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0-BRANE MATRIX DYNAMICS FOR QCD PURPOSES: REGGE TRAJECTORIES

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Abstract. The energy spectrum of two 0-branes for fixed angular momentum in 2+1 dimensions is calculated by the Rayleigh-Ritz method. The basis function used for each angular momentum consists of 80 eigenstates of the harmonic oscillator problem on the corresponding space. It is seen that the spectrum exhibits a definite linear Regge trajectory behavior. It is argued how this behavior supports the picture by which the bound-states of quarks and QCD-strings are governed by the quantum mechanics of matrix coordinates.

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1. Introduction

The string theoretic description of gauge theories is an old idea [20, 23, 24], still stimulating research works in theoretical physics [14, 17, 21]. Depending on the amount of momentum transfer, the hadron-hadron scattering processes have shown two different behaviors [2, Ch.14], [22]. At very large momentum transfers the interactions are among the point-like substructures, and qualitative similarities to electron-hadron scattering emerge. At high energies and small momentum transfers the Regge trajectories are exchanged. The exchanged linear trajectories are the first motivation for the string picture of strong interaction. However, the fairly good fitting between the linear Regge trajectories and the mass of QCD bound-states has not been explained yet [17], partially due to the lack of a consistent formulation of string theory in 3+1 dimensions

According to string theory, 0-branes are point-like objects to which the strings can end [18, 19]. It is known that in a specific regime the dynamics of N 0-branes is governed by the matrix quantum mechanics resulting from dimensional reduction of $\mathrm{U}(N)$ Yang-Mills theory to $0+1$ dimension [25]. In this regime, the dynamics of 0-branes and the strings stretched between them is encoded in the elements of matrix coordinates resulted from the dimensional reduction of non-Ableian gauge theory.

By the picture mentioned above, it sounds reasonable that the dynamics of 0-branes is used to model the bound-states of quarks and QCD-strings. This picture is the main theme of a series of works, and it is shown that the dynamics of 0-branes can reproduce some known features and expectations in hadron physics, including the potentials between static and fast decaying quarks, and also the Regge behavior in the scattering amplitudes [5, 6]. The symmetry aspects of the picture and its relation to special relativity agenda were studied in [7]. In particular, it is argued that maybe the full featured formulation of non-Ableian gauge theories is possible on non-commutative matrix spaces [7].

In the present note the aim is to see whether the 0-brane matrix dynamics can reproduce the linear Regge trajectories observed in hadron physics. Early studies on spectrum of 0-brane bound-states are reported in [3, 4, 10, 12]. In [13, 26] the study of spectrum based on the variational method is presented. In the present work, based on the results by [12, 13], for the case of two bosonic 0-branes in 2+1 dimensions the energy eigenvalues are calculated for states with given angular momentum, ranging from 0 to 42. The spectrum is calculated by the Rayleigh-Ritz variational method, and the basis function for each angular momentum consists of 80 eigenstates of the harmonic oscillator problem on the configuration space of the 0-branes. It is seen that apart from two lowest angular momenta, the energy versus angular momentum can be fitted with straight-line at each level. The spectrum may be interpreted as the one for massive 0-branes in 2+1 dimensions, or in a Matrix theory perspective [1], as for massless particles in 3+1 dimensions but in the light-cone frame. Based on the latter way of interpretation, the linear relation can be turned as the one between mass-squared and angular momentum, just reminiscent the observed one in QCD bound-states.

Based on the above observation about the spectrum, this may be suggested that, the quantum mechanics of matrix coordinates can reconcile string picture and QCD in 3+1 dimensions. In particular, according to this picture the bound-states of quarks and QCD-strings are governed by the quantum mechanics of matrix coordinates [5–7].

The scheme of the rest of this paper is the following. In Section 2, the basic notions for the 0-brane matrix dynamics are presented, together with a demonstration of a bound-state classical solution. In Section 3, the quantum theory is developed. The eigen-functions of the angular momentum together with the complete solution for harmonic oscillator on the 0-branes' configuration space are presented. This solution is used to construct the basis function used in the Rayleigh-Ritz variational method of Section 4. The light-cone interpretation of the results is also presented in Section 4.

2. Matrix Dynamics of 0-Branes

The dynamics of N 0-branes is given by a $\mathrm{U}(N)$ Yang-Mills theory dimensionally reduced to $0+1$ dimensions [12, 19], given by (in units $\hbar = c = 1$)

$$L \;=\; m_0 \mathrm{Tr}\Big(\frac{1}{2}(D_t X_i)^2 \;+\; \frac{1}{4\,l_s^2}\,[X_i, X_j]^2\Big), \qquad D_t \;=\; \partial_t \;-\; \mathrm{i}[A_0, \cdot] \quad (1)$$

where $i, j = 1, ..., d$, l_s as the fundamental string length, $m_0 = (g_s l_s)^{-1}$ with g_s supposedly small string coupling, i.e., $m_0 \gg l_s^{-1}$. X's are in adjoint representation of $\mathrm{U}(N)$ with the usual expansion $X_i = x_{i\,a} T_a$, $a = 1, ..., N^2$. The theory is invariant under the gauge symmetry

$$\vec{X} \to \vec{X}' = U \vec{X} U^\dagger, \qquad A_0 \to A_0' = U A_0 U^\dagger + \mathrm{i}\, U \partial_t U^\dagger \quad (2)$$

where U is an arbitrary time-dependent $N \times N$ unitary matrix. Under these transformations one can check that

$$D_t \vec{X} \to D_t' \vec{X}' = U (D_t \vec{X}) U^\dagger, \qquad D_t D_t \vec{X} \to D_t' D_t' \vec{X}' = U (D_t D_t \vec{X}) U^\dagger. \quad (3)$$

For each direction there are N^2 variables and it is understood that the extra $N^2 - N$ degrees of freedom are representing the dynamics of oriented strings stretched between N 0-branes. The center-of-mass of 0-branes is represented by the trace of the X matrices.

In the quantum theory the off-diagonal elements of matrices play an essential role. In particular, it is shown that in the quantum theory the off-diagonal elements cause the interaction between 0-branes. For the case of classically static 0-branes it is shown that the fluctuations of the off-diagonal elements develop a linear potential, just as the case for QCD-strings stretched between quarks [5].

The canonical momenta are given by

$$P_i = \frac{\partial L}{\partial X_i} = m_0\, D_t X_i \quad (4)$$

by which the Hamiltonian is constructed

$$H = \mathrm{Tr}\left(\frac{P_i^2}{2\,m_0} - \frac{m_0}{4\,l_s^2}\,[X_i, X_j]^2\right). \quad (5)$$

As the time-derivative of the dynamical variable A_0 is absent, its equation of motion introduces a constraint, the so-called Gauss law

$$G_a := \sum_i [X_i, P_i]_a = \mathrm{i} \sum_{i,b,c} f_{abc}\, x_{i\,b}\, p_{i\,c} \equiv 0. \quad (6)$$

In the present work we take the two dimensional case ($d = 2$) for a pair of 0-branes. It would be quite useful to separate the pure gauge variables from the others. For the case of SU(2) theory in 2+1 dimensions, following [10, 13] we use the decomposition

$$x_{i\,a} = (\Psi)_{a\,b}(\Lambda)_{b\,j}(\eta)_{j\,i} \tag{7}$$

in which the matrix Ψ is an element of group of SU(2). Accordingly the gauge transformations of the variable $x_{i\,a}$ are captured by Ψ through ordinary gauge group left multiplications. Parameterizing the SU(2) group elements by the three Euler angles, the matrix Ψ is represented by [8]

$$\Psi = R_z(\alpha)R_x(\gamma)R_z(\beta) \tag{8}$$

in which R_a is the rotation matrix about the ath axis. Analogously, the matrix η is an element of the SO(2) group parameterized by the angle ϕ, capturing the effect of rotation in the two dimensional space. The matrix Λ takes the form [13]

$$\Lambda = \begin{pmatrix} r\cos\theta & 0 \\ 0 & r\sin\theta \\ 0 & 0 \end{pmatrix}. \tag{9}$$

We mention that the only variable with dimension of length is r. Also, apart from pure gauge variables α, β, and γ, the two dimensional configuration space is spanned by the polar coordinates (r, ϕ), and the extra variable θ appears as an internal degree of freedom.

By the decomposition, the three constraints (6) take the form [13]

$$\begin{aligned} G_1 &= \sin\alpha\cot\gamma\, p_\alpha - \sin\alpha\csc\gamma\, p_\beta - \cos\alpha\, p_\gamma \\ G_2 &= \cos\alpha\cot\gamma\, p_\alpha - \cos\alpha\csc\gamma\, p_\beta + \sin\alpha\, p_\gamma \\ G_3 &= -p_\alpha \end{aligned} \tag{10}$$

in which p_α, p_β, and p_γ are the conjugate momenta of the pure gauge variables α, β, and γ. By the constraints (6), using the explicit forms (10), we have to set

$$p_\alpha = p_\beta = p_\gamma \equiv 0. \tag{11}$$

By imposing the constraints, setting $l_s = 1$ the Hamiltonian takes the form [12, 13]

$$H = \frac{1}{2\mu}\left(p_r^2 + \frac{p_\theta^2}{r^2} + \frac{p_\phi^2}{r^2\cos^2(2\theta)}\right) + \frac{\mu}{8}r^4\sin^2(2\theta) \tag{12}$$

in which $\mu = m_0/2$, as the reduced mass appearing in the relative motion of two 0-branes. It is easy to check that the canonical momentum of ϕ, p_ϕ, is conserved. So as expected, the two dimensional angular momentum is a constant of motion.

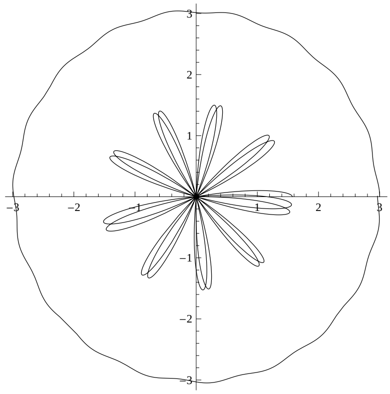

Figure 1. The plots of a numerical solution of (13). The outer curve is representing the radial coordinate as a function of the polar angle ϕ. The inner one, which is scaled ten times to make it visible, is $\theta(\phi)$. The solution is by the conditions: $\mu = 1/2$, $l_s = 1$, $p_\phi = 1.42$, $r(0) = 3$, $\theta(0) = 0.157$ rad, $\dot{r}(0) = \dot{\theta}(0) = 0$, $\phi(0) = 0$.

The equations of motion by (12) are

$$\begin{aligned}
&\mu(\ddot{r} - r\dot{\theta}^2) - \frac{p_\phi^2}{\mu r^3 \cos^2(2\theta)} + \frac{\mu}{2} r^3 \sin^2(2\theta) = 0 \\
&\mu(r\ddot{\theta} + 2\dot{r}\dot{\theta}) + \frac{2p_\phi^2 \sin(2\theta)}{\mu r^3 \cos^3(2\theta)} + \frac{\mu}{2} r^3 \sin(2\theta)\cos(2\theta) = 0 \\
&\dot{\phi} = \frac{p_\phi}{\mu r^2 \cos^2(2\theta)}.
\end{aligned} \tag{13}$$

It is easy to check that $\theta(t) \equiv 0$, by which the potential is set to zero, the equations for (r, ϕ) would come to the form of a free particle in polar coordinate. As an illustration that the above equations can develop bound-states, the plots of a numerical solution are presented in Fig. 1. In the figure, the outer curve is $r(\phi)$ as the path of the relative motion of 0-branes in the polar coordinate setup (r, ϕ), while the inner curve is a ten times scaled of $\theta(\phi)$, as the internal degree of freedom causing the effective attractive force between 0-branes. Evidently, this solution represents an almost circular path for the relative motion of 0-branes.

3. Quantum Dynamics

In passing to quantum theory, the constraints in operator form define the physically acceptable states as

$$\hat{G}\,|\psi\rangle = 0. \tag{14}$$

By the replacements

$$p_\alpha \to -\mathrm{i}\,\frac{\partial}{\partial\alpha}, \qquad p_\beta \to -\mathrm{i}\,\frac{\partial}{\partial\beta}, \qquad p_\gamma \to -\mathrm{i}\,\frac{\partial}{\partial\gamma} \tag{15}$$

one would find, as expected, that the physical wave-functions do not depend on the pure gauge degrees of freedom α, β, and γ. The Laplacian operator can be constructed using the metric g_{ij}

$$\nabla^2 \equiv \frac{1}{\sqrt{g}}\partial_i(\sqrt{g}g^{ij}\,\partial_j) \tag{16}$$

in which $g = \det g$, explicitly found to be $\frac{1}{4}r^5\sin\gamma\sin(4\theta)$ [13]. So, in the coordinate setup $(r, 4\,\theta, \phi)$, with $0 \le \theta \le \pi/4$ and $0 \le \phi \le 2\pi$, the Hamiltonian acting on the wave-function $\psi(r,\theta,\phi)$, takes the form [12, 13]

$$H = -\frac{1}{2\mu}\left(\frac{1}{r^5}\partial_r\left(r^5\partial_r\right) + \frac{1}{r^2}\nabla^2_\Omega\right) + \frac{\mu}{8}r^4\sin^2(2\theta) \tag{17}$$

in which

$$\nabla^2_\Omega = \frac{1}{\sin(4\theta)}\partial_\theta\left(\sin(4\theta)\partial_\theta\right) + \frac{\partial^2_\phi}{\cos^2(2\theta)}. \tag{18}$$

Using the scaling $\psi \to r^{-3/2}\psi$, the Hamiltonian comes to the form

$$H = -\frac{1}{2\mu}\left(\frac{1}{r^2}\partial_r\left(r^2\partial_r\right) + \frac{1}{r^2}(\nabla^2_\Omega - {}^{15}\!/\!_4)\right) + \frac{\mu}{8}r^4\sin^2(2\theta). \tag{19}$$

By the introduced separation of variables, the two dimensional angular momentum is $L_z = -\mathrm{i}\,\frac{\partial}{\partial\phi}$ [13], and obviously commutes with the Hamiltonian, $[\hat{L}_z, \hat{H}] = 0$. So one can construct states with given energy and angular momentum.

3.1. Angular Momentum Spectrum

Here the aim is to find the eigenfunctions and eigenvalues of the operator ∇^2_Ω

$$\nabla^2_\Omega\,\mathcal{Y}_\lambda(\theta,\phi) = \lambda\,\mathcal{Y}_\lambda(\theta,\phi) \tag{20}$$

for which we assume as usual

$$\mathcal{Y}_\lambda(\theta,\phi) = g_\lambda(\theta)\frac{\mathrm{e}^{\mathrm{i}m_z\phi}}{\sqrt{2\pi}} \tag{21}$$

with m_z as the quantum number associated to the angular momentum in the two dimensional configuration space. Although the pure gauge degrees of freedom have been separated out, it is known that a remaining discrete gauge transformation would cause that only even integer values are accepted for m_z [13]. In particular, the shifts $\alpha \to \pi + \alpha$ and $\phi \to 2\pi + \phi$ would make equal changes to the original variables, namely $x_{i\,a} \to -x_{i\,a}$, if m_z has an odd value. So, to construct absolute gauge invariant physical states, the quantum number m_z has to be even, setting

$$m_z = 2\,m, \qquad m = 0, \pm 1, \pm 2, \ldots. \tag{22}$$

Using the change of variable $x = \cos(4\theta)$, one has

$$\frac{\mathrm{d}}{\mathrm{d}x}\left((1-x^2)\frac{\mathrm{d}g_\lambda}{\mathrm{d}x}\right) - \frac{m^2}{2(1+x)}g_\lambda(x) = \frac{\lambda}{16}\,g_\lambda(x). \tag{23}$$

As the spectrum is invariant under the change $m \to -m$, from now on we take $m \geq 0$. Using the replacement $g_\lambda(x) = (1+x)^{m/2}Q_\lambda(x)$

$$(1-x^2)Q''(x) + \big(m - (m+2)x\big)Q'(x) - \left(\lambda + \frac{m(m+2)}{4}\right)Q(x) = 0 \tag{24}$$

which is known to have Jacobi polynomials of order $n = l - m \geq 0$, $\mathcal{P}_n^{(0,m)}(x)$, as solutions [9]. By this the eigenvalue λ is found

$$\lambda = -16(l - m/2)(l - m/2 + 1), \qquad m \leq l = 0, 1, \ldots \tag{25}$$

for the normalized eigenfunction

$$\mathcal{Y}_l^m(\theta,\phi) = \sqrt{\frac{2l-m+1}{2^{m+1}}}(1+\cos(4\theta))^{m/2}\mathcal{P}_{l-m}^{(0,m)}(\cos(4\theta))\frac{\mathrm{e}^{2im\phi}}{\sqrt{2\pi}}. \tag{26}$$

The Jacobi polynomials of our interest satisfy the following recurrence relation, which comes mostly helpful when the matrix elements of the Hamiltonian (19) are evaluated in the angular momentum basis

$$\frac{2(l+1)(l-m+1)}{(2l-m+1)(2l-m+2)}\mathcal{P}_{l-m+1}^{(0,m)}(x) + \frac{2l(l-m)}{(2l-m)(2l-m+1)}\mathcal{P}_{l-m-1}^{(0,m)}(x) + \frac{m^2}{(2l-m)(2l-m+2)}\mathcal{P}_{l-m}^{(0,m)}(x) = x\,\mathcal{P}_{l-m}^{(0,m)}(x). \tag{27}$$

3.2. Harmonic Oscillator Solution

As we are going to evaluate the spectrum of the Hamiltonian (19) by the variational Rayleigh-Ritz method [15], a set of basis functions is needed, for which we shall take those of harmonic oscillator. For a harmonic oscillator with kinetic term as in (19) and unit frequency ($\omega = 1$), taking

$$\psi_{E,l,m}(r,\theta,\phi) = R_{E,l,m}(r)\;\mathcal{Y}_l^m(\theta,\phi) \tag{28}$$

the radial equation would come to the form

$$-\frac{1}{2\mu}\left(R''_{E,l,m} - \frac{J_l^m(J_l^m+1)}{r^2}R_{E,l,m}\right) + \frac{1}{2}\mu r^2 R_{E,l,m} = E\,R_{E,l,m} \tag{29}$$

in which

$$J_l^m = 4\,l - 2\,m + 3/2. \tag{30}$$

It is known that the above has normalized solutions in terms of the Laguerre polynomials

$$R_{k,l,m}(r) = \sqrt{\frac{2\,k!\,\mu^{J_l^m+3/2}}{\Gamma(k+J_l^m+3/2)}}\; r^{J_l^m}\,\mathrm{e}^{-\mu r^2/2}\,L_k^{(J_l^m+1/2)}(\mu r^2) \tag{31}$$

with ($k = 0, 1, 2, \ldots$), and

$$E_{k,l,m} = 2k + J_l^m + 3/2 = 2k + 4l - 2m + 3. \tag{32}$$

To calculate the matrix elements of the Hamiltonian (19), the following recurrence relations for Laguerre polynomia1ls would appear mostly useful [9]

$$\begin{gathered} L_k^{(\alpha+1)}(x) - L_{k-1}^{(\alpha+1)}(x) = L_k^{(\alpha)}(x) \\ (2k+\alpha+1-x)L_k^{(\alpha)}(x) = (k+1)L_{k+1}^{(\alpha)}(x) + (k+\alpha)L_{k-1}^{(\alpha)}(x) \\ (k+\alpha)L_{k-1}^{(\alpha)}(x) - kL_k^{(\alpha)}(x) = x\,L_{k-1}^{(\alpha+1)}(x). \end{gathered} \tag{33}$$

The following identity for the integral of Laguerre polynomials is known as well [16]

$$\begin{aligned} &\int_0^\infty z^p\,L_{k'}^{(p-\tau')}(z)L_k^{(p-\tau)}(z)\mathrm{d}z = (-1)^{k'+k}\,\tau'!\,\tau! \\ &\times \sum_{\sigma=\max\{{k'-\tau' \atop k-\tau}\}}^{\min\{{k' \atop k}\}} \frac{(p+\sigma)!}{\sigma!(k'-\sigma)!(k-\sigma)!(\sigma+\tau'-k')!(\sigma+\tau-k)!}. \end{aligned} \tag{34}$$

Of course if $\max\{{k'-\tau' \atop k-\tau}\} > \min\{{k' \atop k}\}$ the integral is zero.

4. Rayleigh-Ritz Method and Spectrum

To find the eigenvalues of the Hamiltonian (19) we use the Rayleigh-Ritz variational method, in which a basis function is needed to approximate the exact eigenfunctions. Here we take the basis function to be a collection of eigenstates of harmonic oscillator obtained in previous part. As we are interested to find eigenvalues with given angular momentum m_z, the basis function is taken (recall $m_z = 2\,m$)

$$\Big\{\psi_{k,l,m_z/2}(r,\theta,\phi)\Big\}, \quad l = \frac{m_z}{2}, \ldots, \frac{m_z}{2} + n_{\max}, \quad k = 0, \ldots, n'_{\max} \tag{35}$$

in which $n_{\max}$ and $n'_{\max}$ determine the level of truncations. By this choice, the number of the members of the basis function is equal to $(n_{\max}+1)(n'_{\max}+1)$.

m_z	E_1	E_2	E_3	E_4	E_5	E_6	m_z	E_1	E_2	E_3	E_4	E_5	E_6
0	2.66	4.54	5.95	7.15	8.25	9.09	22	13.5	14.9	16.5	18.3	20.4	22.8
2	4.13	5.31	6.22	7.16	8.34	9.79	24	14.4	15.9	17.6	19.5	21.6	24.1
4	5.39	6.13	6.89	7.91	9.22	10.9	26	15.4	16.9	18.7	20.6	22.9	25.4
6	6.44	6.99	7.83	8.95	10.4	12.1	28	16.3	17.9	19.7	21.8	24.1	26.7
8	7.33	7.96	8.89	10.1	11.6	13.4	30	17.3	18.9	20.8	22.9	25.3	28.0
10	8.18	8.95	9.97	11.3	12.8	14.7	32	18.2	19.9	21.9	24.1	26.5	29.2
12	9.03	9.95	11.1	12.4	14.1	16.1	34	19.2	21.0	23.0	25.2	27.7	30.5
14	9.90	10.9	12.2	13.6	15.4	17.4	36	20.2	22.0	24.0	26.4	28.9	31.8
16	10.8	11.9	13.3	14.8	16.6	18.8	38	21.1	23.0	25.1	27.5	30.1	33.1
18	11.7	12.9	14.3	16.0	17.9	20.1	40	22.1	24.0	26.2	28.6	31.3	34.3
20	12.6	13.9	15.4	17.2	19.1	21.4	42	23.1	25.0	27.3	29.8	32.5	35.6

Table 1. The first six energy eigenvalues for given m_z by the Rayleigh-Ritz method, in units $g_s^{1/3}\, l_s^{-1}$. For each m_z basis function consists of 80 elements.

Before to proceed, it would be useful to determine how the spectrum depends on the initial parameters l_s and g_s (recall $m_0 = 1/(g_s l_s)$, and $\mu = m_0/2$). By the re-scalings [12]

$$X_i \to g_s^{1/3}\, l_s\, X_i, \qquad P_i \to g_s^{-1/3}\, l_s^{-1}\, P_i \tag{36}$$

in the Hamiltonian (5) one finds that the eigenvalues have the form $E = \kappa\, g_s^{1/3}\, l_s^{-1}$, with κ as dimensionless number (recall we have set $\hbar = c = 1$).

In calculation of the matrix elements of the Hamiltonian (19) one could avoid explicit integrations over r and θ variables, simply by using the recurrence relations (27) and (33), and the integral identity (34).

For the energy eigenvalues reported in Table 1 we have set $n_{\max} = n'_{\max} = 8$, making 80 elements for the basis function for each m_z.

Apart from two lowest m_z's, the values given in Table 1 together with the straight-line data fittings are plotted in Fig. 2. The results of the fittings are presented in equation (37), with the brackets indicating the standard error for each given value

$$\begin{aligned}
E_1 &= 3.474\,[0.059] + 0.462\,[0.002]\, m_z\\
E_2 &= 3.953\,[0.031] + 0.500\,[0.001]\, m_z\\
E_3 &= 4.632\,[0.020] + 0.539\,[0.001]\, m_z\\
E_4 &= 5.535\,[0.027] + 0.579\,[0.001]\, m_z\\
E_5 &= 6.754\,[0.038] + 0.616\,[0.001]\, m_z\\
E_6 &= 8.277\,[0.047] + 0.654\,[0.002]\, m_z.
\end{aligned} \tag{37}$$

By the present standard errors one finds that all the percentage errors are less than 2%. Further, all the statistical P-values for the straight-line fittings are less than 10^{-22}, leaving almost no room for the null hypothesis.

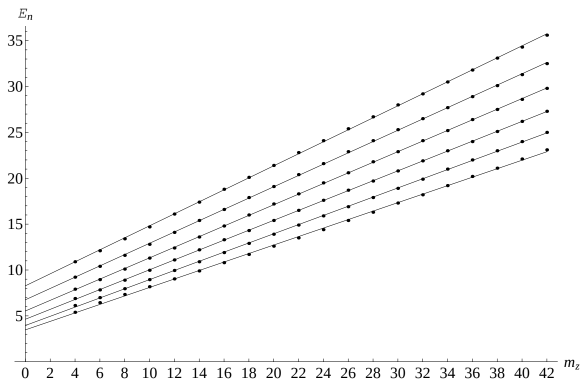

Figure 2. The plots of energy eigenvalues versus m_z, according to Table 1, together with the straight-line fittings given in equation (37).

4.1. Light-Cone Interpretation

The obtained spectrum may be interpreted as the one for massive 0-branes in 2+1 dimensions, or in a Matrix theory perspective [1], as for massless particles in 3+1 dimensions but in the light-cone frame. For the latter way of interpretation, the relative motion of bound-state constituents (defined by $\vec{P}_\perp = \sum_i \vec{k}_{\perp i} = 0$) is related to the masses' constituents and the potential W in transverse directions as following [11]

$$H := P^- = \sum_{i=1,2} \frac{\vec{k}^2_{\perp i} + {m_i}^2}{2\,p_i^+} + \frac{W}{2\,P^+}$$

in which $P^+ = \sum_i p_i^+$, and p_i^+'s appear as the masses of constituents but in the transverse directions of the light-cone frame. For the case of interest, setting $m_i = 0$ and $p_1^+ = p_2^+$ one simply has the relation between Hamiltonian eigenvalues for relative motion ($\vec{P}_\perp = 0$) and the mass squared of bound-state using the key relation $\mathcal{M}^2 = 2\,P^- P^+$ [1, 11]. So in the light-cone frame the linear relation between the Hamiltonian eigenvalues and the angular momentum turns as the linear relation between the mass squared and the angular momentum, just reminiscent the observed one in hadron physics.

Acknowledgements

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