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QUANTIZED VERSION OF THE THEORY OF AFFINE BODY

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Abstract. In the previous lecture we have introduce and discussed the concept of affinely-rigid, i.e., homogeneously deformable body. Some symmetry problems and possible applications were discussed. We referred also to our motivation by Euler ideas. Below we describe the general principles of the quantization of this theory in the Schrödinger language. The special stress is laid on highly-symmetric, in particular affinely-invariant, models and the Peter-Weyl analysis of wave functions.

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1. Introduction

Let us consider quantum-mechanical system in configuration space Q – the n-dimensional differential manifold. In Schrödinger theory pure states are described by complex scalar densities Ψ of weight $1/2$ [13]. The scalar product is given by

$$(\Psi, \Phi) = \int \overline{\Psi}\Phi = \int \overline{\Psi}(q)\Phi(q)\mathrm{d}q^1 \ldots \mathrm{d}q^n.$$

Usually Q is a Riemannian or pseudo-Riemannian space (Q, Γ). Classical kinetic energy is then given by

$$T = \frac{1}{2}\Gamma_{\mu\nu}\frac{\mathrm{d}q^\mu}{\mathrm{d}t}\frac{\mathrm{d}q^\nu}{\mathrm{d}t}.$$

The metric Γ gives rise to the natural volume measure

$$\mathrm{d}\mu_\Gamma(q) = \sqrt{|\det[\Gamma_{\mu\nu}]|}\mathrm{d}q^1 \ldots \mathrm{d}q^n.$$

Wave densities Ψ are then represented by scalar functions ψ

$$\Psi(q) = \psi(q)\sqrt[4]{|\det \Gamma_{\mu\nu}|}.$$

Scalar product becomes then [12]

$$(\Psi|\Phi) = \langle\psi|\varphi\rangle = \int \overline{\psi}(q)\varphi(q)\mathrm{d}\mu_{\Gamma}(q).$$

Classical and quantum kinetic energies are given by

$$\mathcal{T} = \frac{1}{2}\Gamma^{\mu\nu}p_{\mu}p_{\nu}, \qquad \Gamma^{\mu\alpha}\Gamma_{\alpha\nu} = \delta^{\mu}{}_{\nu}, \qquad p_{\mu} = \frac{\partial T}{\partial \dot{q}^{\mu}} = \Gamma_{\mu\nu}\frac{\mathrm{d}q^{\nu}}{\mathrm{d}t}$$

$$\mathbf{T} = -\frac{\hbar^2}{2}\Delta(\Gamma)$$

where $\Delta(\Gamma)$ is the Laplace-Beltrami operator

$$\Delta(\Gamma) = \frac{1}{\sqrt{|\Gamma|}}\sum_{\mu,\nu}\partial_{\mu}\sqrt{|\Gamma|}\Gamma^{\mu\nu}\partial_{\nu} = \Gamma^{\mu\nu}\nabla_{\mu}\nabla_{\nu}$$

$$\mathbf{H} = \mathbf{T} + \mathbf{V}, \qquad (\mathbf{V}\psi)(q) = V(q)\psi(q).$$

If Q is multiply-connected, we can admit wave functions on $\overline{Q}$, the covering manifold of Q. But $\overline{\psi}\psi$ should be projectable to Q. This is the case with rigid body, affinely-rigid body and many other systems [1–3, 5, 6, 13, 22, 23].

We assume Q to be a Lie group G. It is endowed with the Haar measure μ. But usually $\mu_{\Gamma} = \mu$. Namely, in practical problems $\Gamma_{\mu\nu}$ is left- or right-invariant. But then μ_{Γ} is so as well. But the invariant measure on G is unique up to normalization constant, so we can admit $\mu_{\Gamma} = \mu$.

Let E_{μ}, E^{μ} be elements of mutually dual bases in G', G'^{*} (Lie algebra and co-algebra of G), and q^{μ} – first kind canonical coordinates on G

$$g(q) = \exp(q^{\mu}E_{\mu}).$$

Lie-algebraic velocities

$$\begin{aligned}
\Omega &= \frac{\mathrm{d}g}{\mathrm{d}t}g^{-1} = \Omega^{\mu}E_{\mu} = \left(\Omega^{\mu}{}_{\nu}(q)\frac{\mathrm{d}g^{\nu}}{\mathrm{d}t}\right)E_{\mu}\\
\widehat{\Omega} &= g^{-1}\frac{\mathrm{d}g}{\mathrm{d}t} = \widehat{\Omega}^{\mu}E_{\mu} = \left(\widehat{\Omega}^{\mu}{}_{\nu}(q)\frac{\mathrm{d}g^{\nu}}{\mathrm{d}t}\right)E_{\mu} = g^{-1}\Omega g.
\end{aligned}$$

Left- and right-invariant kinetic energies have the form

$$T_{\text{left}} = \frac{1}{2}\mathcal{L}_{\mu\nu}(q)\widehat{\Omega}^{\mu}\widehat{\Omega}^{\nu}, \qquad T_{\text{right}} = \frac{1}{2}\mathcal{R}_{\mu\nu}(q)\Omega^{\mu}\Omega^{\nu}$$

where $[\mathcal{L}_{\mu\nu}(q)]$, $[\mathcal{R}_{\mu\nu}(q)]$ – constant, non-singular and symmetric matrices.

Legendre transformation is given non-holonomically by

$$\widehat{\Sigma}_\mu = \frac{\partial T_{\rm left}}{\partial \widehat{\Omega}^\mu} = \mathcal{L}_{\mu\nu}\widehat{\Omega}^\nu, \qquad \Sigma_\mu = \frac{\partial T_{\rm right}}{\partial \Omega^\mu} = \mathcal{R}_{\mu\nu}\Omega^\nu.$$

They are Hamiltonian generators, i.e., momentum mappings of right and left transformations. Hamiltonians are given by

$$\begin{aligned} H_{\rm left} &= \mathcal{T}_{\rm left} + \mathcal{V}(q) = \frac{1}{2}\mathcal{L}^{\mu\nu}\widehat{\Sigma}_\mu\widehat{\Sigma}_\nu + \mathcal{V}(q) \\ H_{\rm right} &= \mathcal{T}_{\rm right} + \mathcal{V}(q) = \frac{1}{2}\mathcal{R}^{\mu\nu}\Sigma_\mu\Sigma_\nu + \mathcal{V}(q) \end{aligned}$$

where $[\mathcal{L}^{\mu\nu}(q)]$, $[\mathcal{R}^{\mu\nu}(q)]$ denote inverses of $[\mathcal{L}_{\mu\nu}(q)]$, $[\mathcal{R}_{\mu\nu}(q)]$.
When the structure constants are $C^\lambda{}_{\mu\nu}$

$$[E_\mu, E_\nu] = E_\lambda C^\lambda{}_{\mu\nu}$$

then the Poisson brackets are

$$\{\Sigma_\mu, \Sigma_\nu\} = \Sigma_\lambda C^\lambda{}_{\mu\nu}, \qquad \left\{\widehat{\Sigma}_\mu, \widehat{\Sigma}_\nu\right\} = -\widehat{\Sigma}_\lambda C^\lambda{}_{\mu\nu}, \qquad \left\{\Sigma_\mu, \widehat{\Sigma}_\nu\right\} = 0.$$

Left and right regular translations in $L^2(G,\mu)$ are given by

$$(\mathbf{L}(k)\Psi)(g) = \Psi(kg), \qquad (\mathbf{R}(k)\Psi)(g) = \Psi(gk).$$

They are unitary

$$\langle \mathbf{L}(k)\Psi|\mathbf{L}(k)\varphi\rangle = \langle\Psi|\varphi\rangle = \langle \mathbf{R}(k)\Psi|\mathbf{R}(k)\varphi\rangle$$

and represent G

$$\mathbf{R}(kl) = \mathbf{R}(k)\mathbf{R}(l), \qquad \mathbf{L}(kl) = \mathbf{L}(l)\mathbf{L}(k).$$

Generators are defined as usual

$$\begin{aligned} (\mathbf{L}_\mu f)(g) &= \left.\frac{\partial}{\partial q^\mu} f(k(q)g)\right|_{q=0} \\ (\mathbf{R}_\mu f)(g) &= \left.\frac{\partial}{\partial q^\mu} f(gk(q))\right|_{q=0}. \end{aligned}$$

Commutation rules have the form

$$[\mathbf{L}_\mu, \mathbf{L}_\nu] = -\mathbf{L}_\varkappa C^\varkappa{}_{\mu\nu}, \qquad [\mathbf{R}_\mu, \mathbf{R}_\nu] = \mathbf{R}_\varkappa C^\varkappa{}_{\mu\nu}, \qquad [\mathbf{L}_\mu, \mathbf{R}_\nu] = 0$$

$\mathbf{L}_\mu$, $\mathbf{R}_\mu$ are anti-self-adjoint

$$\langle \mathbf{L}_\mu\Psi|\varphi\rangle = -\langle\Psi|\mathbf{L}_\mu\varphi\rangle, \qquad \langle \mathbf{R}_\mu\Psi|\varphi\rangle = -\langle\Psi|\mathbf{R}_\mu\varphi\rangle.$$

Classical Poisson brackets in terms of $\mathbf{L}_\mu$, $\mathbf{R}_\mu$ are expressed as follows

$$\begin{aligned}\{A,B\} &= \Sigma_\lambda C^\lambda{}_{\mu\nu}\frac{\partial A}{\partial\Sigma_\mu}\frac{\partial B}{\partial\Sigma_\nu} - \frac{\partial A}{\partial\Sigma_\mu}\mathbf{L}_\mu B + \frac{\partial B}{\partial\Sigma_\mu}\mathbf{L}_\mu A\\ &= -\widehat{\Sigma}_\lambda C^\lambda{}_{\mu\nu}\frac{\partial A}{\partial\widehat{\Sigma}_\mu}\frac{\partial B}{\partial\widehat{\Sigma}_\nu} - \frac{\partial A}{\partial\widehat{\Sigma}_\mu}\mathbf{R}_\mu B + \frac{\partial B}{\partial\widehat{\Sigma}_\mu}\mathbf{R}_\mu A.\end{aligned}$$

In particular, when f depends only on q^μ, we have

$$\{\Sigma_\mu, f\} = -\mathbf{L}_\mu f, \qquad \{\widehat{\Sigma}_\mu, f\} = -\mathbf{R}_\mu f.$$

Exponential expression for $\mathbf{L}(k)$, $\mathbf{R}(k)$ read

$$F\left(k(q)g\right) = \exp\left(q^\mu\mathbf{L}_\mu\right)F, \qquad F\left(gk(q)\right) = \exp\left(q^\mu\mathbf{R}_\mu\right)F.$$

This is true for the restricted class of smooth F-s, but the left-hand sides are generally well defined. One can show that

$$\mathbf{L}_\mu = \Sigma^\alpha{}_\mu\frac{\partial}{\partial q^\alpha}, \qquad \mathbf{R}_\mu = \widehat{\Sigma}^\alpha{}_\mu\frac{\partial}{\partial q^\alpha}$$

where

$$\Sigma^\alpha{}_\mu\Omega^\mu{}_\beta = \delta^\alpha{}_\beta, \qquad \widehat{\Sigma}^\alpha{}_\mu\widehat{\Omega}^\mu{}_\beta = \delta^\alpha{}_\beta.$$

Quantum operators of physical Σ, $\widehat{\Sigma}$-quantities are dependent on the Planck constant

$$\boldsymbol{\Sigma}_\mu := \frac{\hbar}{\mathrm{i}}\mathbf{L}_\mu = \frac{\hbar}{\mathrm{i}}\Sigma^\alpha{}_\mu(q)\frac{\partial}{\partial q^\alpha}, \qquad \widehat{\boldsymbol{\Sigma}}_\mu := \frac{\hbar}{\mathrm{i}}\mathbf{R}_\mu = \frac{\hbar}{\mathrm{i}}\widehat{\Sigma}^\alpha{}_\mu(q)\frac{\partial}{\partial q^\alpha}.$$

Obviously, they are self-adjoint. The quantum Poisson bracket

$${}_Q\{\mathbf{F},\mathbf{G}\} = \frac{1}{\mathrm{i}\hbar}\left[\mathbf{F},\mathbf{G}\right] = \frac{1}{\mathrm{i}\hbar}\left(\mathbf{FG} - \mathbf{GF}\right)$$

for $\boldsymbol{\Sigma}_\mu$, $\widehat{\boldsymbol{\Sigma}}_\mu$ has the same algebraic structure as classical

$$\{\boldsymbol{\Sigma}_\mu, \boldsymbol{\Sigma}_\nu\}_Q = \boldsymbol{\Sigma}_\lambda C^\lambda{}_{\mu\nu}, \quad \left\{\widehat{\boldsymbol{\Sigma}}_\mu, \widehat{\boldsymbol{\Sigma}}_\nu\right\}_Q = -\boldsymbol{\Sigma}_\lambda C^\lambda{}_{\mu\nu}, \quad \left\{\boldsymbol{\Sigma}_\mu, \widehat{\boldsymbol{\Sigma}}_\nu\right\}_Q = 0.$$

Quantum operators of kinetic energy are given by

$$\begin{aligned}\mathbf{T}_{\text{left}} &= \frac{1}{2}\mathcal{L}^{\mu\nu}\widehat{\boldsymbol{\Sigma}}_\mu\widehat{\boldsymbol{\Sigma}}_\nu = -\frac{\hbar^2}{2}\mathcal{L}^{\mu\nu}\mathbf{R}_\mu\mathbf{R}_\nu\\ \mathbf{T}_{\text{right}} &= \frac{1}{2}\mathcal{R}^{\mu\nu}\boldsymbol{\Sigma}_\mu\boldsymbol{\Sigma}_\nu = -\frac{\hbar^2}{2}\mathcal{R}^{\mu\nu}\mathbf{L}_\mu\mathbf{L}_\nu.\end{aligned}$$

If G is semi-simple, then these models coincide when the Killing tensor

$$\gamma_{\mu\nu} = C^\alpha{}_{\beta\mu}C^\beta{}_{\alpha\nu}$$

is used as the metric tensor at the identity of G. Then

$$\Gamma_{\mu\nu}(q) = \gamma_{\alpha\beta}\Sigma^{\alpha}{}_{\mu}(q)\Sigma^{\beta}{}_{\nu}(q) = \gamma_{\alpha\beta}\widehat{\Sigma}^{\alpha}{}_{\mu}(q)\widehat{\Sigma}^{\beta}{}_{\nu}(q).$$

More precisely, this is true when

$$G = G_1 \times \ldots \times G_p = \times_{k=1}^{N} G_k$$

where $\Gamma(k)$ are simple and

$$\Gamma = \sum_{k=1}^{N} c_k \pi_k{}^{*}\Gamma(k) = c_1\pi_1{}^{*}\Gamma(1) + \ldots + c_N\pi_N{}^{*}\Gamma(N)$$

$\pi_k = G \to G_k$ is the natural projection, $\Gamma(k)$ is the Killing metric on G_k and c_k are constants.

2. Quantization of Affine Bodies

Let us now go to the general case of the quantized affinely-rigid body [4, 7, 10, 11, 15, 17, 18, 21]. In the classical part it was stated that the configuration space is $\mathrm{LI}(U,V) \times M$, where M is the physical space, V, U are translation spaces of the physical and material spaces M, N, and $\mathrm{LI}(U,V)$ is the manifold of linear isomorphisms from U onto V. The induced coordinates in the configuration space are $\left(x^i, \varphi^i{}_K\right)$. Any choice of coordinates identifies $Q \simeq \mathrm{GAffI}(N,M)$ with $\mathrm{LI}(U,V) \times M$, and consequently, with $\mathrm{GL}(n,\mathbb{R})\widetilde{\times}\mathbb{R}^n$. The most natural measures on $\mathrm{GL}(n,\mathbb{R})\widetilde{\times}\mathbb{R}^n$ and $\mathrm{GL}(n,\mathbb{R})$ seem to be a, l, where

$$\begin{aligned} \mathrm{d}a(\varphi,x) &= \mathrm{d}x^1\ldots\mathrm{d}x^n\mathrm{d}\varphi^1{}_1\ldots\mathrm{d}\varphi^n{}_n = \mathrm{d}l(\varphi)\mathrm{d}x^1\ldots\mathrm{d}x^n \\ \mathrm{d}l(\varphi) &= \mathrm{d}\varphi^1{}_1\ldots\mathrm{d}\varphi^n{}_n. \end{aligned}$$

They are not Haar measures. The latter ones are given by α, λ, where

$$\begin{aligned} \mathrm{d}\alpha(\varphi,x) &= (\det\varphi)^{-n-1}\mathrm{d}a(\varphi,x) \\ \mathrm{d}\lambda(\varphi) &= (\det\varphi)^{-n}\mathrm{d}l(\varphi). \end{aligned}$$

In practical calculations it is convenient to express them in terms of the two-polar decomposition

$$\mathrm{d}\lambda(\varphi) = \mathrm{d}\lambda(L,q,R) = \prod_{i\neq j}\left|\mathrm{sh}\left(q^i - q^j\right)\right|\mathrm{d}\nu(L)\mathrm{d}\nu(R)\mathrm{d}q^1\ldots\mathrm{d}q^n$$

where ν is the Haar measure on $\mathrm{SO}(n,\mathbb{R})$, or equivalently – on the manifold of orthonormal frames.

Similarly, one can show that

$$\mathrm{d}l(\varphi) = \mathrm{d}l(L,Q,R) = \prod_{i\neq j}\left|\left(Q^i + Q^j\right)\left(Q^i - Q^j\right)\right|\mathrm{d}\nu(L)\mathrm{d}\nu(R)\mathrm{d}Q^1\ldots\mathrm{d}Q^n.$$

We shall use the shortened notation

$$P_\lambda = \prod_{i\neq j} \left|\operatorname{sh}\left(q^i - q^j\right)\right|, \qquad P_l = \prod_{i\neq j} \left|\left(Q^i + Q^j\right)\left(Q^i - Q^j\right)\right|.$$

Then

$$\mathrm{d}\lambda(\varphi) = P_\lambda \mathrm{d}\nu(L)\mathrm{d}\nu(R)\mathrm{d}q^1 \ldots \mathrm{d}q^n, \qquad \mathrm{d}l(\varphi) = P_l \mathrm{d}\nu(L)\mathrm{d}\nu(R)\mathrm{d}Q^1 \ldots \mathrm{d}Q^n.$$

Switching out the dilatational variable, i.e., reducing to the subgroup $\mathrm{SL}(n, \mathbb{R})$, we obtain

$$\mathrm{d}\lambda_{\mathrm{SL}}(\varphi) = P_\lambda \mathrm{d}\nu(L)\mathrm{d}\nu(R)\delta\left(q^1 + \ldots + q^n\right)\mathrm{d}q^1 \ldots \mathrm{d}q^n.$$

The indices μ in Σ_μ, $\widehat{\Sigma}_\mu$ become now two-indices like $\left({}^a{}_b\right)$, $\left({}^A{}_B\right)$. Therefore, the laboratory and co-moving representations of affine spin become now operators

$$\mathbf{\Sigma}^a{}_b := \frac{\hbar}{\mathrm{i}}\mathbf{L}^a{}_b = \frac{\hbar}{\mathrm{i}}\varphi^a{}_K \frac{\partial}{\partial \varphi^b{}_K}, \qquad \widehat{\mathbf{\Sigma}}^A{}_B := \frac{\hbar}{\mathrm{i}}\mathbf{R}^A{}_B = \frac{\hbar}{\mathrm{i}}\varphi^m{}_B \frac{\partial}{\partial \varphi^m{}_A}.$$

Similarly, the spin and vorticity operators are given by

$$\mathbf{S}^a{}_b = \mathbf{\Sigma}^a{}_b - g^{ac} g_{bd} \mathbf{\Sigma}^d{}_c, \qquad \mathbf{V}^A{}_B = \widehat{\mathbf{\Sigma}}^A{}_B - \eta^{AC}\eta_{BD}\widehat{\mathbf{\Sigma}}^D{}_C.$$

When using the Lebesgue measure l, we must replace $\mathbf{\Sigma}_\mu$, $\widehat{\mathbf{\Sigma}}_\mu$ by

$$\mathbf{\Sigma}(l)^a{}_b = \mathbf{\Sigma}^a{}_b + \frac{\hbar n}{2\mathrm{i}}\delta^a{}_b, \qquad \widehat{\mathbf{\Sigma}}(l)^A{}_B = \widehat{\mathbf{\Sigma}}^A{}_B + \frac{\hbar n}{2\mathrm{i}}\delta^A{}_B.$$

Similarly, for the linear momentum in spatial and material representations we have

$$\mathbf{P}_a = \frac{\hbar}{\mathrm{i}}\frac{\partial}{\partial x^a}, \qquad \widehat{\mathbf{P}}_A = \frac{\hbar}{\mathrm{i}}\varphi^a{}_A \frac{\partial}{\partial x^a}.$$

They are interrelated through φ

$$\widehat{\mathbf{P}}_A = \varphi^a{}_A \mathbf{P}_a, \qquad \mathbf{P}_a = \varphi^{-1A}{}_a \widehat{\mathbf{P}}_A.$$

One can also introduce the translational and total affine momentum of the body with respect to some fixed spatial origin $\mathfrak{O} \in M$

$$\mathbf{\Lambda}[\mathfrak{O}]^i{}_j = x^i \mathbf{P}_j, \qquad \mathbf{J}[\mathfrak{O}]^i{}_j = \mathbf{\Lambda}[\mathfrak{O}]^i{}_j + \mathbf{\Sigma}^i{}_j.$$

They generate $\mathrm{GAff}(M)$ acting, e.g., through

$$\left(x^a, \varphi^a{}_A\right) \mapsto \left(L^a{}_b x^b, L^a{}_b \varphi^b{}_A\right).$$

Let us introduce the operator of canonical momentum conjugate to the "centre of mass" of logarithmic deformation invariants

$$\mathbf{p} = \frac{\hbar}{\mathrm{i}}\frac{\partial}{\partial q} = \mathbf{\Sigma}^a{}_a = \widehat{\mathbf{\Sigma}}^A{}_A.$$

The deviatoric, i.e., shear components of $\mathbf{\Sigma}$, $\widehat{\mathbf{\Sigma}}$ are given by

$$\boldsymbol{\sigma}^a{}_b := \mathbf{\Sigma}^a{}_b - \frac{1}{n}\,\mathbf{p}\,\delta^a{}_b, \qquad \widehat{\boldsymbol{\sigma}}^A{}_B := \widehat{\mathbf{\Sigma}}^A{}_B - \frac{1}{n}\,\mathbf{p}\,\delta^A{}_B.$$

Just like in classical theory, spin and vorticity are Hamiltonian generators of the left and right rotations of φ. For any functions F, H of the L, R-arguments we have

$$\begin{aligned} F\left(W(\mu)L\right) &= \left(\exp\left(\frac{\mathrm{i}}{2\hbar}\mu^i{}_j\mathbf{S}^j{}_i\right)F\right)(L) \\ H\left(T(\nu)R\right) &= \left(\exp\left(-\frac{\mathrm{i}}{2\hbar}\nu^A{}_B\mathbf{V}^B{}_A\right)H\right)(R) \end{aligned}$$

where the coefficients $\mu^i{}_j$, $\nu^A{}_B$ are g, η-skew-symmetric

$$\mu^i{}_j = -g^{ik}g_{jl}\mu^l{}_k, \qquad \nu^A{}_B = -\eta^{AC}\eta_{BD}\nu^D{}_C$$

$W(\mu), T(\nu)$ are finite transformations from $\mathrm{SO}(V,g)$, $\mathrm{SO}(U,\eta)$.

Other factors of the two-polar decomposition are unaffected. From the point of view of "rigid bodies" L, R are "spatially" rotated respectively in V and U. The corresponding "co-moving" components

$$\boldsymbol{\varrho}^a{}_b = L^a{}_iL^j{}_b\mathbf{S}^i{}_j, \qquad \boldsymbol{\tau}^a{}_b = -R^B{}_bR^a{}_A\mathbf{V}^A{}_B$$

generate right, i.e., material, rotations of the L, R-rigid bodies. Namely, for any $\omega^a{}_b$ satisfying

$$\omega^a{}_b = -\delta^{ac}\delta_{bd}\omega^d{}_c$$

the corresponding $Z(\omega) \in \mathrm{SO}(n,\mathbb{R})$ acts on the L, R dependence as follows

$$\begin{aligned} F\left(LZ(\omega)\right) &= \left(\exp\left(\frac{\mathrm{i}}{2\hbar}\omega^a{}_b\boldsymbol{\varrho}^b{}_a\right)F\right)(L) \\ H\left(RZ(\omega)\right) &= \left(\exp\left(-\frac{\mathrm{i}}{2\hbar}\omega^a{}_b\boldsymbol{\tau}^b{}_a\right)H\right)(R). \end{aligned}$$

Just like in classical theory one achieves a partial diagonalization of the kinetic energy in terms of operators

$$\mathbf{M}^a{}_b = -\boldsymbol{\varrho}^a{}_b - \boldsymbol{\tau}^a{}_b, \qquad \mathbf{N}^a{}_b = \boldsymbol{\varrho}^a{}_b - \boldsymbol{\tau}^a{}_b.$$

In geodetic affinely-invariant models in two dimensions these quantities are constants of motion. For $n > 2$ it is no longer the case, but the Casimir invariants built of $\boldsymbol{\varrho}^a{}_b$, $\boldsymbol{\tau}^a{}_b$ are constants of motion. They are so even in non-geodetic case if the potential energy depends only on deformation invariants.

Casimir operators are given by

$$C(k) = \mathbf{\Sigma}^a{}_b\mathbf{\Sigma}^b{}_c\ldots\mathbf{\Sigma}^r{}_s\mathbf{\Sigma}^s{}_a = \widehat{\mathbf{\Sigma}}^A{}_B\widehat{\mathbf{\Sigma}}^B{}_C\ldots\widehat{\mathbf{\Sigma}}^R{}_S\widehat{\mathbf{\Sigma}}^S{}_A.$$

In particular for $k=2$

$$C(2) = \mathbf{\Sigma}^{a}{}_{b}\mathbf{\Sigma}^{b}{}_{a} = \widehat{\mathbf{\Sigma}}^{A}{}_{B}\widehat{\mathbf{\Sigma}}^{B}{}_{A}.$$

For skew-symmetric tensor operators like $\mathbf{S}^{a}{}_{b}$, $\mathbf{V}^{a}{}_{b}$ we change the normalization

$$\|\mathbf{S}\|^2 = -\frac{1}{2}\mathbf{S}^{a}{}_{b}\mathbf{S}^{b}{}_{a}, \qquad \|\mathbf{V}\|^2 = -\frac{1}{2}\mathbf{V}^{A}{}_{B}\mathbf{V}^{B}{}_{A}.$$

In analogy to classical formulas one can show that [18–20, 24, 25, 32, 34–36, 40, 41, 43, 44, 48, 49]

$$\begin{aligned}
\mathbf{T}^{\mathrm{aff-aff}}_{\mathrm{int}} &= \frac{1}{2A}\mathbf{C}(2) - \frac{B}{2A(A+nB)}\mathbf{p}^2\\
\mathbf{T}^{\mathrm{met-aff}}_{\mathrm{int}} &= \frac{1}{2\alpha}\mathbf{C}(2) + \frac{1}{2\beta}\mathbf{p}^2 + \frac{1}{2\mu}\|\mathbf{S}\|^2\\
\mathbf{T}^{\mathrm{aff-met}}_{\mathrm{int}} &= \frac{1}{2\alpha}\mathbf{C}(2) + \frac{1}{2\beta}\mathbf{p}^2 + \frac{1}{2\mu}\|\mathbf{V}\|^2
\end{aligned}$$

with the same meaning of symbols as in the classical part

$$\alpha = I + A, \qquad \beta = -\frac{(I+A)(I+A+nB)}{B}, \qquad \mu = \frac{I^2 - A^2}{I}.$$

In certain formulas it is convenient to separate the shear and dilatational phenomena

$$\begin{aligned}
\mathbf{T}^{\mathrm{aff-aff}}_{\mathrm{int}} &= \frac{1}{2A}\mathbf{C}_{\mathrm{SL}(n,\mathbb{R})}(2) + \frac{1}{2n(A+nB)}\mathbf{p}^2\\
\mathbf{T}^{\mathrm{met-aff}}_{\mathrm{int}} &= \frac{1}{2(I+A)}\mathbf{C}_{\mathrm{SL}(n,\mathbb{R})}(2) + \frac{1}{2n(I+A+nB)}\mathbf{p}^2 + \frac{I}{2(I^2-A^2)}\|\mathbf{S}\|^2\\
\mathbf{T}^{\mathrm{aff-met}}_{\mathrm{int}} &= \frac{1}{2(I+A)}\mathbf{C}_{\mathrm{SL}(n,\mathbb{R})}(2) + \frac{1}{2n(I+A+nB)}\mathbf{p}^2 + \frac{I}{2(I^2-A^2)}\|\mathbf{V}\|^2.
\end{aligned}$$

The Peter-Weyl decomposition of wave functions is given by

$$\Psi(\varphi) = \Psi(L,D,R) = \sum_{\alpha,\beta\in\Omega}\sum_{m,n=1}^{N(\alpha)}\sum_{k,l=1}^{N(\beta)} D^{\alpha}_{mn}(L) f^{\alpha\beta}_{\substack{nk\\ml}}(D) D^{\beta}_{kl}\left(R^{-1}\right).$$

Here Ω is the set of equivalence classes of irreducible unitary representations of $\mathrm{SO}(n,\mathbb{R})$ and $N(\alpha)$ is their dimension.

The two-polar decomposition is non-unique. Let $W \in \mathrm{SO}(n,\mathbb{R})$ has in every row and column exactly one ± 1 element and nulls besides. Then

$$LWDW^{-1}R^{-1} = LD_{\mathrm{perm}}R^{-1}$$

where $D_{\rm perm}$ is diagonal and differs from D by the permutation of diagonal elements. So, we must have

$$f^{\alpha\beta}_{\substack{nk\\ml}}\left(q^{\pi_W(1},\ldots,q^{n)}\right)=\sum_{r=1}^{N(\alpha)}\sum_{s=1}^{N(\beta)}D^{\alpha}_{nr}(W)f^{\alpha\beta}_{\substack{rs\\ml}}\left(q^1,\ldots,q^n\right)D^{\beta}_{sk}(W)$$

for any matrix W of the above form.

The same is true on the subsets $M^{(k;p_1,\ldots,p_k)}\subset \mathrm{SO}(n,\mathbb{R})\times\mathbb{R}\times\mathrm{SO}(n,\mathbb{R})$, where there is a coincidence between some of $\left(q^1,\ldots,q^n\right)$. Then W contains some continuous part. The special and simplest case is the total degeneracy when $D=\lambda I_n$. Then L, R separately are not determined and only LR^{-1} is well defined.

If α, β, m, l are kept fixed, then we can omit the symbols m, l and just write

$$\Psi(\varphi)=\Psi^{\alpha\beta}_{ml}(L,D,R)=\sum_{n=1}^{N(\alpha)}\sum_{k=1}^{N(\beta)}D^{\alpha}_{mn}(L)f^{\alpha\beta}_{nk}(D)D^{\beta}_{kl}\left(R^{-1}\right).$$

Obviously, D^α are $N(\alpha)\times N(\alpha)$ quadratic matrices and $f^{\alpha\beta}$ are $N(\alpha)\times N(\beta)$ matrices depending on deformation invariants $D(q)$.

Let us fix our attention on the physical case $n=3$. Then $\omega^a{}_b$ is expressed by the rotation vector $\overline{k}$ – canonical coordinates of the first kind

$$\begin{aligned}\omega^a{}_b=-\varepsilon^a{}_{bc}k^c,&\qquad k^a=-\frac{1}{2}\varepsilon^a{}_b{}^c\omega^b{}_c\\ k\in[0,\pi]-\mathrm{SO}(3,\mathbb{R}),&\qquad k\in[0,2\pi]-\mathrm{SU}(2)=\overline{\mathrm{SO}(3,\mathbb{R})}\end{aligned}$$

where $\overline{n}=\overline{k}/k$ is a rotation axis.

The generated finite rotations are given by exponentials

$$W(\overline{k})=\exp\left(k^aE_a\right)=\sum_{m=0}^{\infty}\frac{1}{m!}\left(k^aE_a\right)^m,\qquad \left(E_a\right)^b{}_c=-\varepsilon_a{}^b{}_c$$

or explicitly

$$\begin{aligned}W(\overline{k})\overline{u}&=\overline{u}+\overline{k}\times\overline{u}+\frac{1}{2}\overline{k}\times\left(\overline{k}\times\overline{u}\right)+\ldots\\ W(\overline{k})^a{}_b&=\cos k\;\delta^a{}_b+(1-\cos k)\frac{k^ak_b}{k^2}+\sin k\;\varepsilon^a{}_{bc}\frac{k^c}{k}.\end{aligned}$$

Generators of the left and right translations are given by

$$\begin{aligned}\mathbf{L}_a&=\frac{k_a}{k}\frac{\partial}{\partial k}-\frac{1}{2}\mathrm{ctg}\frac{k}{2}\varepsilon_{ab}{}^ck^b\mathbf{D}_c+\frac{1}{2}\mathbf{D}_a\\ \mathbf{R}_a&=\frac{k_a}{k}\frac{\partial}{\partial k}-\frac{1}{2}\mathrm{ctg}\frac{k}{2}\varepsilon_{ab}{}^ck^b\mathbf{D}_c-\frac{1}{2}\mathbf{D}_a\end{aligned}$$

where $\mathbf{D}$ are generators of inner automorphisms

$$\mathbf{D}_a = \mathbf{L}_a - \mathbf{R}_a = \varepsilon_{ab}{}^c k^b \frac{\partial}{\partial k^c}.$$

The following holds

$$W(\pi\overline{n}) = W(-\pi\overline{n}) = W(\pi\overline{n})^{-1}$$

so, for any $\overline{n}$ $W(\pi\overline{n})$ are square roots of identity.
The covering group $\mathrm{Spin}(3) \simeq \mathrm{SU}(2)$ is parameterized by $\overline{k}$ with $k \in [0, 2\pi]$

$$u(\overline{k}) = \exp\left(k^a e_a\right) = \cos\frac{k}{2} I_2 - \frac{k^a}{k}\sin\frac{k}{2}\mathrm{i}\sigma_a$$

$e_a = \sigma_a/2\mathrm{i}$ – generators of $\mathrm{SU}(2)$.
Now $u(\pi\overline{n}) \neq u(-\pi\overline{n})$, but $u(2\pi\overline{n}) = -u(\overline{n})$.
Casimir invariants are given by

$$\mathbf{C}_{\mathrm{SO}(V,g)}(2) = \mathbf{S}_1^2 + \mathbf{S}_2^2 + \mathbf{S}_3^2, \qquad \mathbf{C}_{\mathrm{SO}(U,\eta)}(2) = \mathbf{V}_1^2 + \mathbf{V}_2^2 + \mathbf{V}_3^2.$$

For $n = 3$ the family of Casimirs begins and terminates at $p = 2$. The Haar measure is proportional to

$$\mathrm{d}\mu(\overline{k}) = \frac{4}{k^2}\sin^2\frac{k}{2}\mathrm{d}_3\overline{k} = 4\sin^2\frac{k}{2}\sin\vartheta\mathrm{d}k\mathrm{d}\vartheta\mathrm{d}\varphi$$

for both $\mathrm{SO}(3,\mathbb{R})$ and $\mathrm{SU}(2)$. But if we wish to normalize the measure to unity, then some normalization constant must appear. Otherwise $\mathrm{SU}(2)$ has the twice larger volume than $\mathrm{SO}(3,\mathbb{R})$, what is, by the way, relatively sensible.
The Peter-Weyl theorem becomes then

$$\Psi(\varphi) = \Psi(L, D, R) = \sum_{s,j=0}^{\infty} \sum_{m,n=-s}^{s} \sum_{k,l=-j}^{j} D^s_{mn}(L) f^{sj}_{\substack{nk\\ml}}(D) D^j_{kl}\left(R^{-1}\right)$$

or with fixed values of m, l, s, j

$$\Psi^{sj}_{ml}(L, D, R) = \sum_{n=-s}^{s} \sum_{k=-j}^{j} D^s_{mn}(L) f^{sj}_{nk}(D) D^j_{kl}\left(R^{-1}\right).$$

They satisfy eigenequations of rotational Casimirs

$$\|\mathbf{S}^2\|\Psi^{sj}_{ml} = \hbar^2 s(s+1)\Psi^{sj}_{ml}, \qquad \|\mathbf{V}^2\|\Psi^{sj}_{ml} = \hbar^2 j(j+1)\Psi^{sj}_{ml}.$$

And traditionally one uses eigenstates of $\|\mathbf{S}^2\|$, $\|\mathbf{V}^2\|$

$$\mathbf{S}_3\Psi^{sj}_{ml} = \hbar m\Psi^{sj}_{ml}, \qquad \mathbf{V}_3\Psi^{sj}_{ml} = \hbar l\Psi^{sj}_{ml}.$$

Similarly for $\boldsymbol{\varrho}_3$, $\boldsymbol{\tau}_3$

$$\boldsymbol{\varrho}_3 \Psi^{sj}_{\substack{ml\\nk}} = \hbar n \Psi^{sj}_{\substack{ml\\nk}}, \qquad \boldsymbol{\tau}_3 \Psi^{sj}_{\substack{ml\\nk}} = \hbar k \Psi^{sj}_{\substack{ml\\nk}}.$$

On $\mathrm{GL}^+(3,\mathbb{R})$ s, j are non-negative integers and m, l, n, k jump by one from $-s$, $-j$ to s, j. But something similar may be done on $\overline{\mathrm{GL}^+(3,\mathbb{R})}$. One begins with $\mathrm{SU}(2) \times \mathbb{R}^3 \times \mathrm{SU}(2)$ – the analog of the two-polar representation

$$\Psi(u,q,v) = \sum_{s,j \in \mathbb{N}/2 \cup \{0\}}^{\infty} \sum_{m,n=-s}^{s} \sum_{k,l=-j}^{j} D^s_{mn}(u) f^{sj}_{\substack{nk\\ml}}(q) D^j_{kl}\left(v^{-1}\right)$$

where with fixed s, j other quantum numbers jump by one under the summation sign. But the summation must be restricted only to two disjoint subspaces: one with both s, j half-integer and the other one with integers. In any case this must be so if $\overline{\psi}\psi$ is to be projectable onto $\mathrm{GL}^+(3,\mathbb{R})$ (incidentally, it is not quite clear if it must be so) [45–47, 50–53].

3. Affine and Euclidean Models of Kinetic Energy in Terms of the Two-Polar Splitting

Let us quote the explicit expressions for the highly (affinely) invariant kinetic energy operators.

For models of internal kinetic energy left- and right-affinely invariant we have

$$\begin{aligned}\mathbf{T}^{\mathrm{aff-aff}}_{\mathrm{int}} &= -\frac{\hbar^2}{2A}\mathbf{D}_\lambda + \frac{\hbar^2 B}{2A(A+nB)}\frac{\partial^2}{\partial q^2} \\ &\qquad + \frac{1}{32A}\sum_{a,b}\frac{(\mathbf{M}^a{}_b)^2}{\mathrm{sh}^2\frac{q^a-q^b}{2}} - \frac{1}{32A}\sum_{a,b}\frac{(\mathbf{N}^a{}_b)^2}{\mathrm{ch}^2\frac{q^a-q^b}{2}}\end{aligned}$$

where, however, something classically unexpected appears

$$\mathbf{D}_\lambda = \frac{1}{P_\lambda}\sum_a \frac{\partial}{\partial q^a} P_\lambda \frac{\partial}{\partial q^a} = \sum_a \frac{\partial^2}{\partial (q^a)^2} + \sum_a \frac{\partial \ln P_\lambda}{\partial q^a}\frac{\partial}{\partial q^a}.$$

The "naively" expected term $\sum_a \partial^2/\partial(q^a)^2$ appears when we substitute

$$\varphi = \sqrt{P_\lambda}\,\Psi.$$

But this is for the price of additional "bad" potential $\widetilde{\mathbf{V}}$

$$-\frac{\hbar^2}{2A}\widetilde{D} = -\frac{\hbar^2}{2A}\sum_a \frac{\partial^2}{\partial(q^a)^2} + \widetilde{\mathbf{V}}$$

$$\widetilde{\mathbf{V}} = -\frac{\hbar^2}{2A}\frac{1}{P_\lambda{}^2} + \frac{\hbar^2}{4A}\frac{1}{P_\lambda}\sum_a\left(\frac{\partial P_\lambda}{\partial q^a}\right)^2.$$

For the internal models right-affinely, left-metrically invariant, and conversely, left-affinely, right-metrically invariant we have respectively

$$\mathbf{T}^{\mathrm{met-aff}}_{\mathrm{int}} = -\frac{\hbar^2}{2\lambda}\mathbf{D}_\lambda - \frac{\hbar^2}{2\beta}\frac{\partial^2}{\partial q^2} + \frac{1}{32\alpha}\sum_{a,b}\frac{\left(\mathbf{M}^a{}_b\right)^2}{\mathrm{sh}^2\frac{q^a-q^b}{2}} - \frac{1}{32\alpha}\sum_{a,b}\frac{\left(\mathbf{N}^a{}_b\right)^2}{\mathrm{ch}^2\frac{q^a-q^b}{2}} + \frac{1}{2\mu}\|\mathbf{S}\|^2$$

$$\mathbf{T}^{\mathrm{aff-met}}_{\mathrm{int}} = -\frac{\hbar^2}{2\lambda}\mathbf{D}_\lambda - \frac{\hbar^2}{2\beta}\frac{\partial^2}{\partial q^2} + \frac{1}{32\alpha}\sum_{a,b}\frac{\left(\mathbf{M}^a{}_b\right)^2}{\mathrm{sh}^2\frac{q^a-q^b}{2}} - \frac{1}{32\alpha}\sum_{a,b}\frac{\left(\mathbf{N}^a{}_b\right)^2}{\mathrm{ch}^2\frac{q^a-q^b}{2}} + \frac{1}{2\mu}\|\mathbf{V}\|^2$$

where α, β, μ are given by the previous formulas.
For the doubly isotropic d'Alembert model with the scalar inertia I we obtain

$$\mathbf{T}^{\mathrm{d.A.}}_{\mathrm{int}} = -\frac{\hbar^2}{2I}\mathbf{D}_l + \frac{1}{8I}\sum_{a,b}\frac{\left(\mathbf{M}^a{}_b\right)^2}{\left(Q^a-Q^b\right)^2} + \frac{1}{8I}\sum_{a,b}\frac{\left(\mathbf{N}^a{}_b\right)^2}{\left(Q^a+Q^b\right)^2}$$

with $\mathbf{D}_l$ given by

$$\mathbf{D}_l = \frac{1}{P_l}\sum_a\frac{\partial}{\partial Q^a}P_l\frac{\partial}{\partial Q^a} = \sum_a\frac{\partial^2}{\partial\left(Q^a\right)^2} + \sum_a\frac{\partial\ln P_l}{\partial Q^a}\frac{\partial}{\partial Q^a}.$$

Then again the substitution

$$\varphi = \sqrt{P_l}\,\Psi \tag{1}$$

eliminates the first-order derivatives but introduces a hardly treatable potential

$$\widetilde{V}_l = -\frac{\hbar}{2I}\frac{1}{P_l^2} + \frac{\hbar^2}{4I}\frac{1}{P_l}\sum_a\left(\frac{\partial P_l}{\partial Q^a}\right)^2.$$

Although the kinetic energy operator may be in the d'Alembert case expressed by the usual Laplace operator

$$\mathbf{T}^{\mathrm{d.A.}} = -\frac{\hbar^2}{2I}\Delta^{n^2} = -\frac{\hbar^2}{2I}\sum_{i,A}\frac{\partial^2}{\partial\left(\varphi^i{}_A\right)^2}$$

this is useless because the geodetic models predict infinite motion, and to be physically admissible, they must be modified by the potential term $V\left(Q^1,\ldots,Q^n\right)$. And then only curvilinear coordinates, e.g., polar or two-polar ones are useful and everything goes back to the previous treatment.

Similarly, for affinely-invariant models one can modify $\mathbf{T}^{\mathrm{aff-aff}}$, $\mathbf{T}^{\mathrm{met-aff}}$, and $\mathbf{T}^{\mathrm{aff-met}}$ by the doubly isotropic potential correction $\mathbf{V}\left(q^1,\ldots,q^n\right)$.

The matrix generators of D^α will be denoted by M^α, so that for

$$W(\omega)=\exp\left(\frac{1}{2}\,\omega^a{}_b E^b{}_a\right)$$

we have

$$D^\alpha(\omega)=\exp\left(\frac{1}{2}\,\omega^a{}_b M^{\alpha b}{}_a\right).$$

If $n=3$, then

$$D^j(\omega)=\exp\left(\omega^a M^j{}_a\right).$$

Obviously, then

$$\left[M^j{}_a, M^j{}_b\right]=-\varepsilon_{ab}{}^c M^j{}_c.$$

Let us introduce hermitian matrices of angular momenta

$$S^{\alpha a}{}_b=\frac{\hbar}{\mathrm{i}}M^{\alpha a}{}_b,\qquad S^j{}_a=\frac{\hbar}{\mathrm{i}}M^j{}_a.$$

Their Poisson brackets have the form

$$\frac{1}{\mathrm{i}\hbar}\left[S^j{}_a, S^j{}_b\right]=\varepsilon_{ab}{}^c S^j{}_c.$$

The advantage of their use is that differential operators $\boldsymbol{\rho}^a{}_b$, $\boldsymbol{\tau}^a{}_b$, $\mathbf{M}^a{}_b$, $\mathbf{N}^a{}_b$ are algebraized. Let us introduce the symbols

$$\overrightarrow{S^{\alpha}}{}^a{}_b f^{\alpha\beta}:=S^{\alpha a}{}_b f^{\alpha\beta},\qquad \overleftarrow{S}^{\beta a}{}_b f^{\alpha\beta}:=f^{\alpha\beta}S^{\beta a}{}_b.$$

The affinely-invariant and even rotationally-invariant Schrödinger equation

$$\mathbf{H}\Psi=E\Psi$$

splits then into family of equations

$$H^{\alpha\beta}f^{\alpha\beta}=E^{\alpha\beta}f^{\alpha\beta}$$

where for any $\alpha,\beta\in\Omega$, $f^{\alpha\beta}$ is again the $N(\alpha)\times N(\beta)$ matrix depending on q^a. The problem is $N(\alpha)\times N(\beta)$-fold degenerate.

$\mathbf{H}^{\alpha\beta}$ is an $N(\alpha)\times N(\beta)$ matrix the elements of which are differential operators

$$\mathbf{H}^{\alpha\beta}=\mathbf{T}^{\alpha\beta}+\mathbf{V}.$$

D^α are irreducible, therefore the Casimir matrices

$$C^\alpha(p)^a{}_z := \underbrace{S^{\alpha a}{}_b S^{\alpha b}{}_c \ldots S^{\alpha u}{}_w S^{\alpha w}{}_z}_{p \text{ factors}}$$

reduce on them to ones proportional to $I_{N(\alpha)}$

$$C^\alpha(p) = \left(\frac{\hbar}{\mathrm{i}}\right)^p C(\alpha, p)\mathrm{I}_{N(\alpha)}.$$

One can show that the Schrödinger equations reduce to the above family with the following quantum counterparts of the classical kinetic energy

$$\begin{aligned}
\mathbf{T}^{\alpha\beta} f^{\alpha\beta} &= -\frac{\hbar^2}{2A}\mathbf{D}_\lambda f^{\alpha\beta} + \frac{1}{32A}\sum_{a,b}\frac{\left(\overleftarrow{S^\beta}{}^a{}_b - \overrightarrow{S^\alpha}{}^a{}_b\right)^2}{\mathrm{sh}^2\frac{q^q-q^b}{2}} f^{\alpha\beta} \\
&\quad - \frac{1}{32A}\sum_{a,b}\frac{\left(\overleftarrow{S^\beta}{}^a{}_b + \overrightarrow{S^\alpha}{}^a{}_b\right)^2}{\mathrm{ch}^2\frac{q^q-q^b}{2}} f^{\alpha\beta} + \frac{\hbar^2 B}{2A(A+nB)}\frac{\partial^2}{\partial q^2} f^{\alpha\beta}
\end{aligned}$$

$$\begin{aligned}
\mathbf{T}^{\alpha\beta} f^{\alpha\beta} &= -\frac{\hbar^2}{2\alpha}\mathbf{D}_\lambda f^{\alpha\beta} + \frac{1}{32\alpha}\sum_{a,b}\frac{\left(\overleftarrow{S^\beta}{}^a{}_b - \overrightarrow{S^\alpha}{}^a{}_b\right)^2}{\mathrm{sh}^2\frac{q^q-q^b}{2}} f^{\alpha\beta} \\
&\quad - \frac{1}{32\alpha}\sum_{a,b}\frac{\left(\overleftarrow{S^\beta}{}^a{}_b + \overrightarrow{S^\alpha}{}^a{}_b\right)^2}{\mathrm{ch}^2\frac{q^q-q^b}{2}} f^{\alpha\beta} - \frac{\hbar^2}{2\beta}\frac{\partial^2}{\partial q^2} f^{\alpha\beta} - \frac{\hbar^2}{2\mu}C(\alpha,2) f^{\alpha\beta}
\end{aligned} \tag{2}$$

$$\begin{aligned}
\mathbf{T}^{\alpha\beta} f^{\alpha\beta} &= -\frac{\hbar^2}{2\alpha}\mathbf{D}_\lambda f^{\alpha\beta} + \frac{1}{32\alpha}\sum_{a,b}\frac{\left(\overleftarrow{S^\beta}{}^a{}_b - \overrightarrow{S^\alpha}{}^a{}_b\right)^2}{\mathrm{sh}^2\frac{q^q-q^b}{2}} f^{\alpha\beta} \\
&\quad - \frac{1}{32\alpha}\sum_{a,b}\frac{\left(\overleftarrow{S^\beta}{}^a{}_b + \overrightarrow{S^\alpha}{}^a{}_b\right)^2}{\mathrm{ch}^2\frac{q^q-q^b}{2}} f^{\alpha\beta} - \frac{\hbar^2}{2\beta}\frac{\partial^2}{\partial q^2} f^{\alpha\beta} - \frac{\hbar^2}{2\mu}C(\beta,2) f^{\alpha\beta}.
\end{aligned} \tag{3}$$

One must not confuse the representation labels α, β with the inverses of the multiplicative constants. We apologise for this inconvenience. It is seen that there is no very essential difference between those three expressions; only one in multiplicative constants and with the use of spin and vorticity Casimirs. Those formulas are valid for any spatial dimension n. In the directly physical case $n = 3$ we have obviously $\alpha = s = 0, 1/2, 1, \ldots \in \mathbb{N}/2 \cup \{0\}$, $\beta = j = 0, 1/2, 1, \ldots \in \mathbb{N}/2 \cup \{0\}$ when we admit half-integer values of angular momenta and vorticity. If we admit only integer values, then obviously $s, j \in \mathbb{N} \cup \{0\}$. Obviously, in three dimensions we have $C(2,2) = s(s+1)$, $C(j) = j(j+1)$. Then the constant terms in the formulas (2), (3) are simply given by $\hbar^2 s(s+1)/2\mu$, $\hbar^2 j(j+1)/2\mu$. Those corrections to the affine-affine model are very interesting and have the structure interesting for any physicist. The term $\hbar^2 s(s+1)/2\mu$ is interesting as the rotational connection to the situation when the purely deformative part is established and later on excited to quicker rotations. From this point of view the correction term $\hbar^2 j(j+1)/2\mu$ in (3) is perhaps even more interesting because it may be interpreted as a kind of internal quantum term following from the $\mathrm{SO}(3,\mathbb{R})$-group or its covering $\mathrm{SU}(2)$. This might be something like the isospin. To combine them, i.e., to obtain some combination of terms $\hbar^2 s(s+1)/2\mu$, $\hbar^2 j(j+1)/2\mu$, we should modify more deeply the primary affine-affine model, e.g., to use the quantization of kinetic energies like (171), (172) from the classical part of this text.

Let us observe that the use of the two-polar description together with the Weyl-Peter theorem enables one to simplify the expression for the scalar product, reducing it to the integration over the q^i-variables and the series summation over discrete variables. Namely, if we take two wave functions Ψ_1, Ψ_2 with the deformation profiles f_1, f_2, then one can easily show that

$$\langle \Psi_1 \mid \Psi_2 \rangle = \sum_{\alpha,\beta\in\Omega} \frac{1}{N(\alpha)N(\beta)} \int \sum_{n,m=1}^{N(\alpha)} \sum_{k,l=1}^{N(\beta)} \overline{f_1}{}^{\alpha\beta}_{\substack{nk\\ml}} f_2{}^{\alpha\beta}_{\substack{nk\\ml}} P_\lambda \mathrm{d}q^1 \dots \mathrm{d}q^n.$$

When we restrict ourselves to the subspace of wave functions with fixed labels α, β, m, l and use the simplified $N(\alpha)\times N(\beta)$-matrix amplitudes

$$\Psi^{\alpha\beta}\left(L; q^1,\dots,q^n; R\right) = D^\alpha(l) f^{\alpha\beta}\left(q^1,\dots,q^n\right) D^\beta(R^{-1}) \tag{4}$$

this scalar product may be reduced to

$$\begin{aligned}\left\langle {\Psi_1}^{\alpha\beta} \mid {\Psi_2}^{\alpha\beta} \right\rangle \;=\; & \frac{1}{N(\alpha)N(\beta)} \int \mathrm{Tr}\left(f_1^{\alpha\beta+}\left(q^1,\dots,q^n\right) f_2^{\alpha\beta}\left(q^1,\dots,q^n\right)\right) \\ & \cdot P_\lambda\left(q^1,\dots,q^n\right) \mathrm{d}q^1 \dots \mathrm{d}q^n.\end{aligned}$$

For the general case (4) may be written as

$$\langle \Psi_1 \mid \Psi_2 \rangle = \sum_{\alpha,\beta\in\Omega} \frac{1}{N(\alpha)N(\beta)} \int \mathrm{Tr}\left(f_1^{\alpha\beta+} f_2^{\alpha\beta}\right) P_\lambda \mathrm{d}q^1 \dots \mathrm{d}q^n. \tag{5}$$

where, obviously

$$\mathrm{Tr}\left(f_1^{\alpha\beta+} f_2^{\alpha\beta}\right) = \sum_{n,m=1}^{N(\alpha)} \sum_{k,l=1}^{N(\beta)} \overline{f_1}{}^{\alpha\beta}_{\substack{nk\\ml}} f_2{}^{\alpha\beta}_{\substack{nk\\ml}}.$$

The weight factor P_λ may be eliminated from (5) by (1).

Let us mention again the usual d'Alembert models. Now for the isotropic inertial tensor and for the doubly isotropic potential energy we can also state that the

Schrödinger equation

$$\mathbf{H}\Psi = E\Psi$$

reduces to the family

$$\mathbf{H}^{\alpha\beta} f^{\alpha\beta} = E^{\alpha\beta} f^{\alpha\beta}$$

where

$$\begin{aligned}\mathbf{H}^{\alpha\beta} f^{\alpha\beta} &= -\frac{\hbar^2}{2I}\mathbf{D}_l f^{\alpha\beta} + \frac{1}{8I}\sum_{a,b}\frac{\left(\overleftarrow{S^{\beta}}^{a}{}_{b} - \overrightarrow{S^{\alpha}}^{a}{}_{b}\right)}{\left(Q^a - Q^b\right)^2} f^{\alpha\beta}\\ &+\frac{1}{8I}\sum_{a,b}\frac{\left(\overleftarrow{S^{\beta}}^{a}{}_{b} + \overrightarrow{S^{\alpha}}^{a}{}_{b}\right)^2}{\left(Q^a + Q^b\right)^2} f^{\alpha\beta} + V\left(Q^1, \ldots, Q^n\right) f^{\alpha\beta}.\end{aligned}$$

It is clear that without the potential term, i.e., when dealing with the geodetic model, all motions are infinite and there are no elastic vibrations, just like in the corresponding classical theory.

We have seen that in classical mechanics the geodetic affinely-invariant models on $\mathrm{SL}(n, \mathbb{R})$ may describe elastic vibrations. Moreover, there exists a sharp threshold between finite vibrations and infinite escaping motions. It is given by some relationship between spin and vorticity. In $\mathrm{GL}(n, \mathbb{R})$ the same qualitative picture may be obtained by introducing some stabilizing dilatational potential. By analogy something similar exists in quantum theory. Let us consider this again in the special, particularly simple model in $n = 2$. The Haar measure on $\mathrm{GL}(2, \mathbb{R})$ may be expressed as

$$\mathrm{d}\lambda\left(\alpha; q^1, q^2; \beta\right) = \left|\mathrm{sh}\left(q^1 - q^2\right)\right| \mathrm{d}\alpha\, \mathrm{d}\beta\, \mathrm{d}q^1 \mathrm{d}q^2$$

where, as usual q^1, q^2 are logarithmic deformation invariants and α, β are polar angles parametrizing respectively L and R in the two-polar decomposition. As usual we introduce new variables

$$q = \frac{1}{2}\left(q^1 + q^2\right), \qquad x = q^2 - q^1.$$

In certain problems it is also convenient to introduce the mixed angular variables

$$\gamma = \frac{1}{2}\left(\beta - \alpha\right), \qquad \delta = \frac{1}{2}\left(\beta + \alpha\right).$$

Therefore

$$\mathrm{d}\lambda\left(\alpha; q, x; \beta\right) = \left|\mathrm{sh}\, x\right| \mathrm{d}\alpha\, \mathrm{d}\beta\, \mathrm{d}q \mathrm{d}x, \qquad P_\lambda = \left|\mathrm{sh}\, x\right|.$$

According to the Peter-Weyl theorem, or more directly, to the Fourier theorem, we have the following expansion for our wave functions on $\mathrm{GL}(2, \mathbb{R})$

$$\Psi\left(\alpha; q, x; \beta\right) = \sum_{m,n\in\mathbb{Z}} f^{mn}(q, x)\mathrm{e}^{\mathrm{i}m\alpha}\mathrm{e}^{\mathrm{i}n\beta}.$$

For the model $T^{\mathrm{aff-aff}}_{\mathrm{int}}$ we have the following reduced expression for the kinetic energy $\mathbf{T}^{mn}$

$$\begin{aligned}\mathbf{T}^{mn} f^{mn} &= -\frac{\hbar^2}{A}\mathbf{D}_x f^{mn} - \frac{\hbar^2}{4(A+2B)}\frac{\partial^2 f^{mn}}{\partial q^2}\\ &\quad + \frac{\hbar^2 (n-m)^2}{16A^2 \mathrm{sh}^2 \frac{x}{2}} f^{mn} - \frac{\hbar^2 (n+m)^2}{16A^2 \mathrm{ch}^2 \frac{x}{2}} f^{mn}.\end{aligned} \tag{6}$$

For the metric-affine and affine-metric models $T^{\mathrm{met-aff}}_{\mathrm{int}}$, $T^{\mathrm{aff-met}}_{\mathrm{int}}$ we obtain respectively

$$\begin{aligned}\mathbf{T}^{mn} f^{mn} &= -\frac{\hbar^2}{I+A}\mathbf{D}_x f^{mn} - \frac{\hbar^2}{4(I+A+2B)}\frac{\partial^2 f^{mn}}{\partial q^2}\\ &\quad + \frac{\hbar^2 (n-m)^2}{16(I+A)\mathrm{sh}^2 \frac{x}{2}} f^{mn} - \frac{\hbar^2 (n+m)^2}{16(I+A)\mathrm{ch}^2 \frac{x}{2}} f^{mn} + \frac{I\hbar^2 m^2}{I^2 - A^2} f^{mn}\\ \mathbf{T}^{mn} f^{mn} &= -\frac{\hbar^2}{I+A}\mathbf{D}_x f^{mn} - \frac{\hbar^2}{4(I+A+2B)}\frac{\partial^2 f^{mn}}{\partial q^2}\\ &\quad + \frac{\hbar^2 (n-m)^2}{16(I+A)\mathrm{sh}^2 \frac{x}{2}} f^{mn} - \frac{\hbar^2 (n+m)^2}{16(I+A)\mathrm{ch}^2 \frac{x}{2}} f^{mn} + \frac{I\hbar^2 n^2}{I^2 - A^2} f^{mn}\end{aligned}$$

where

$$\mathbf{D}_x f^{mn} = \frac{1}{|\mathrm{sh}\, x|}\frac{\partial}{\partial x}\left(|\mathrm{sh}\, x|\,\frac{\partial f^{mn}}{\partial x}\right).$$

Of course, for the purely geodetic models on $\mathrm{GL}(2,\mathbb{R})$ the spectrum is continuous, because dilatational motion is free. To avoid this fact we must introduce to the Hamiltonian some dilatation-stabilizing potential $V_{\mathrm{dil}}(q)$. This may be either the potential well or some harmonic oscillator with large elastic constant. Of course, the problem is also explicitly separable for any potential of the form

$$V(q,x) = V_{\mathrm{dil}}(q) + V_{\mathrm{sh}}(x).$$

The corresponding solutions of the time-independent Schrödinger equation will be sought in the product form

$$f^{mn}(q,x) = \varphi^{mn}(q)\chi^{mn}(x).$$

It is interesting that there exists a discrete spectrum for χ-terms in $\mathrm{SL}(2,\mathbb{R})$ even in the purely geodetic models without any shear potential $V_{\mathrm{sh}}(x)$. This depends on the mutual relationship between "gyroscopic" quantum numbers m, n. If $|n-m| < |n+m|$, then the attractive ch^{-2}-term becomes dominant at large distances, when $|x| \to \infty$, and the spectrum for χ is then discrete. Conversely, it becomes continuous when $|n-m| > |n+m|$. For the affine-affine model (6) the spectrum is not bounded from below. Conversely, for the affine-metric and metric-affine models the kinetic energy may be bounded from below and so is the spectrum. This happens for certain open range of parameters I, A, B.

Similar phenomena hold for the dimension of space greater than two, because everything follows from the commutation rules (structure constants) of $\mathrm{SL}(n, \mathbb{R})$.

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