \partial k_1$, k_1 is a parameter of the one soliton solution u, and μ is a constant. Then the surface S, generated by U, V, A and B, has the following first and second fundamental forms $(j, k = 1, 2)$

$$\mathrm{d}s_I^2 \equiv g_{jk}\, \mathrm{d}x^j\, \mathrm{d}x^k, \qquad \mathrm{d}s_{II}^2 \equiv h_{jk}\, \mathrm{d}x^j\, \mathrm{d}x^k$$

where

$$g_{11} = \frac{1}{4}\mu^2 \phi_1^2, \qquad g_{12} = g_{21} = \frac{1}{16}\mu^2 \phi_1 \Big(2\,k_1\,u + \phi_1(k_1^2 + 4\,\lambda) \Big)$$

$$g_{22} = \frac{1}{64}\mu^2 \Big(4(k_1^2 + 4\,\phi_1^2)u^2 + 4\,k_1\,(k_1^2 - 4)u\,\phi_1 + 16(\phi_1)_x^2$$

$$+ (k_1^2 + 4\,\lambda)^2 \phi_1^2 + 4\,k_1^2(\lambda + 1)^2 \Big)$$

$$h_{11} = \frac{1}{16}\Delta_2\,\mu^3\,\lambda\,\phi_1^2 \Big(k_1(\lambda + 1) - 2\,u\,\phi_1 \Big)$$

$$h_{12} = h_{21} = \frac{1}{64}\Delta_2\,\mu^3 \phi_1^2 \Big(8\,(\phi_1)_x u_x$$

$$+ \big(k_1(\lambda + 1) - 2u\phi_1 \big) \big(2(2\lambda^2 - u^2) + k_1^2(\lambda + 1) \big) \Big)$$

$$h_{22} = \frac{1}{256}\Delta_2\,\mu^3\,\phi_1 \Big(8\,(\phi_1)_x \big\{ 2\,k_1\,u\,u_x + (k_1^2 + 4\,\lambda)\big(\phi_1\,u_x - u(\phi_1)_x \big)$$

$$+ 4(u\phi_1)_t \big\} + \big(k_1(\lambda + 1) - 2\,u\,\phi_1 \big)\big\{ 16\,(\phi_1)_{xt} - 4\,k_1\,u(u^2 + 2\,\lambda)$$

$$+ \phi_1(k_1^2 + 4\,\lambda)\big(2(u^2 + 2\,\lambda^2) + k_1^2(\lambda + 1) \big) \big\} \Big).$$

The Gaussian and mean curvatures are

$$K = \frac{1}{65536}\Delta_2^6\,\phi_1^{\,5}\,\mu^{10} \Big(4(\phi_1)_x^2 + \Delta_3^2 \Big)\Big(4\Delta_3^2 \lambda(\phi_1)_{xt} - 16\phi_1(\phi_1)_x^2 u_x^2$$

$$+ 2\Delta_3 u_x(\phi_1)_x \Big(4\phi_1 u^2 + 2\lambda k_1 u - \phi_1\big((\lambda + 2)k_1^2 + 4\lambda^2 \big) \Big)$$

$$+ \Delta_3 \Big\{ 8\lambda\phi_1(\phi_1)_x u_t - 2\lambda u(\phi_1)_x \Big((\phi_1)_x(k_1^2 + 4\lambda) - 4(\phi_1)_t \Big)$$

$$- (\Delta_3/4)\Big(2u^2\phi_1\Big(2u^2 - (3\lambda + 2)k_1^2 - 12\lambda^2 \Big)$$

$$+ 4\lambda k_1 u(u^2 + 2\lambda) + k_1^2\phi_1\big((1 + \lambda)k_1^2 + 4\lambda^2 \big) \Big) \Big\} \Big)$$

$$H = \frac{1}{2048}\, \Delta_2^3\, \mu^5\, \phi_1^2 \Big(8\, \phi_1\, (\phi_1)_x \Big(4(u\phi_1)_t - 2\, k_1\, u\, u_x - (k_1^2 + 4\,\lambda)(\phi_1 u)_x \Big)$$

$$+ \Delta_3 \Big\{ 16\, \phi_1\, \phi_{xt} + 2\, k_1\, u\, \phi_1 \Big(2\, u^2 - 8\,\lambda^2 - 2k_1^2 \Big)$$

$$+ 16\,\lambda\, (\phi_1)_x^2 + 2\, u^2 \Big(2\, k_1^2\, \lambda + \phi_1^2(3\, k_1^2 + 20\,\lambda) \Big) + 4\,\lambda\, k_1^2(\lambda + 1)^2$$

$$- k_1^2\, \phi_1^2(k_1^2 + 4\,\lambda) \Big\} - 24\, k_1 \lambda\, \Delta_3\, u\, \phi_1 \Big)$$

where $x^1 = x$, $x^2 = t$. Here Δ_2 and Δ_3 are in the following form

$$\Delta_2 = \frac{8}{\mu^2 \phi_1 \Big(\big(2\, u\, \phi_1 - k_1(\lambda + 1) \big)^2 + 4\, \phi_x^2 \Big)^{1/2}}, \qquad \Delta_3 = k_1(\lambda + 1) - 2u\phi_1.$$

When u, given by equation (17), is substituted into K and H, they take the following forms

$$K = \frac{1}{\mu^2\, \eta_0 \left(4\, \eta_4^2 + \eta_3^2 \right)^2} \sum_{l=1}^{7} Q_l\, (\mathrm{sech}\, \xi_1)^l$$

$$H = \frac{1}{4\, \mu\, \eta_0 \left(4\, \eta_4^2 + \eta_3^2 \right)^{3/2}} \sum_{m=0}^{7} Z_m\, (\mathrm{sech}\, \xi_1)^m$$

where

$$\eta_0 = \mathrm{sech}\, \xi_1 \left(1 - k_1\, \eta_1 \tanh \xi_1 \right), \qquad \eta_1 = (3\, k_1^2\, t + 4\, x)/8$$

$$\eta_2 = 3\, k_1^3\, \pi\, t(k_1^2\, t + 8\, x)/16, \qquad \eta_3 = k_1(\pi - k_1)/\pi - 2\, k_1 \eta_0$$

$$\eta_4 = -(k_1/2)(k_1 \eta_1 \mathrm{sech}\, \xi_1 + 2\eta_0 \tanh \xi_1)$$

$$Q_1 = -k_1^6\, (\pi - k_1)^2\, (\pi(\pi + 4\, k_1) + 12)\, /\pi^4$$

$$Q_2 = k_1^7\, \left(4 + \pi^2 \right) (\pi - k_1)^2\, \eta_1\, \sinh \xi_1 /\pi^4$$

$$Q_3 = 4\, k_1^6(\pi - k_1)\, \left(k_1\, \pi\left(\pi - 2\, k_1 \right) + 6\, (\pi - k_1) \right) /\pi^4$$

$$Q_4 = -4Q_2 \left(k_1\, \pi^2 - 6\, k_1 + 8\, \pi \right) / \left((\pi - k_1)(4 + \pi^2) \right)$$

$$Q_5 = 4\, k_1^7 \left(12 + 14k_1^2\eta_1^2 + \pi k_1(2\eta_1^2 + 3) \right) /\pi^3$$

$$Q_6 = -4\, k_1^8 \eta_1 \sinh \xi_1 \left(24 + \pi k_1(4\eta_1^2 + 3) \right) /\pi^3, \qquad Q_7 = -48\, k_1^9 \eta_1^2 /\pi^3$$

$$Z_0 = 4\, k_1^4(k_1 - \pi)^3 /\pi^4, \qquad Z_1 = 0$$

$$Z_2 = 4\, k_1^4(k_1 - \pi)(6\, k_1 + \pi(k_1^2 - 5))/\pi^3$$

$$Z_3 = -24\, k_1^5 \eta_1 \sinh \xi_1 (k_1 - \pi)^2 /\pi^3$$

$$Z_4 = 4\, k_1^5 \left(k_1\pi(2 + 3\eta_1^2) + 15k_1^2\eta_1^2 + 14 \right) /\pi^2$$

$$Z_5 = -8\, k_1^6 \eta_1 \sinh \xi_1 \left(k_1\pi(1 + 2\eta_1^2) + 14 \right) /\pi^2, \qquad Z_6 = -56\, k_1^7\, \eta_1^2 /\pi^2.$$

3.2. The Parameterized Form of the mKdV Surfaces

In the previous section, we constructed mKdV surfaces. We found the first and second fundamental forms, Gaussian and mean curvatures of the surfaces. But we did not find the position vectors of these surfaces. In this section, we explore the position vector

$$\mathbf{y} = (y_1(x,t), y_2(x,t), y_3(x,t))$$

of the mKdV surfaces for a given solution of the mKdV equation and the corresponding Lax pairs. In order to find immersion function F explicitly, we need to find the solution (Φ) of the Lax equations given by equation (2). Consider the one soliton solution of mKdV equation $u = k_1 \operatorname{sech} \xi_1$, where $\alpha = k_1^2/4$, $\xi_1 = k_1(k_1^2 t + 4x)/8 + k_0$, and k_0 and k_1 are arbitrary constants. We solve the Lax equations [equation (2)] for the given U and V by equations (16), respectively and a solution of the mKdV equation. The components of the 2×2 matrix Φ are

$$\Phi_{11} = -\frac{\Delta_4}{k_1}\Big(A_1(2\,\lambda\mathrm{i} - k_1 \tanh\xi_1) \cdot \operatorname{Exp}\big(\mathrm{i}(k_1^2 + 4\lambda^2)t/8\big) \cdot \Xi_1$$
$$-\mathrm{i}\,k_1^2 B_1 \operatorname{sech}\xi_1 \cdot \operatorname{Exp}\big(-\mathrm{i}(k_1^2 + 4\lambda^2)t/8\big) \cdot \Xi_2\Big)$$

$$\Phi_{12} = -\frac{\Delta_4}{k_1}\Big(A_2(2\,\lambda\mathrm{i} - k_1 \tanh\xi_1) \cdot \operatorname{Exp}\big(\mathrm{i}(k_1^2 + 4\lambda^2)t/8\big) \cdot \Xi_1$$
$$-\mathrm{i}\,k_1^2 B_2 \operatorname{sech}\xi_1 \cdot \operatorname{Exp}\big(-\mathrm{i}(k_1^2 + 4\lambda^2)t/8\big) \cdot \Xi_2\Big)$$

$$\Phi_{21} = \Delta_4\Big(\mathrm{i}\,A_1 \operatorname{sech}\xi_1 \cdot \operatorname{Exp}\big(\mathrm{i}(k_1^2 + 4\lambda^2)t/8\big) \cdot \Xi_1$$
$$+B_1(2\,\lambda\mathrm{i} + k_1 \tanh\xi_1) \cdot \operatorname{Exp}\big(-\mathrm{i}(k_1^2 + 4\lambda^2)t/8\big) \cdot \Xi_2\Big)$$

$$\Phi_{22} = \Delta_4\Big(\mathrm{i}\,A_2 \operatorname{sech}\xi_1 \cdot \operatorname{Exp}\big(\mathrm{i}(k_1^2 + 4\lambda^2)t/8\big) \cdot \Xi_1$$
$$+B_2(2\,\lambda\mathrm{i} + k_1 \tanh\xi_1) \cdot \operatorname{Exp}\big(-\mathrm{i}(k_1^2 + 4\lambda^2)t/8\big) \cdot \Xi_2\Big)$$

$$(19)$$

where

$$\Xi_1 = (\tanh\xi_1 + 1)^{\mathrm{i}\lambda/2k_1}(\tanh\xi_1 - 1)^{-\mathrm{i}\lambda/2k_1}$$
$$\Xi_2 = (\tanh\xi_1 - 1)^{\mathrm{i}\lambda/2k_1}(\tanh\xi_1 + 1)^{-\mathrm{i}\lambda/2k_1}$$
$$\Delta_4 = \sqrt{k_1/(k_1^2 + 4\lambda^2)}.$$

Here we find the determinant of the matrix Φ as

$$\det(\Phi) = (A_1 B_2 - A_2 B_1) \neq 0.$$

3.2.1. Immersion function of the mKdV surfaces obtained using k_0 deformation

We find the immersion function F of the mKdV surface obtained using k_0 deformation by using the following equation

$$F = \nu\, \Phi^{-1} \frac{\partial \Phi}{\partial k_0} + \begin{pmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{pmatrix}$$

from which we obtain the position vector, where the components of Φ are given by equations (19), respectively. Here we choose $A_1 = -k_1 B_2 \operatorname{Exp}(-\lambda\pi/k_1)$, $A_2 = k_1 B_1 \operatorname{Exp}(-\lambda\pi/k_1)$, $r_{11} = r_{22} = 0$, $r_{12} = -r_{21}$ to write F in the form $F = -\mathrm{i}(\sigma_1 y_1 + \sigma_2 y_2 + \sigma_3 y_3)$. Hence we obtain a family of mKdV surfaces parameterized by

$$\begin{aligned}
y_1 &= W_6 \cdot \operatorname{sech}^2(\xi_1)\Big(W_3 \cdot \cosh(\xi_1)\cos(\Omega_1) \\
&\quad + W_4 \cdot \sinh(\xi_1)\sin(\Omega_1) + 4\lambda\, W_8\big(2W_1 \cosh(2\xi_1) + W_7\big)\Big) \\
y_2 &= \frac{1}{W_5}\operatorname{sech}^2(\xi_1)\Big(W_{10} \cdot \sinh(\xi_1)\cos(\Omega_1) \\
&\quad - W_{11} \cdot \cosh(\xi_1)\sin(\Omega_1) + W_9 \cdot \cosh^2(\xi_1)\Big) \\
y_3 &= W_6 \cdot \operatorname{sech}^2(\xi_1)\Big(W_{13} \cdot \cosh(\xi_1)\cos(\Omega_1) \\
&\quad - W_{12} \cdot \sinh(\xi_1)\sin(\Omega_1) + 2\lambda\, W_2\big(2W_1 \cosh(2\xi_1) + W_7\big)\Big)
\end{aligned} \tag{20}$$

where

$$\begin{aligned}
&\Omega_1 = \big(k_1^2(\lambda+1)/4 + \lambda^2\big)t + \lambda\, x + 2\lambda\, k_0/k_1 \\
&\xi_1 = k_1(k_1^2 t + 4x)/8 + k_0 \\
&W_1 = (k_1^2 + 4\lambda^2)/8, \qquad W_2 = B_1^2 - B_2^2, \qquad W_3 = W_2\, k_1^3 \\
&W_4 = 2\lambda\, W_2\, k_1^2, \qquad\qquad W_5 = k_1^2 + 4\,\lambda^2, \qquad W_6 = \nu/\big(W_5\, k_1\,(B_1^2 + B_2^2)\big) \\
&W_7 = 5k_1^2/4 + \lambda^2, \qquad\quad W_8 = B_1 B_2, \qquad\qquad W_9 = W_5\, r_{21}, \quad W_{10} = 2\,\lambda\nu\, k_1 \\
&W_{11} = \nu k_1^2, \qquad\qquad\qquad W_{12} = 4\,\lambda\, W_8 k_1^2, \qquad W_{13} = -2W_8\, k_1^3.
\end{aligned}$$

Thus the position vector $\mathbf{y} = (y_1(x,t), y_2(x,t), y_3(x,t))$ of the surface is given by equations (20).

3.2.2. Immersion function of the mKdV surfaces obtained using k_1 deformation

We find the immersion function F of the mKdV surface obtained using k_1 deformation by using the following equation

$$F = \nu \, \Phi^{-1} \frac{\partial \Phi}{\partial k_1} + \begin{pmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{pmatrix}$$

from which we obtain the position vector, where the components of Φ are given by equations (19), respectively. Here we choose

$$A_1 = -k_1 B_2 \operatorname{Exp}(-\lambda\,\pi/k_1), \qquad A_2 = k_1 B_1 \operatorname{Exp}(-\lambda\pi/k_1)$$

$$r_{11} = -r_{22} = \frac{\nu\,(\pi\lambda + k_1)\,(B_2^2 - B_1^2)}{k_1^2(B_1^2 + B_2^2)}$$

$$r_{12} = -\frac{r_{21}\,k_1^2(B_1^2 + B_2^2) + 2\nu B_1 B_2(\pi\lambda + k_1)}{k_1^2(B_1^2 + B_2^2)}$$

in order to write F in the form $F = -i(\sigma_1 y_1 + \sigma_2 y_2 + \sigma_3 y_3)$. Hence we obtain a family of mKdV surfaces parameterized by

$$
\begin{aligned}
y_1 =\ & W_{14} \cdot \operatorname{sech}^2(\xi_1)\Big(W_{15}\Big(2\Omega_2 \cdot \sinh(\xi_1) - (16/3)\cosh(\xi_1)\Big)\sin(\Omega_1) \\
& + W_{16} \cdot \Omega_2 \cdot \cosh(\xi_1)\cos(\Omega_1) \\
& + W_8\Big(2\Omega_3 \cdot \cosh(2\xi_1) + 2k_1^2\lambda \sinh(2\xi_1) + \Omega_4\Big)\Big) \\
y_2 =\ & W_{14} \cdot \operatorname{sech}^2(\xi_1)\Big(W_{17}\Big(2\Omega_2 \cdot \sinh(\xi_1) - (16/3)\cosh(\xi_1)\Big)\cos(\Omega_1) \\
& - W_{18} \cdot \Omega_2 \cdot \cosh(\xi_1)\sin(\Omega_1) + W_{19}\Big(\cosh(2\xi_1) + 1\Big)\Big) \\
y_3 =\ & W_{14} \cdot \operatorname{sech}^2(\xi_1)\Big(W_{20}\Big(2\Omega_2 \cdot \sinh(\xi_1) - (16/3)\cosh(\xi_1)\Big)\sin(\Omega_1) \\
& - W_{21} \cdot \Omega_2 \cdot \cosh(\xi_1)\cos(\Omega_1) \\
& + (W_2/2)\Big(2\Omega_3 \cdot \cosh(2\xi_1) + 2k_1^2\lambda \cdot \sinh(2\xi_1) + \Omega_4\Big)\Big)
\end{aligned}
\tag{21}
$$

where

$$\Omega_2 = t\,k_1^3 + 4\,x\,k_1/3, \qquad \Omega_3 = \left(4\,\lambda^2 + k_1^2\right)\left(k_1^3(\lambda+1)t - 4\,\lambda\,k_0\right)$$

$$\Omega_4 = \frac{tk_1^3}{4}\left(4\,\lambda^2(\lambda+1) + k_1^2(7\,\lambda+1)\right) + \lambda\left(k_1^2(2\,x\,k_1 - k_0) - 4\,\lambda^2\,k_0\right)$$

$$W_{14} = W_6/k_1, \qquad W_{15} = 3\lambda\,W_2\,k_1^2/8, \qquad W_{16} = 3\,W_2\,k_1^3/8$$

$$W_{17} = \lambda\,k_1^2(B_1^2 + B_2^2), \qquad W_{18} = k_1^3(B_1^2 + B_2^2)$$

$$W_{19} = 4\left(r_{21}k_1^2(B_1^2 + B_2^2)/\nu + W_8(k_1 + \lambda\,\pi)\right)W_5/3$$

$$W_{20} = -3\,\lambda\,W_8\,k_1^2/4, \qquad W_{21} = W_8 k_1^3.$$

Thus the position vector **y** of the surface is given by equations (21).

3.3. Graph of Some of the mKdV Surfaces

3.3.1. Graph of some of the mKdV surfaces from k_0 deformation

Example 8. *Taking $\lambda = 2.7$, $\nu = 1$, $B_1 = 1$, $B_2 = 1$, $k_0 = 0.3$, $k_1 = 1.5$ and $r_{21} = 1$ in the equations provided by equations* (20)*, we get the surface given in Fig. 1.*
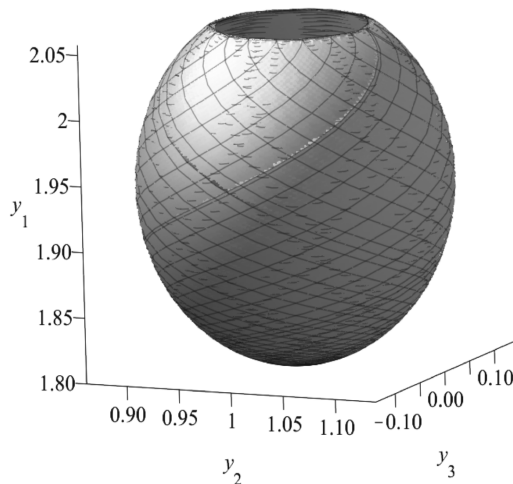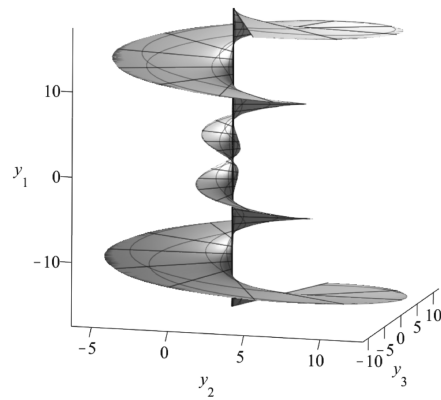

Figure 1. $(x,t) \in [-5,5] \times [-5,5]$

3.3.2. Graph of some of the mKdV surfaces from k_1 deformation

Example 9. *Taking $\lambda = 0.03$, $\nu = 1$, $B_1 = 1$, $B_2 = 1$, $k_0 = 0$, $k_1 = -0.1$ and $r_{21} = 1$ in the equations provided by equations (21), we get the surface given in Fig. 2.*
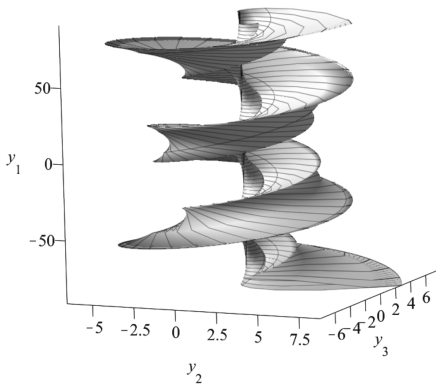
**Figure 2.** $(x, t) \in [-3000, 3000] \times [-3000, 3000]$

Figure 3. $(x, t) \in [-100, 100] \times [-100, 100]$

Example 10. *Taking $\lambda = 0$, $\nu = 1$, $B_1 = 1$, $B_2 = 1$, $k_0 = 0$, $k_1 = 0.7$ and $r_{21} = 1$ in the equations provided by equations (21), we get the surface given in Fig. 3.*

Example 11. *Taking $\lambda = -0.8$, $\nu = 1$, $B_1 = 1$, $B_2 = 1$, $k_0 = 0$, $k_1 = -0.2$ and $r_{21} = 1$ in the equations provided by equations (21), we get the surface given in Fig. 4.*

Example 12. *Taking $\lambda = -0.8$, $\nu = 1$, $B_1 = 1$, $B_2 = 1$, $k_0 = 5$, $k_1 = -0.2$ and $r_{21} = 1$ in the equations provided by equations (21), we get the surface given in Fig. 5.*
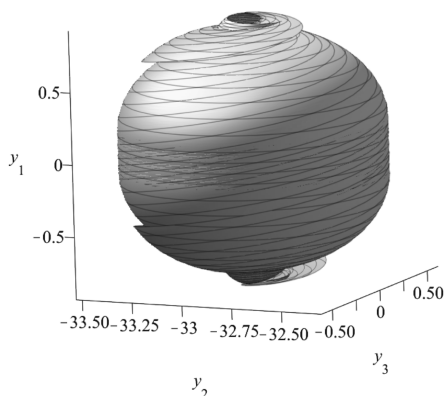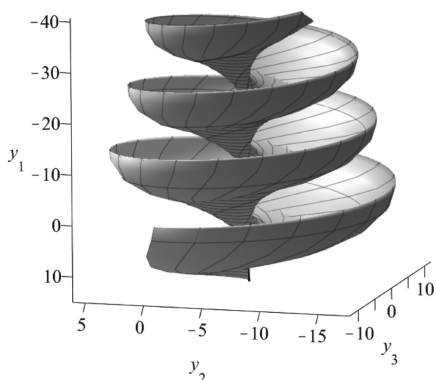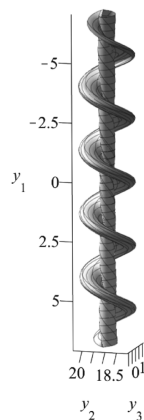
Figure 4. $(x,t) \in$
$[-20, 20] \times [-20, 20]$
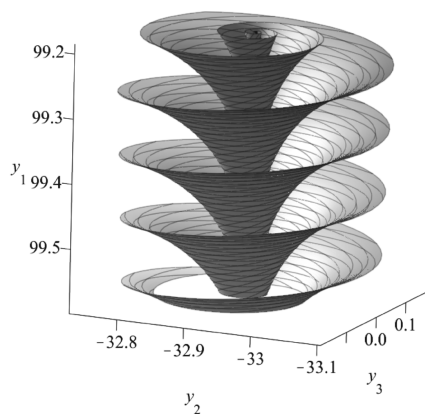
Figure 5. $(x,t) \in$
$[-20, 20] \times [-20, 20]$

Example 13. *Taking $\lambda = -0.1$, $\nu = 1$, $B_1 = 1$, $B_2 = 1$, $k_0 = -4$, $k_1 = -0.2$ and $r_{21} = 1$ in the equations provided by equations (21), we get the surface given in* Fig. 6.

Figure 6. $(x,t) \in$
$[-500, 500] \times [-500, 500]$

Figure 7. $(x,t) \in$
$[-80, 80] \times [-80, 80]$

Example 14. *Taking $\lambda = 0.4$, $\nu = 1$, $B_1 = 1$, $B_2 = 1$, $k_0 = 0$, $k_1 = 0.2$ and $r_{21} = 1$ in the equations provided by equations (21), we get the surface given in* Fig. 7.

4. Sine-Gordon Surfaces

Let $u(x, t)$ be the solution of the sine-Gordon equation

$$u_{xt} = \sin u. \tag{22}$$

The matrix Lax pairs U and V of the SG equation given by equation (22) are

$$U = \frac{i}{2} \begin{pmatrix} \lambda & -u_x \\ -u_x & -\lambda \end{pmatrix}, \qquad V = \frac{1}{2\lambda} \begin{pmatrix} -i \cos u & \sin u \\ -\sin u & i \cos u \end{pmatrix} \tag{23}$$

were λ is a spectral constant. One soliton solution of the SG equation given in equation (22) has the following form

$$u = 4 \arctan\left(e^{\xi_2}\right). \tag{24}$$

Here $\xi_2 = (x/k_3 + k_3 t + k_2)$, where k_2 and k_3 are arbitrary constants. We develop SG surfaces using deformation of parameter k_2 in the following Proposition.

Proposition 15. *Let u, provided by equation (24), satisfy the* SG *equation given in equation (22). The corresponding* $\mathfrak{su}(2)$ *valued Lax pairs U and V of the* SG *equation are provided by equations (23). The* $\mathfrak{su}(2)$ *valued matrices A and B are obtained as*

$$A = \frac{i\mu}{2} \begin{pmatrix} 0 & -(\phi_2)_x \\ -(\phi_2)_x & 0 \end{pmatrix}, \qquad B = \frac{\mu}{2\lambda} \begin{pmatrix} i \sin(u)\,\phi_2 & \cos(u)\,\phi_2 \\ -\cos(u)\,\phi_2 & -i \sin(u)\,\phi_2 \end{pmatrix}$$

where $A = \mu\,(\partial U/\partial k_2)$, $B = \mu\,(\partial V/\partial k_2)$, $\phi_2 = \partial u/\partial k_2$, k_2 is a parameter of the one soliton solution u, and μ is a constant. Then the surface S, generated by U, V, A and B, has the following first and second fundamental forms $(j, k = 1, 2)$

$$ds_I^2 \equiv g_{jk}dx^j\,dx^k = \mu^2 \operatorname{sech}^2\xi_2 \left(\frac{1}{k_3^2} \tanh^2 \xi_2\,dx^2 + \frac{1}{\lambda^2}\,dt^2\right)$$

$$ds_{II}^2 \equiv h_{jk}dx^j\,dx^k = \frac{2\mu}{\lambda k_3} \operatorname{sech}^2\xi_2 \left(\lambda^2 \tanh^2 \xi_2\,dx^2 + k_3^2\,dt^2\right)$$

and the corresponding Gaussian and mean curvatures are

$$K = \left(\frac{2\lambda k_3}{\mu}\right)^2, \qquad H = \frac{2\lambda k_3}{\mu}$$

where $x^1 = x$, $x^2 = t$.

These surfaces given in Proposition 15 are also sphere in $\mathbb{R}^3$.

Similar to k_2 deformation, the following proposition gives new SG surfaces by using deformation of the other parameter k_3 of the one soliton solution given in equation (24).

Proposition 16. *Let u, provided by equation (24), satisfy the SG equation given in equation (22). The corresponding $\mathfrak{su}(2)$ valued Lax pairs U and V of the SG equation are given by equation (23) respectively. The $\mathfrak{su}(2)$ valued matrices A and B are obtained as*

$$A = \frac{\mathrm{i}\,\mu}{2} \begin{pmatrix} 0 & -(\phi_3)_x \\ -(\phi_3)_x & 0 \end{pmatrix}, \qquad B = \frac{\mu}{2\,\lambda} \begin{pmatrix} \mathrm{i}\sin(u)\,\phi_3 & \cos(u)\,\phi_3 \\ -\cos(u)\,\phi_3 & -\mathrm{i}\sin(u)\,\phi_3 \end{pmatrix}$$

where $A = \mu\,(\partial U/\partial k_3)$, $B = \mu\,(\partial V/\partial k_3)$, $\phi_3 = \partial\,u/\partial k_3$, k_3 is a parameter of the one soliton solution u, and μ is a constant. Then the surface S, generated by U, V, A and B, has the following first and second fundamental forms $(j, k = 1, 2)$

$$\mathrm{d}s_I^2 \equiv g_{jk}\mathrm{d}x^j\,\mathrm{d}x^k, \qquad \mathrm{d}s_{II}^2 \equiv h_{jk}\mathrm{d}x^j\,\mathrm{d}x^k \tag{25}$$

and the corresponding Gaussian and mean curvatures are

$$K = -4m_0^2 \frac{\sinh\xi_2}{\xi_3\,(k_3\,\cosh\xi_2 - \xi_3\,\sinh\xi_2)} \tag{26}$$

$$H = -m_0 \frac{(k_3\,\cosh\xi_2 - 2\,\xi_3\,\sinh\xi_2)}{\xi_3\,(k_3\,\cosh\xi_2 - \xi_3\,\sinh\xi_2)} \tag{27}$$

where

$$\xi_2 = (x/k_3 + k_3\,t + k_2), \qquad \xi_3 = x - t\,k_3^2, \qquad m_0 = \frac{\lambda\,k_3^3}{\mu}$$

$$g_{11} = \frac{\mu^2}{k_3^6}\mathrm{sech}^4\xi_2\Big(k_3\cosh\xi_2 - \xi_3\sinh\xi_2\Big)^2, \qquad g_{22} = \left(\frac{\mu\,\xi_3}{\lambda\,k_3^2}\right)^2 \mathrm{sech}^2\xi_2$$

$$h_{11} = \frac{2\mu\,\lambda}{k_3^3}\,\sinh\xi_2\,\mathrm{sech}^4\xi_2\Big(k_3\,\cosh\xi_2 - \xi_3\sinh\xi_2\Big)$$

$$h_{22} = -\frac{2\mu\,\xi_3}{\lambda\,k_3}\mathrm{sech}^2\xi_2.$$

4.1. SG surfaces are not the Critical Points of Functionals

Let us consider a polynomial Lagrange as functions of the curvatures H and K

$$\mathcal{E} = a_{N0}\,H^N + \ldots + a_{11}\,H\,K + a_{21}\,H^2\,K + \ldots + a_{01}\,K + \ldots \tag{28}$$

where a_{ij} with $i, j = 0, 1, 2, \cdots N$ are all constants. Here H and K are mean and Gauss curvatures of the SG surfaces given in equations (26) and (27) with the metrics equation (25). When we use equation (28) in the Euler-Lagrange equation [equation (9)] the direct calculation yields all the constants $a_{nl} = 0$ and $p = 0$, where $n, l = 0, 1, 2, \ldots$ and $N = 3, 4, 5, \ldots$. This leads to an important theorem on SG surfaces.

Theorem 17. *The* SG *surfaces obtained in* Proposition 16 *are not the critical points of functional $\mathcal{E}$ in the generalized shape equation given by equation* (9) *when $\mathcal{E}$ is a polynomial function of H and K.*

4.2. The Parameterized Form of the SG Surfaces

In this section, we find the position vector $\mathbf{y}$ of the SG surfaces obtained using k_2 and k_3 deformation. For this purpose, first we need to solve the Lax equations given by equation (2) using the given Lax pairs. Using one soliton solution given by equation (24) and the Lax pairs U and V provided by equations (23) we solve the Lax equations. The components of the solution 2×2 matrix Φ are obtained as

$$
\begin{aligned}
\Phi_{11} &= -\frac{1}{2}\Xi_4\Big(A_1 \cdot \Xi_3 \cdot \mathrm{Exp}(-\xi_4) + 2B_1\ \mathrm{Exp}(\xi_4)\Big) \\
\Phi_{12} &= -\frac{1}{2}\Xi_4\Big(A_2 \cdot \Xi_3 \cdot \mathrm{Exp}(-\xi_4) + 2B_2 \cdot \mathrm{Exp}(\xi_4)\Big) \\
\Phi_{21} &= \Xi_4\Big(\frac{1}{2}A_1 \cdot \mathrm{Exp}(-\xi_4) - B_1 \cdot \overline{\Xi}_3 \cdot \mathrm{Exp}(\xi_4)\Big) \\
\Phi_{22} &= \Xi_4\Big(\frac{1}{2}A_2 \cdot \mathrm{Exp}(-\xi_4) - B_2 \cdot \overline{\Xi}_3 \cdot \mathrm{Exp}(\xi_4)\Big)
\end{aligned}
\tag{29}
$$

where $\overline{\Xi}_3$ denotes the complex conjugate of Ξ_3 and

$$
\begin{aligned}
\Xi_3 &= \lambda\, k_3 \cosh(\xi_2) + \mathrm{i}\sinh(\xi_2), \qquad \Xi_4 = \mathrm{sech}(\xi_2) \\
\xi_4 &= \frac{\mathrm{i}}{2\lambda}\left(k_3\,(tk_3 - \xi_2)\,\lambda^2 + t\right).
\end{aligned}
$$

Here we find the determinant of the matrix Φ as

$$
\det(\Phi) = \frac{1}{2}\left(\lambda^2\,k_3^2 + 1\right)(A_1 B_2 - A_2 B_1) \neq 0.
$$

4.2.1. Immersion function of the SG surface obtained using k_2 deformation

We find the immersion function F of the SG surface obtained using k_2 deformation by using the following equation

$$
F = \nu\,\Phi^{-1}\frac{\partial\Phi}{\partial k_2} + \begin{pmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{pmatrix}
$$

from which we obtain the position vector, where the components of Φ are given by equations (29), respectively. Here we choose $A_1 = -2\,B_2$, $A_2 = 2\,B_1$, $r_{11} = r_{22} = 0$, $r_{12} = -r_{21}$ to write F in the form $F = -\mathrm{i}(\sigma_1 y_1 + \sigma_2 y_2 + \sigma_3 y_3)$. Hence

we obtain components of the position vector of the SG surfaces as

$$
\begin{aligned}
y_1 = {}& D_5 \cdot \operatorname{sech}^2(\xi_2)\Big(D_1 \cdot \cosh(\xi_2)\cos(\zeta_1) \\
& - D_2 \cdot \sinh(\xi_2)\sin(\zeta_1) + D_3\big(1 + D_4 \cdot \cosh^2(\xi_2)\big)\Big) \\
y_2 = {}& \frac{1}{D_6} \cdot \operatorname{sech}^2(\xi_2)\Big(D_7 \cdot \sinh(\xi_2)\cos(\zeta_1) \\
& - \nu \cdot \cosh(\xi_2)\sin(\zeta_1) + D_8 \cdot \cosh^2(\xi_2)\Big) \\
y_3 = {}& D_5 \cdot \operatorname{sech}^2(\xi_2)\Big(D_9 \cdot \cosh(\xi_2)\cos(\zeta_1) \\
& - D_3 \cdot \sinh(\xi_2)\sin(\zeta_1) + D_2\big(1 + D_4 \cdot \cosh^2(\xi_2)\big)\Big)
\end{aligned}
\tag{30}
$$

where

$$
\begin{aligned}
&\xi_2 = (x/k_3 + k_3\,t + k_2)\,, \qquad \zeta_2 = t/\lambda - (x + k_2 k_3)\lambda \\
&D_1 = -(B_1^2 - B_2^2), \qquad D_2 = -\lambda\,k_3\,D_1, \qquad D_3 = \lambda\,k_3\,D_9 \\
&D_4 = D_6/2, \qquad D_5 = \nu/((B_1^2 + B_2^2)\cdot D_6), \qquad D_6 = \lambda^2\,k_3^2 + 1 \\
&D_7 = -\nu\,\lambda\,k_3, \qquad D_8 = D_6 \cdot r_{21}, \qquad D_9 = 2\,B_1\,B_2.
\end{aligned}
$$

Thus the position vector $\mathbf{y}$ of the SG surface is given by equations (30).

4.2.2. Immersion function of the SG surface obtained using k_3 deformation

We find the immersion function F of the SG surface obtained using k_3 deformation by using the following equation

$$
F = \nu\,\Phi^{-1}\frac{\partial \Phi}{\partial k_3} + \begin{pmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{pmatrix}
$$

from which we obtain the position vector, where the components of Φ are given by equations (29), respectively. Here we choose $A_1 = -2\,B_2$, $A_2 = 2\,B_1$, $r_{11} = r_{22} = -\nu\,k_3\,\lambda^2/\left(\lambda^2\,k_3^2 + 1\right)$, $r_{12} = -r_{21}$ to write F in the form $F = -\mathrm{i}(\sigma_1 y_1 + \sigma_2 y_2 + \sigma_3 y_3)$. Hence we obtain components of the position vector of the SG

surfaces

$$
\begin{aligned}
y_1 = {}& -D_{10} \cdot \operatorname{sech}^2(\xi_2)\Big(D_2\big(\xi_3 \sinh(\xi_2) - k_3 \cosh(\xi_2)\big) \sin(\zeta_1) \\
& + D_1 \cdot \xi_3 \cdot \cosh(\xi_2) \cos(\zeta_1) \\
& - D_3\big(D_{11} \cdot \cosh^2(\xi_2) - \xi_3 + (k_3/2) \sinh(2\xi_2)\big)\Big) \\
y_2 = {}& D_{12} \cdot \operatorname{sech}^2(\xi_2)\Big(- D_7\big(\xi_3 \sinh(\xi_2) - k_3 \cosh(\xi_2)\big) \cos(\zeta_1) \\
& + \nu \cdot \xi_3 \cdot \cosh(\xi_2) \sin(\zeta_1) + D_{13} \, \cosh^2(\xi_2)\Big) \\
y_3 = {}& D_{10} \cdot \operatorname{sech}^2(\xi_2)\Big(D_3\big(\xi_3 \sinh(\xi_2) - k_3 \cosh(\xi_2)\big) \sin(\zeta_1) \\
& - D_9 \cdot \xi_3 \cdot \cosh(\xi_2) \cos(\zeta_1) \\
& + D_2\big(D_{11} \cdot \cosh^2(\xi_2) - \xi_3 + (k_3/2) \sinh(2\xi_2)\big)\Big)
\end{aligned} \tag{31}
$$

where

$$
\begin{aligned}
\xi_2 &= (x/k_3 + k_3\, t + k_2)\,, & \xi_3 &= x - t\, k_3^2 \\
\zeta_1 &= t/\lambda - (x + k_2 k_3)\lambda, & D_{10} &= D_5/k_3^2 \\
D_{11} &= k_2\, k_3\, D_6/2, & D_{12} &= 1/[D_6 \cdot k_3^2], & D_{13} &= D_8 \cdot k_3^2.
\end{aligned}
$$

Thus the position vector $\mathbf{y}$ of the SG surface is given by equations (31).

### 4.3. Graph of Some of the SG Surfaces

4.3.1. Graph of some of the SG surfaces from k_2 deformation

Example 18. *Taking $\lambda = 2$, $\nu = 1$, $B_1 = 2$, $B_2 = 2$, $k_2 = 0$, $k_3 = 3$ and $r_{21} = 1$ in the equations provided by equations* (30), *we get the surface given in* Fig. 8.
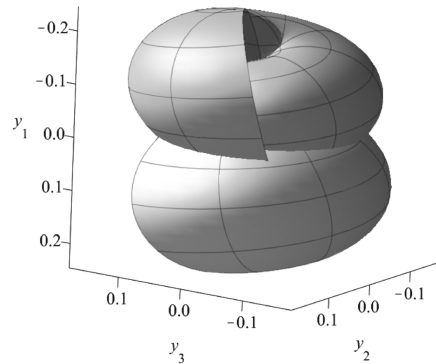
4.3.2. Graph of some of the SG surfaces from k_3 deformation

Example 19. *Taking $\lambda = 4$, $\nu = 1$, $B_1 = 1$, $B_2 = 1$, $k_2 = 0$, $k_3 = 2$ and $r_{21} = 0$ in the equations provided by equations* (31), *we get the surface given in* Fig. 9.

Example 20. *Taking $\lambda = 1.4$, $\nu = 1$, $B_1 = 1$, $B_2 = 1$, $k_2 = 0$, $k_3 = 1$ and $r_{21} = 1$ in the equations provided by equations* (31), *we get the surface given in* Fig. 10.
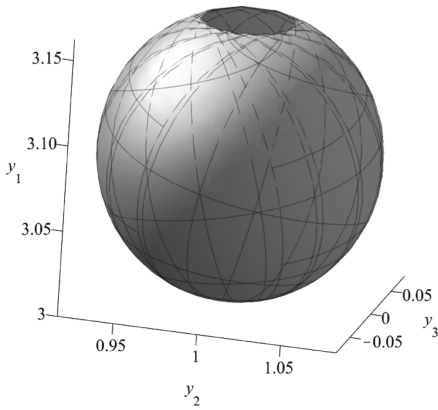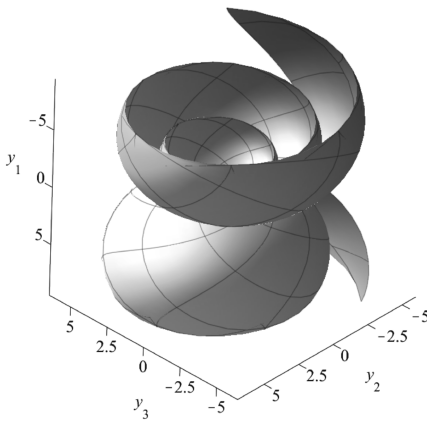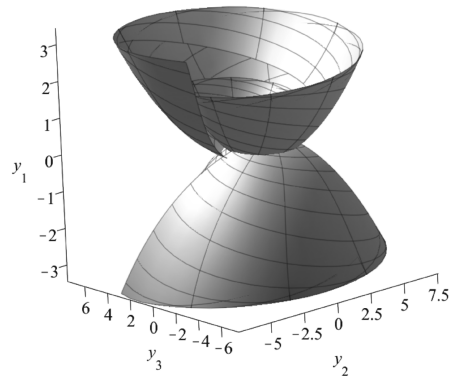
Figure 8. $(x,t) \in$ [−8, 8] × [−8, 8]

Figure 9. $(x,t) \in$ [−4, 4] × [−4, 4]

Example 21. *Taking* $\lambda = 0.4$, $\nu = 1$, $B_1 = 1$, $B_2 = 1$, $k_2 = 0$, $k_3 = 1$ *and* $r_{21} = 1$ *in the equations provided by equations* (31), *we get the surface given in* Fig. 11.

Figure 10. $(x,t) \in$ [−10, 10] × [−10, 10]

Figure 11. $(x,t) \in$ [−5, 5] × [−5, 5]

5. Conclusion

In this work, we introduce a new deformation, namely, deformation of parameters of solution of integrable equations to develop surfaces from integrable equations. Using this deformation, we construct surfaces from modified mKdV and

SG equations. For each integrable equation, we considered two types of deformations arising from parameters of soliton solutions of the corresponding integrable equation. One of them gives surfaces on spheres and the other one gives highly complicated surfaces in $\mathbb{R}^3$. We obtain the quantities such as first and second fundamental forms, Gaussian and mean curvatures of the mKdV and SG surfaces. We find the position vector of mKdV and SG surfaces using the immersion function (F). Furthermore, we provide the graph of interesting mKdV and SG surfaces. The SG surfaces that we obtained are not the critical points of functional where the Lagrange function is a polynomial function of the Gaussian and mean curvatures of the SG surfaces.

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