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BOUR SURFACE COMPANIONS IN SPACE FORMS

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Abstract. In this paper, we give explicit parametrizations for Bour type surfaces in various three-dimensional space forms, using Weierstrass-type representations. We also determine classes and degrees of some Bour type zero mean curvature surfaces in three-dimensional Minkowski space.

MSC: 53A35, 53C42
Keywords: Bour type surface, class, constant mean curvature surface, degree, zero mean curvature surface

1. Introduction

Minimal surfaces in three-dimensional Euclidean space $\mathbb{R}^3$ isometric to rotational surfaces were first introduced by Bour [2] in 1862. All such minimal surfaces are given via the well-known Weierstrass representation for minimal surfaces by choosing suitable data depending on a parameter m, as shown by Schwarz [15]. They are called Bour's minimal surfaces $\mathfrak{B}_m$ of value m. Furthermore, when m is an integer greater than one, $\mathfrak{B}_m$ become algebraic, that is, there is an implicit polynomial equation satisfied by the three coordinates of $\mathfrak{B}_m$, see also [5, 13, 18].

Kobayashi [9] gave an analogous Weierstrass-type representation for conformal spacelike surfaces with mean curvature identically zero, called maximal surfaces, in three-dimensional Minkowski space $\mathbb{R}^{2,1}$. We remark that Magid [12] gave a Weierstrass-type representation for timelike surfaces with mean curvature identically zero, called timelike minimal surfaces, in $\mathbb{R}^{2,1}$, see also [8].

On the other hand, Lawson [10] showed that there is an isometric correspondence between constant mean curvature (CMC for short) surfaces in Riemannian space forms, and Palmer [14] showed that there is an analogous correspondence between spacelike CMC surfaces in Lorentzian space forms. In particular, minimal surfaces in $\mathbb{R}^3$ correspond to CMC 1 surfaces in three-dimensional hyperbolic space $\mathbb{H}^3$, and maximal surfaces in $\mathbb{R}^{2,1}$ correspond to CMC 1 surfaces in three-dimensional de Sitter space $\mathbb{S}^{2,1}$. Thus it is natural to expect existence of corresponding Weierstrass-type representations in these cases. Bryant [3] gave such a representation formula for CMC 1 surfaces in $\mathbb{H}^3$, and Umehara, Yamada [16] applied it. Similarly, Aiyama and Akutagawa [1] gave a representation formula for CMC 1 surfaces in $\mathbb{S}^{2,1}$. However, analogues of Bour's surfaces in other three-dimensional space forms had not yet been studied.

In Sections 2 and 3 of this paper, in order to show that several maximal and timelike minimal Bour's surfaces of value m are algebraic, we review Weierstrass-type representations for maximal surfaces and timelike minimal surfaces in $\mathbb{R}^{2,1}$, and give explicit parametrizations for spacelike and timelike minimal Bour's surfaces of value m. In Section 4, we introduce Bour type CMC 1 surfaces in $\mathbb{H}^3$ and $\mathbb{S}^{2,1}$, and show several properties of those surfaces. Finally, in Section, 5 we calculate the degrees, classes and implicit equations of the maximal and timelike minimal Bour's surfaces of values 2, 3, 4 in $\mathbb{R}^{2,1}$ in terms of their coordinates. We remark that in the cases of $\mathbb{H}^3$ and $\mathbb{S}^{2,1}$, all surfaces are algebraic in some sense, because the Lorentz ($\mathbb{R}^{3,1}$) norm of all elements in $\mathbb{H}^3 \subset \mathbb{R}^{3,1}$ or $\mathbb{S}^{2,1} \subset \mathbb{R}^{3,1}$ is constant. However, we have the following three remaining problems:

Problem.

- What is the class of maximal and timelike minimal Bour's surfaces of general value m in $\mathbb{R}^{2,1}$?
- Are there any other implicit equations for CMC 1 Bour type surfaces? If there exist implicit equations, what are the corresponding degrees and classes?

2. Spacelike Maximal Bour Type Surfaces in $\mathbb{R}^{2,1}$

Let $\mathbb{R}^{n,1} := \big(\{x = (x_1, \cdots, x_n, x_0)^t\,;\, x_i \in \mathbb{R}\}, \langle\cdot,\cdot\rangle\big)$ be the $(n+1)$-dimensional Lorentz-Minkowski (for short, Minkowski) space with Lorentz metric $\langle x, y\rangle = x_1y_1 + \cdots + x_ny_n - x_0y_0$. Then the three-dimensional hyperbolic space $\mathbb{H}^3$ and three-dimensional de Sitter space $\mathbb{S}^{2,1}$ are defined as follows

$$\mathbb{H}^3 := \{x \in \mathbb{R}^{3,1}\,;\, \langle x, x\rangle = -1,\ x_0 > 0\} \cong \left\{F\bar{F}^t\,;\, F \in \mathrm{SL}(2,\mathbb{C})\right\}$$
$$\mathbb{S}^{2,1} := \{x \in \mathbb{R}^{3,1}\,;\, \langle x, x\rangle = 1\} \cong \left\{F\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\bar{F}^t\,;\, F \in \mathrm{SL}(2,\mathbb{C})\right\}.$$

A vector $x \in \mathbb{R}^{n,1}$ is called spacelike if $\langle x, x\rangle > 0$, timelike if $\langle x, x\rangle < 0$, and lightlike if $x \neq 0$ and $\langle x, x\rangle = 0$. A surface in $\mathbb{R}^{n,1}$ is called spacelike (respectively timelike, lightlike) if the induced metric on the tangent planes is a positive definite Riemannian (respectively Lorentzian, degenerate) metric.

Kobayashi [9] has found a Weierstrass-type representation for spacelike conformal maximal surfaces in $\mathbb{R}^{2,1}$.

Theorem 1. *Let g be a meromorphic function and let ω be a holomorphic function defined on a simply connected open subset $\mathcal{U} \subset \mathbb{C}$ such that ω does not vanish on $\mathcal{U}$. Then*

$$f(z) = \mathrm{Re} \int \begin{pmatrix} \left(1+g^2\right)\omega \\ \mathrm{i}\left(1-g^2\right)\omega \\ 2g\omega \end{pmatrix} \mathrm{d}z$$

is a spacelike conformal immersion with mean curvature identically 0 *(i.e., spacelike conformal maximal surface). Conversely, any spacelike conformal maximal surface can be described in this manner.*

Remark 2. *A pair of a meromorphic function g and a holomorphic function ω (g, ω) is called* Weierstrass data *for a maximal surface. In* Section 4 *we also call (g, ω) the Weierstrass data for* CMC 1 *surfaces in $\mathbb{H}^3$ and $\mathbb{S}^{2,1}$.*

We call maximal surfaces $\mathfrak{B}_m$ $(m \in \mathbb{Z}_{\geq 2} := \{n \in \mathbb{Z};\, n \geq 2\})$ given by $(g, \omega) = (z, z^{m-2})$ the spacelike Bour's maximal surfaces $\mathfrak{B}_m$ of value m (spacelike $\mathfrak{B}_m$, for short). Several properties of spacelike $\mathfrak{B}_m$ can be found in the first author's paper [6]. The parametrization of spacelike $\mathfrak{B}_m$ is

$$\mathfrak{B}_m(u,v) = \mathrm{Re}\begin{pmatrix} \frac{1}{m-1}\sum_{k=0}^{m-1}\binom{m-1}{k}u^{m-1-k}(\mathrm{i}v)^k + \frac{1}{m+1}\sum_{k=0}^{m+1}\binom{m+1}{k}u^{m+1-k}(\mathrm{i}v)^k \\ \frac{\mathrm{i}}{m-1}\sum_{k=0}^{m-1}\binom{m-1}{k}u^{m-1-k}(\mathrm{i}v)^k - \frac{\mathrm{i}}{m+1}\sum_{k=0}^{m+1}\binom{m+1}{k}u^{m+1-k}(\mathrm{i}v)^k \\ \frac{2}{m}\sum_{k=0}^{m}\binom{m}{k}u^{m-k}(\mathrm{i}v)^k \end{pmatrix} \tag{1}$$

with a Gauss map $n = \left(\dfrac{2u}{u^2+v^2-1}, \dfrac{2v}{u^2+v^2-1}, \dfrac{u^2+v^2+1}{u^2+v^2-1}\right)$, where $z = u + \mathrm{i}v$. The left two pictures in Figure 1 are two examples of spacelike $\mathfrak{B}_m$.

3. Timelike Minimal Bour Type Surfaces in $\mathbb{R}^{2,1}$

Next, we give the Weierstrass-type representation for timelike minimal surfaces in $\mathbb{R}^{2,1}$, which was obtained by Magid [12] (see also [8]).

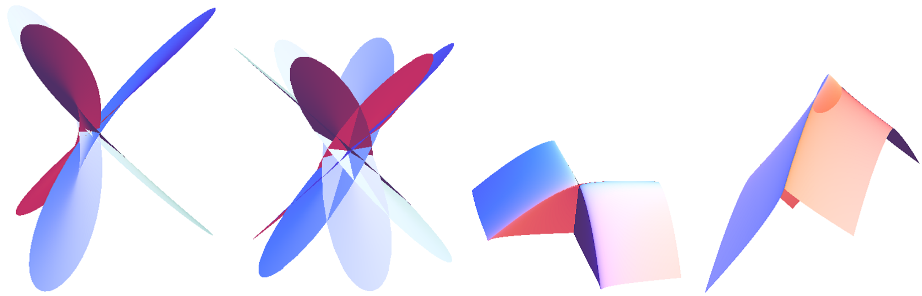

Figure 1. Left two pictures: spacelike $\mathfrak{B}_3$ and $\mathfrak{B}_6$ in $\mathbb{R}^{2,1}$. Right two pictures: timelike $\mathfrak{B}_3$ and $\mathfrak{B}_6$ in $\mathbb{R}^{2,1}$.

Theorem 3. *Let $g_1(u)$, $\omega_1(u)$ (respectively $g_2(v)$, $\omega_2(v)$) be smooth functions depending on only u (respectively v) on a connected orientable 2-manifold with local coordinates u, v. Then*

$$\hat{f}(u,v) = \int \begin{pmatrix} 2g_1\omega_1 \\ \left(1-g_1^2\right)\omega_1 \\ -\left(1+g_1^2\right)\omega_1 \end{pmatrix} \mathrm{d}u + \int \begin{pmatrix} 2g_2\omega_2 \\ \left(1-g_2^2\right)\omega_2 \\ \left(1+g_2^2\right)\omega_2 \end{pmatrix} \mathrm{d}v$$

is a timelike surface with mean curvature identically 0 (i.e., timelike minimal surface). Conversely, any timelike minimal surface can be described in this manner.

The timelike minimal surfaces given by $(g_1(u), \omega_1(u)) = \left(u, u^{m-2}\right)$, $(g_2(v), \omega_2(v)) = \left(v, v^{m-2}\right)$ are called timelike Bour surfaces $\mathfrak{B}_m$ of value m (timelike $\mathfrak{B}_m$, for short) in $\mathbb{R}^{2,1}$, where $m \in \mathbb{Z}_{\geq 2}$. The parametrization of timelike $\mathfrak{B}_m$ is

$$\mathfrak{B}_m(u,v) = \begin{pmatrix} \frac{2}{m}\left(u^m + v^m\right) \\ \frac{1}{m-1}\left(u^{m-1} + v^{m-1}\right) - \frac{1}{m+1}\left(u^{m+1} + v^{m+1}\right) \\ -\frac{1}{m-1}\left(u^{m-1} - v^{m-1}\right) - \frac{1}{m+1}\left(u^{m+1} - v^{m+1}\right) \end{pmatrix} \tag{2}$$

with Gauss map $n = \left(\dfrac{uv-1}{1+uv}, \dfrac{u+v}{1+uv}, \dfrac{u-v}{1+uv}\right)$.

The right two pictures in Figure 1 are two examples of timelike $\mathfrak{B}_m$.

4. CMC 1 Bour Type Surfaces in $\mathbb{H}^3$ and $\mathbb{S}^{2,1}$

In this section we consider CMC 1 surfaces in $\mathbb{H}^3$ and $\mathbb{S}^{2,1}$. Here we identify elements in $\mathbb{H}^3$ and $\mathbb{S}^{2,1}$ with $\mathrm{SL}_2\mathbb{C}$ matrix forms as in Section 2. In this setting Bryant [3] showed the following representation formula for CMC 1 surfaces in $\mathbb{H}^3$, and Aiyama and Akutagawa [1] showed the following Bryant-type representation formula for CMC 1 surfaces in $\mathbb{S}^{2,1}$.

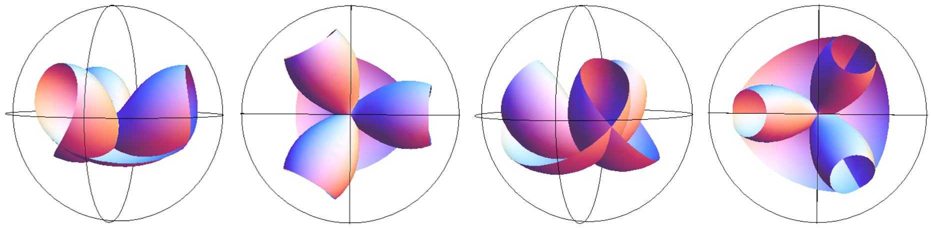

Figure 2. Left two pictures: $\mathfrak{B}_3$ cousins in $\mathbb{H}^3$. Right two pictures: their dual cousins in $\mathbb{H}^3$ (in the Poincaré ball model for $\mathbb{H}^3$).

Theorem 4. *Let $F \in \mathrm{SL}(2,\mathbb{C})$ be a solution of the equation*

$$\frac{\mathrm{d}F}{\mathrm{d}z} = F \begin{pmatrix} g & -g^2 \\ 1 & -g \end{pmatrix} \omega, \qquad F|_{z=z_0} \in \mathrm{SL}(2,\mathbb{C}) \tag{3}$$

for some z_0 in a given domain, where (g,ω) is Weierstrass data. Then the surface $f = F\bar{F}^t$ (respectively $f = F \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \bar{F}^t$) is a conformal CMC 1 *immersion into $\mathbb{H}^3$ (respectively $\mathbb{S}^{2,1}$). Conversely, any conformal* CMC 1 *immersion in $\mathbb{H}^3$ (respectively $\mathbb{S}^{2,1}$) can be described in this way.*

We call CMC 1 surfaces in $\mathbb{H}^3$ and $\mathbb{S}^{2,1}$ given by the Weierstrass data $(g,\omega) = (z, z^{m-2})$ the Bour type CMC 1 cousins $\mathfrak{B}_m$ of value m ($\mathfrak{B}_m$ cousin, for short). We now describe F explicitly

Theorem 5. *Let $F(z) = \begin{pmatrix} a(z) & b(z) \\ c(z) & d(z) \end{pmatrix} \in \mathrm{SL}(2,\mathbb{C})$ be a solution of* equation (3) *with $(g,\omega) = (z, z^{m-2}dz)$ and with initial condition $F(0) = \mathrm{Id}$. Then*

$$\begin{aligned} a(z) &= m^{\frac{1}{m}} \Gamma\left(\frac{m+1}{m}\right) z^{\frac{m-1}{2}} \mathrm{Bessel\,I}\left(-\frac{m-1}{m}, \frac{2}{m} z^{\frac{m}{2}}\right) \\ b(z) &= -\, m^{\frac{1}{m}} \Gamma\left(\frac{m+1}{m}\right) z^{\frac{m+1}{2}} \mathrm{Bessel\,I}\left(\frac{m+1}{m}, \frac{2}{m} z^{\frac{m}{2}}\right) \\ c(z) &= m^{\frac{-1}{m}} \Gamma\left(\frac{m-1}{m}\right) z^{\frac{m-1}{2}} \mathrm{Bessel\,I}\left(\frac{m-1}{m}, \frac{2}{m} z^{\frac{m}{2}}\right) \\ d(z) &= -\, m^{\frac{-1}{m}} \Gamma\left(\frac{m-1}{m}\right) z^{\frac{m+1}{2}} \mathrm{Bessel\,I}\left(-\frac{m+1}{m}, \frac{2}{m} z^{\frac{m}{2}}\right) \end{aligned} \tag{4}$$

where Γ denotes the Gamma function and Bessel I *represents the modified Bessel function. The definition of* Bessel I *can be found in standard textbooks, for example, see* [7].

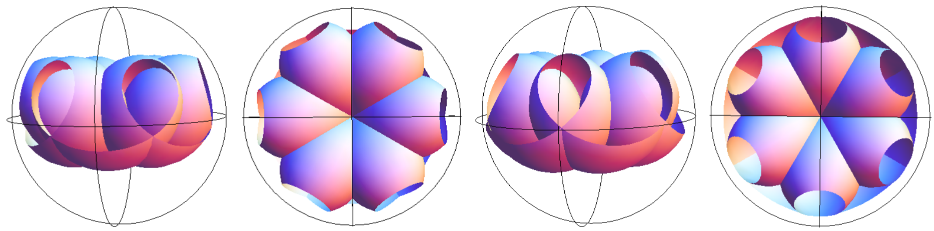

Figure 3. Left two pictures: $\mathfrak{B}_6$ cousins in $\mathbb{H}^3$. Right two pictures: their dual cousins in $\mathbb{H}^3$.

Proof: Equation (3) gives

$$X'' - \frac{\omega'}{\omega}X' - g'\omega X = 0, \qquad X = a(z),\ c(z) \tag{5}$$

$$Y'' - \frac{(g^2\omega)'}{g^2\omega}Y' - g'\omega Y = 0, \qquad Y = b(z),\ d(z) \tag{6}$$

which are given in [16]. Here we solve equation (5). Inserting $(g, \omega) = (z, z^{m-2})$ into equation (5), we have

$$X'' - \frac{m-2}{z}X' - z^{m-2}X = 0, \qquad m \in \mathbb{Z}_{\geq 2}. \tag{7}$$

We give two independent power series solutions of the differential equation (7) by the Frobenius method. The indicial equation at $z = 0$ is $\rho(\rho - 1) - (m-2)\rho = 0$. So we see that the characteristic exponents of the equation (7) are 0 and $m - 1$. Then we have a solution of the form

$$z^{m-1}\sum_{p=0}^{\infty} a_p z^p$$

where the coefficients a_p are inductively given by

$$\begin{aligned} a_{mk+l} &= 0, \qquad l = 0, \cdots, m \\ a_{mk+m+1} &= \frac{a_{m(k-1)+m-1}}{(m-2)k(mk+m-1)} \\ &= \frac{\Gamma(\frac{m-1}{m}+k)}{m^2\Gamma(\frac{m-1}{m}+k+1)}a_{m(k-1)+m-1}, \qquad l \geq m+1. \end{aligned}$$

Therefore we obtain a solution of the differential equation (7)

$$z^{\frac{m-1}{2}}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!\Gamma(\frac{m-1}{m}+k+1)}\left(\frac{z^{\frac{m}{2}}}{m}\right)^{2k+\frac{m-1}{m}} = z^{\frac{m-1}{2}}\mathrm{Bessel\,I}\left(\frac{m-1}{m}, \frac{2}{m}z^{\frac{m}{2}}\right).$$

Similarly, we obtain another independent solution as

$$z^{\frac{m-1}{2}}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!\Gamma(\frac{1-m}{m}+k+1)}\left(\frac{z^{\frac{m}{2}}}{m}\right)^{2k-\frac{m-1}{m}}=z^{\frac{m-1}{2}}\,\mathrm{Bessel\,I}\left(\frac{1-m}{m},\frac{2}{m}z^{\frac{m}{2}}\right).$$

So we have two independent solutions of equation (5). Next, we find two independent solutions of equation (6). Inserting $(g,\omega)=(z,z^{m-2})$ into equation (6), we have

$$Y''-\frac{m}{z}Y'-z^{m-2}Y=0,\qquad m\in\mathbb{Z}_{\geq 2}.$$

Similarly to the way we solved equation (5), we have two independent solutions

$$z^{\frac{m+1}{2}}\,\mathrm{Bessel\,I}\left(\frac{m+1}{m},\frac{2}{m}z^{\frac{m}{2}}\right),\qquad z^{\frac{m+1}{2}}\,\mathrm{Bessel\,I}\left(-\frac{m+1}{m},\frac{2}{m}z^{\frac{m}{2}}\right).$$

Using the initial conditions, we have the solution F as in equations (4). ■

Remark 6. *If F is a solution of* equation (3)*, the surface*

$$f^\sharp=(F^{-1})\overline{(F^{-1})}^t\quad\left(\textit{respectively } f^\sharp=(F^{-1})\begin{pmatrix}1&0\\0&-1\end{pmatrix}\overline{(F^{-1})}^t\right)$$

is also a CMC 1 *surface in* $\mathbb{H}^3$ (*respectively* $\mathbb{S}^{2,1}$). *This was proven in* [17] (*respectively* [11]). *The surface $f^\sharp$ is called the* CMC 1 *dual of f.*

Using the explicit parametrization of the $\mathfrak{B}_m$ cousin, we can easily show the following corollary, which implies the rotational symmetric property of the $\mathfrak{B}_m$ cousins in $\mathbb{H}^3$ and $\mathbb{S}^{2,1}$.

Corollary 7. *Let $F(z)\in\mathrm{SL}_2\mathbb{C}$ be the form as in* Theorem 5 *with complex coordinate z. Then*

$$F(\mathrm{e}^{\mathrm{i}\frac{2\pi}{m}}\cdot z)=\begin{pmatrix}a(z)&\mathrm{e}^{\mathrm{i}\frac{2\pi}{m}}\cdot b(z)\\\mathrm{e}^{-\mathrm{i}\frac{2\pi}{m}}\cdot c(z)&d(z)\end{pmatrix}.$$

Writing $\mathfrak{B}_m$ cousin in $\mathbb{H}^3$ or $\mathbb{S}^{2,1}$ as $f(z)=(x_1(z),x_2(z),x_3(z),x_0(z))^t$, given by Theorem 5, and setting $f\left(\mathrm{e}^{\mathrm{i}\frac{2\pi}{m}}\cdot z\right)=(\hat{x}_1(z),\hat{x}_2(z),\hat{x}_3(z),\hat{x}_0(z))^t$. By Corollary 7, we have

$$\hat{x}_1(z)=\cos\left(\frac{2\pi}{m}\right)x_1(z)-\sin\left(\frac{2\pi}{m}\right)x_2(z)$$
$$\hat{x}_2(z)=\sin\left(\frac{2\pi}{m}\right)x_1(z)+\cos\left(\frac{2\pi}{m}\right)x_2(z)$$
$$\hat{x}_3(z)=x_3(z),\qquad \hat{x}_0(z)=x_0(z)$$

that is, by rotating z by angle $\frac{2\pi}{m}$, the first and second coordinates are also rotated by the same angle. So like in $\mathbb{R}^3$ and $\mathbb{R}^{2,1}$, $\mathfrak{B}_m$ has symmetry with respect to rotation by angle $\frac{2\pi}{m}$. Its dual $(\mathfrak{B}_m)^\sharp$ also has the same symmetry.

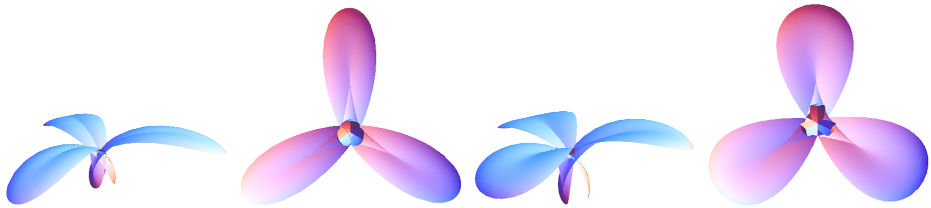

Figure 4. Left two pictures: $\mathfrak{B}_3$ cousins in $\mathbb{S}^{2,1}$. Right two pictures: their dual cousins in $\mathbb{S}^{2,1}$.

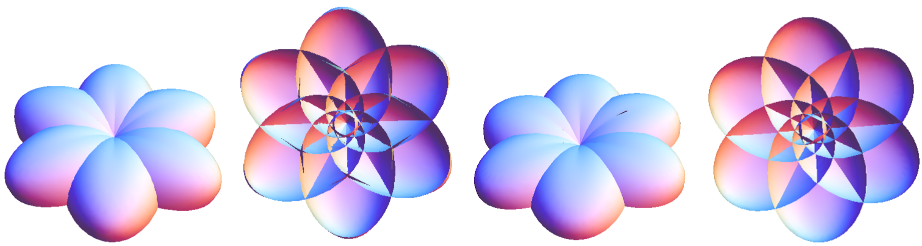

Figure 5. Left two pictures: $\mathfrak{B}_6$ cousins in $\mathbb{S}^{2,1}$. Right two pictures: their dual cousins in $\mathbb{S}^{2,1}$.

In order to see CMC 1 surfaces in $\mathbb{H}^3$, we use a stereographic projection. Consider the map

$$\begin{array}{ccc} \mathbb{H}^3 & \longrightarrow & \mathbb{B}^3 \\ \Cup & & \Cup \\ (x_1, x_2, x_3, x_0)^t & \longmapsto & \left(\dfrac{x_1}{1+x_0}, \dfrac{x_2}{1+x_0}, \dfrac{x_3}{1+x_0} \right)^t \end{array}$$

where $\mathbb{B}^3$ denotes the 3-dimensional unit ball. This is the Poincaré ball model for $\mathbb{H}^3$. The pictures in Fig. 2 and Fig. 3 are two examples of $\mathfrak{B}_m$ cousins projected into $\mathbb{B}^3$.

In order to show graphics of CMC 1 surfaces in $\mathbb{S}^{2,1}$, the hollow ball model is used, see [4] for example. Consider the map

$$\begin{array}{ccc} \mathbb{S}^{2,1} & \longrightarrow & \mathbb{B}^3_{(-\pi,\pi)} \\ \Cup & & \Cup \\ (x_1, x_2, x_3, x_0)^t & \longmapsto & \left(\dfrac{\mathrm{e}^{\arctan(x_0)} \cdot x_1}{\sqrt{1+x_0^2}}, \dfrac{\mathrm{e}^{\arctan(x_0)} \cdot x_2}{\sqrt{1+x_0^2}}, \dfrac{\mathrm{e}^{\arctan(x_0)} \cdot x_3}{\sqrt{1+x_0^2}} \right)^t \end{array}$$

where $\mathbb{B}^3_{(-\pi,\pi)} := \{(y_1, y_2, y_3)^t \in \mathbb{R}^3 \ ; \ \mathrm{e}^{-\pi} < y_1^2 + y_2^2 + y_3^2 < \mathrm{e}^{\pi}\}$. Fig. 4 and Fig. 5 show two examples of $\mathfrak{B}_m$ projected into $\mathbb{B}^3_{(-\pi,\pi)}$.

5. Degree and Class of Bour Type Surfaces in $\mathbb{R}^{2,1}$

For $\mathbb{R}^{2,1}$, the set of roots of a polynomial $Q(x,y,z)=0$ gives an algebraic surface. An algebraic surface f is said to be of *degree* (or *order*) n when $n=\deg(f)$.

The tangent plane at a point (u,v) on a surface $f(u,v)=(x(u,v),y(u,v),z(u,v))$ is given by

$$Xx+Yy-Zz+P=0 \tag{8}$$

where the Gauss map is $n=(X(u,v),Y(u,v),Z(u,v))$ and $P=P(u,v)$. We have inhomogeneous tangential coordinates $a=X/P$, $b=Y/P$, and $c=Z/P$. When we can obtain an implicit equation $\hat{Q}(a,b,c)=0$ of $f(u,v)$ in tangential coordinates, the maximum degree of the equation gives the *class* of $f(u,v)$.

Next, using Groebner and other polynomial elimination methods (in Maple software), we calculate the implicit equations, degrees and classes of spacelike and timelike $\mathfrak{B}_2$, $\mathfrak{B}_3$ and $\mathfrak{B}_4$.

5.1. Degree and Class of Spacelike $\mathfrak{B}_2$, $\mathfrak{B}_3$, $\mathfrak{B}_4$ in $\mathbb{R}^{2,1}$

From (2), the parametrization of $\mathfrak{B}_2$ (maximal Enneper surface) is

$$\mathfrak{B}_2(u,v)=\begin{pmatrix}\frac{1}{3}u^3-uv^2+u\\ u^2v-\frac{1}{3}v^3-v\\ u^2-v^2\end{pmatrix}=\begin{pmatrix}x(u,v)\\ y(u,v)\\ z(u,v)\end{pmatrix}$$

where $u,v\in\mathbb{R}$. In this section, $Q_m(x,y,z)=0$ denotes the irreducible implicit equation that spacelike or timelike $\mathfrak{B}_m$ will satisfy. Then

$$\begin{aligned}Q_2(x,y,z)&=-64z^9+432x^2z^6-432y^2z^6+1215x^4z^3+6318x^2y^2z^3\\&-3888x^2z^5+1215y^4z^3-3888y^2z^5+1152z^7+729x^6-2187x^4y^2\\&-4374x^4z^2+2187x^2y^4+6480x^2z^4-729y^6+4374y^4z^2-6480y^2z^4\\&-729x^4z+1458x^2y^2z+3888x^2z^3-729y^4z+3888y^2z^3-5184z^5\end{aligned}$$

and its degree is $\deg(\mathfrak{B}_2)=9$. Therefore, $\mathfrak{B}_2$ is an algebraic maximal surface. To find the class of the surface $\mathfrak{B}_2$, we obtain $P_2(u,v)=\dfrac{(u^2+v^2-3)(u-v)(u+v)}{3(u^2+v^2-1)}$, where $P_m(u,v)$ denotes the function as in equation (8) for spacelike or timelike $\mathfrak{B}_m$. The inhomogeneous tangential coordinates are $a=\dfrac{6u}{\alpha(u,v)}$, $b=\dfrac{6v}{\alpha(u,v)}$, $c=\dfrac{6(u^2+v^2+1)}{\alpha(u,v)}$, where $\alpha(u,v)=(u^2+v^2-3)(u-v)(u+v)$. In

a, b, c coordinates $\mathfrak{B}_2$ is given by

$$\hat{Q}_2(a,b,c) = 4a^6 + 9a^4 + 9b^4 + 6a^2b^2c^2 + 12b^2c^3 - 3b^4c^2 - 18b^4c$$
$$-4a^4b^2 + 18a^4c - 12a^2c^3 - 4a^2b^4 - 3a^4c^2 + 18a^2b^2 - 4a^2b^4 + 4b^6$$

and in general $\hat{Q}_m(a,b,c) = 0$ denotes the irreducible implicit equation for spacelike or timelike $\mathfrak{B}_m$ in terms of tangential coordinates. Therefore, the class of the spacelike $\mathfrak{B}_2$ is $cl(\mathfrak{B}_2) = 6$. Similarly

$$\mathfrak{B}_3(u,v) = \begin{pmatrix} \frac{u^4}{4} + \frac{v^4}{4} - \frac{3}{2}u^2v^2 + \frac{u^2}{2} - \frac{v^2}{2} \\ u^3v - uv^3 - uv \\ \frac{2}{3}u^3 - 2uv^2 \end{pmatrix} = \begin{pmatrix} x(u,v) \\ y(u,v) \\ z(u,v) \end{pmatrix}$$

$$\mathfrak{B}_4(u,v) = \begin{pmatrix} \frac{1}{3}u^3 - uv^2 + \frac{1}{5}u^5 - 2u^3v^2 + uv^4 \\ -u^2v + \frac{1}{3}v^3 + u^4v - 2u^2v^3 + \frac{1}{5}v^5 \\ \frac{1}{2}u^4 - 3u^2v^2 + \frac{1}{2}v^4 \end{pmatrix} = \begin{pmatrix} x(u,v) \\ y(u,v) \\ z(u,v) \end{pmatrix}$$

and

$$Q_3(x,y,z) = -43046721z^{16} + 272097792x^3z^{12} - 816293376xy^2z^{12}$$
$$+3009871872x^6z^8 + 14834368512x^4y^2z^8 + (69 \text{ other lower order terms})$$
$$Q_4(x,y,z) = -1514571848868138319872z^{25}$$
$$+9244212944751820800000x^4z^{20}$$
$$-2419276165576171875000000x^4y^{12}z^5$$
$$-5546527766851092480000x^2y^2z^{20}$$
$$-30652572326660156250000000x^{12}y^6z^2 + (233 \text{ other lower order terms})$$

and their degrees are $\deg(\mathfrak{B}_3) = 16$, $\deg(\mathfrak{B}_4) = 25$. Therefore, $\mathfrak{B}_3$ and $\mathfrak{B}_4$ are algebraic spacelike maximal surfaces. Furthermore

$$P_3(u,v) = \frac{u(u^2+v^2-2)(u^2-3v^2)}{(u^2+v^2-1)}$$
$$P_4(u,v) = \frac{(3u^2+3v^2-5)(u^2-2uv-v^2)(u^2+2uv-v^2)}{30(u^2+v^2-1)}$$

and the inhomogeneous tangential coordinates are

$$a = \frac{12}{\beta(u,v)}, \quad b = \frac{12v}{u\beta(u,v)}, \quad c = \frac{6(u^2+v^2+1)}{u\beta(u,v)}, \qquad m = 3$$
$$a = \frac{60u}{\gamma(u,v)}, \quad b = \frac{60v}{\gamma(u,v)}, \quad c = \frac{30(u^2+v^2+1)}{\gamma(u,v)}, \qquad m = 4$$

where $\beta(u,v) = (u^2+v^2-2)(u^2-3v^2)$, $\gamma(u,v) = (3u^2+3v^2-5)(u^2-2uv-v^2)(u^2+2uv-v^2)$. Then

$$\hat{Q}_3(a,b,c) = 9a^8 + 72a^6b^2 - 8a^6c^2 + 144a^4b^4 - 168a^4b^2c^2$$
$$-96a^2b^4c^2 + 96a^2b^2c^4 + 64b^6c^2 - 48b^4c^4 - 72a^7$$
$$-288a^5b^2 + 288a^5c^2 + 288a^3b^2c^2 - 192a^3c^4 + 144a^6$$
$$\hat{Q}_4(a,b,c) = -16a^{10} - 8640a^2b^2c^5 - 9000a^4b^4c - 3600a^2b^6c$$
$$+12000a^2b^4c^3 + 570a^4b^4c^2 - 180a^2b^6c^2 + 15b^8c^2 - 900b^8 + 1440a^4c^5$$
$$+1440b^4c^5 - 5400a^4b^4 - 3600a^2b^6 + 900b^8c - 2400b^6c^3 - 416a^6b^4$$
$$-416a^4b^6 + 176a^2b^8 - 16b^{10} + 12000a^4b^2c^3 - 3600a^6b^2c - 180a^6b^2c^2$$
$$-3600a^6b^2 + 176a^8b^2 - 2400a^6c^3 + 900a^8c + 15a^8c^2 - 900a^8.$$

Therefore, $cl(\mathfrak{B}_3) = 8$ and $cl(\mathfrak{B}_4) = 10$.

5.2. Degree and Class of Timelike $\mathfrak{B}_2$, $\mathfrak{B}_3$, $\mathfrak{B}_4$ in $\mathbb{R}^{2,1}$

From (2), the parametrization of $\mathfrak{B}_2$ (timelike Enneper surface) is

$$\mathfrak{B}_2(u,v) = \begin{pmatrix} u^2+v^2 \\ u+v-\frac{1}{3}\left(u^3+v^3\right) \\ -u+v-\frac{1}{3}\left(u^3-v^3\right) \end{pmatrix} = \begin{pmatrix} x(u,v) \\ y(u,v) \\ z(u,v) \end{pmatrix}$$

where $u, v \in \mathbb{R}$. Then

$$Q_2(x,y,z) = -16z^9 - 2916y^4z + 4374x^4y^2 - 6318y2x^2z^3 + 4374x^2y^4$$
$$-15552y^2z^3 - 2916x^4z - 5832x^2y^2z - 20736z^5 + 1152z^7 - 8748x^4z^2$$
$$+8748y^4z^2 + 3888y^2z^5 - 3888x^2z^5 + 15552x^2z^3 + 1215x^4z^3 + 1458x^6$$
$$+216x^2z^6 + 1458y^6 + 1215y^4z^3 + 216y^2z^6 + 12960y^2z^4 + 12960x^2z^4.$$

Its degree is $\deg(\mathfrak{B}_2) = 9$. Hence, $\mathfrak{B}_2$ is an algebraic timelike minimal surface. To find the class of surface $\mathfrak{B}_2$ we obtain $P_2(u,v) = \dfrac{(uv+3)(u^2+v^2)}{3(uv+1)}$, and the inhomogeneous tangential coordinates are $a = -\dfrac{(uv-1)(3uv+3)}{\hat{\alpha}(u,v)}$, $b = -\dfrac{(u+v)(3uv+3)}{\hat{\alpha}(u,v)}$, $c = -\dfrac{(u-v)(3uv+3)}{\hat{\alpha}(u,v)}$, where $\hat{\alpha}(u,v) = (uv+1)(uv+3)(u^2+v^2)$. Then

$$\hat{Q}_2(a,b,c) = 16a^6 + 9a^4 + 36b^4c + 24a^2c^3 + 24b^2c^3 - 24a^2b^2c^2$$
$$-12a^4c^2 - 16a^2b^4 - 12b^4c^2 - 36a^4c + 16a^4b^2 + 9b^4 - 16b^6 - 18a^2b^2.$$

Hence, $cl(\mathfrak{B}_2) = 6$. Similarly

$$\mathfrak{B}_3(u,v) = \begin{pmatrix} \frac{2}{3}\left(u^3+v^3\right) \\ \frac{1}{2}\left(u^2+v^2\right) - \frac{1}{4}\left(u^4+v^4\right) \\ -\frac{1}{2}\left(u^2-v^2\right) - \frac{1}{4}\left(u^4-v^4\right) \end{pmatrix} = \begin{pmatrix} x(u,v) \\ y(u,v) \\ z(u,v) \end{pmatrix}$$

$$\mathfrak{B}_4(u,v) = \begin{pmatrix} \frac{1}{2}\left(u^4+v^4\right) \\ \frac{1}{3}\left(u^3+v^3\right) - \frac{1}{5}\left(u^5+v^5\right) \\ -\frac{1}{3}\left(u^3-v^3\right) - \frac{1}{5}\left(u^5-v^5\right) \end{pmatrix} = \begin{pmatrix} x(u,v) \\ y(u,v) \\ z(u,v) \end{pmatrix}$$

and

$$\begin{aligned} &Q_3(x,y,z) = 43046721z^{16} - 1836660096z^{14} \\ &+5435817984x^6z^4 + 602404356096x^4z^8 \\ &+165112971264x^2z^8 + (69 \text{ other lower order terms}) \\ &Q_4(x,y,z) = 3118369126021466283345445989415649282z^{25} \\ &-380660293703792270916192137379840000x^4z^{20} \\ &-2283961762222753625497152824279040000x^2y^2z^{20} \\ &-380660293703792270916192137379840000y^4z^{20} \\ &-2718338279012676739330717777920000000x^8z^{15} \\ &+(233 \text{ other lower order terms}). \end{aligned}$$

So $\deg(\mathfrak{B}_3) = 16$, $\deg(\mathfrak{B}_4) = 25$. In the tangential coordinates a, b, c

$$\begin{aligned} &\hat{Q}_3(a,b,c) = 81a^6b^2 - 27a^4b^4 - 72a^4b^2c^2 - 45a^2b^6 - 48a^2b^4c^2 - 9b^8 \\ &-8b^6c^2 - 108a^6b + 180a^4b^3 + 432a^4bc^2 - 36a^2b^5 - 288a2b^3c^2 - 288a^2bc^4 \\ &-36b^7 - 144b^5c^2 - 96b^3c^4 + 36a^6 - 108a^4b^2 + 108a^2b^4 - 36b^6 \\ &\hat{Q}_4(a,b,c) = -16a^{10} + 16b^{10} - 450a^8c + 15b^8c^2 - 225b^8 - 720a^4c^5 \\ &-1350a^4b^4 + 900a^2b^6 - 450b^8c - 1200b^6c^3 - 416a^6b^4 + 416a^4b^6 \\ &+176a^2b^8 - 4320a^2b^2c^5 + 4500a^4b^4c - 1800a^2b^6c - 6000a^2b^4c^3 \\ &+570a^4b^4c^2 + 180a^2b^6c^2 + 6000a^4b^2c^3 - 1800a^6b^2c + 180a^6b^2c^2 \\ &-225a^8 - 720b^4c^5 + 900a^6b^2 - 176a^8b^2 + 1200a^6c^3 + 15a^8c^2. \end{aligned}$$

Therefore, $cl(\mathfrak{B}_3) = 8$, $cl(\mathfrak{B}_4) = 10$.

Remark 8. *It is clear that* $\deg(x) = m$, $\deg(y) = m+1$, $\deg(z) = m+1$ *for Bour's algebraic maximal and timelike* $\mathfrak{B}_m$*.*

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