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A CLASSIFICATION OF QUADRATIC HAMILTON-POISSON SYSTEMS IN THREE DIMENSIONS

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Abstract. We classify homogeneous positive semidefinite quadratic Hamilton-Poisson systems on a certain subclass of three-dimensional Lie-Poisson spaces.

1. Introduction

The dual space of a Lie algebra admits a natural Poisson structure, namely the Lie-Poisson structure. Such structures, and more specifically quadratic Hamiltonian systems on these structures, form a natural setting for a variety of dynamical systems. Prevalent examples are Euler's classic equations for the rigid body, its extensions and its generalizations (see, e.g. [15–17, 22, 25, 28, 29]). In particular, a number of Lie-Poisson structures arise naturally in the study of optimal control problems (see e.g. [2–4, 9, 17, 26, 27]). The equivalence of quadratic Hamilton-Poisson systems on Lie-Poisson spaces has been considered only by a few authors ([5, 10, 11, 13, 28, 29]).

In the present paper, we consider quadratic Hamilton-Poisson systems on those three-dimensional Lie-Poisson spaces that admit a global Casimir function. (The spaces that do not admit a global Casimir function exhibit some degeneracies and need to be treated in a somewhat different manner.) Furthermore, we restrict to those systems that are both homogeneous and for which the underlying quadratic form is (positive) semidefinite. Such systems (usually on specific Lie-Poisson spaces) have been considered by several authors ([5–8, 28–30]). We address the equivalence of such systems. A classification (under linear equivalence) is obtained; a complete list of normal forms is exhibited. This is done in two parts. First we classify systems within the context of each three-dimensional Lie-Poisson space (making use of the Bianchi-Behr classification of three-dimensional Lie algebras). Thereafter we consider equivalences of systems on non-isomorphic Lie-Poisson spaces.

1.1. Lie-Poisson Spaces, Quadratic Systems and Linear Equivalence

Let $\mathfrak{g}$ be a (real) Lie algebra. The dual space $\mathfrak{g}^*$ admits a natural Poisson structure

$$\{F, G\}(p) = \Pi_p(\mathrm{d}F(p), \mathrm{d}G(p)) = -p([\mathrm{d}F(p), \mathrm{d}G(p)])$$

called the (*minus*) *Lie-Poisson structure* (cf [19, 22]). Here $p \in \mathfrak{g}^*$, $F, G \in C^\infty(\mathfrak{g}^*)$, and $\mathrm{d}F(p), \mathrm{d}G(p) \in \mathfrak{g}^{**} \cong \mathfrak{g}$. The Poisson space $(\mathfrak{g}^*, \{\cdot,\cdot\})$ is denoted by $\mathfrak{g}^*_-$.

To each function $H \in C^\infty(\mathfrak{g}^*_-)$, we associate a *Hamiltonian vector field* $\vec{H}$ on $\mathfrak{g}^*_-$ specified by $\vec{H}[F] = \{F, H\}$. A function $C \in C^\infty(\mathfrak{g}^*_-)$ is a **Casimir function** if $\{C, F\} = 0$ for all $F \in C^\infty(\mathfrak{g}^*_-)$. Two vector fields $\vec{F}$ and $\vec{G}$ (on $\mathfrak{g}^*_-$ and $\mathfrak{h}^*_-$, respectively) are **compatible** with a diffeomorphism $\phi : \mathfrak{g}^* \to \mathfrak{h}^*$ if $T_p\phi \cdot \vec{F} = \vec{G} \circ \phi$ (i.e., they are ϕ-related). The map ϕ establishes a one-to-one correspondence between the integral curves of $\vec{H}$ and $\vec{F}$. A linear map $\psi : \mathfrak{g}^*_- \to \mathfrak{h}^*_-$ is a *linear Poisson morphism* if $\{F, G\} \circ \psi = \{F \circ \psi, G \circ \psi\}$ for all $F, G \in C^\infty(\mathfrak{g}^*_-)$. Linear Poisson morphisms are exactly the dual maps of Lie algebra homomorphisms.

A quadratic Hamilton-Poisson system (on a Lie-Poisson space) is a pair $(\mathfrak{g}^*_-, H)$, where $H = H_{A,\mathcal{Q}} : \mathfrak{g}^*_- \to \mathbb{R}$, $p \mapsto p(A) + \mathcal{Q}(p)$. Here $A \in \mathfrak{g}$ and $\mathcal{Q}$ is a quadratic form on $\mathfrak{g}^*_-$. (When $\mathfrak{g}^*_-$ is fixed, $(\mathfrak{g}^*_-, H)$ is identified with H.) In this paper we consider only systems that are both *homogeneous* (i.e., $A = 0$) and for which the quadratic form $\mathcal{Q}$ is *positive semidefinite*. Two Hamilton-Poisson systems $(\mathfrak{g}^*_-, H)$ and $((\mathfrak{g}')^*_-, H')$ are said to be (linearly) *equivalent* if the associated Hamiltonian vector fields are compatible with a linear isomorphism. The following Hamilton-Poisson systems are all linearly equivalent to $H_\mathcal{Q}$

- $\mathfrak{E}1)$ $H_\mathcal{Q} \circ \psi$, where $\psi : \mathfrak{g}^*_- \to \mathfrak{g}^*_-$ is a linear Poisson automorphism
- $\mathfrak{E}2)$ $H_{r\mathcal{Q}}$, where $r \neq 0$
- $\mathfrak{E}3)$ $H_\mathcal{Q} + C$, where C is a Casimir function.

Given a basis (E_1, E_2, E_3) for a Lie algebra $\mathfrak{g}$, an element $p = p_1E_1^* + p_2E_2^* + p_3E_3^*$ expressed in the dual basis (E_1^*, E_2^*, E_3^*) will be written as a column matrix $p = (p_i)_{1\leq i\leq 3}$. A system $H_\mathcal{Q}$ (on $\mathfrak{g}^*_-$) is then represented as $H_\mathcal{Q}(p) = p^\top Q\, p$, where Q is a positive semidefinite 3×3 matrix. The equations of motion of a Hamiltonian H (on each of the respective associated Lie-Poisson spaces) take the form

$$\dot{p}_i = -p([E_i, \mathrm{d}H(p)]), \qquad i = 1, \ldots, n$$

or $\vec{H} = \Pi \cdot \nabla H$. Here Π is the Poisson matrix of $\mathfrak{g}^*_-$ (see, e.g. [20]) and ∇H is the naive gradient of H. For the sake of convenience, all linear maps will be identified with their corresponding matrices.

1.2. Three-Dimensional Lie-Poisson Spaces

The classification of three-dimensional Lie algebras is well known. We shall use an adaptation of the Bianchi-Behr enumeration (cf [18, 21, 23]). Any real three-dimensional Lie algebra is isomorphic to one of eleven types (in fact, there are nine algebras and two parametrized infinite families of algebras). In terms of an (appropriate) ordered basis (E_1, E_2, E_3), the commutation operation is given by

$$[E_2, E_3] = n_1 E_1 - \alpha\, E_2, \qquad [E_3, E_1] = \alpha\, E_1 + n_2 E_2, \qquad [E_1, E_2] = n_3 E_3.$$

The (Bianchi-Behr) structure parameters α, n_1, n_2, n_3 for each type are given in Table 1.

Table 1. Bianchi-Behr classification.

Type	Bianchi	α	n_1	n_2	n_3	Representatives
$3\mathfrak{g}_1$	I	0	0	0	0	$\mathbb{R}^3$
$\mathfrak{g}_{2.1} \oplus \mathfrak{g}_1$	III	1	1	-1	0	$\mathfrak{aff}(\mathbb{R}) \oplus \mathbb{R}$, $\mathfrak{g}^1_{3.4}$
$\mathfrak{g}_{3.1}$	II	0	1	0	0	$\mathfrak{h}_3$
$\mathfrak{g}_{3.2}$	IV	1	1	0	0	
$\mathfrak{g}_{3.3}$	V	1	0	0	0	
$\mathfrak{g}^0_{3.4}$	VI_0	0	1	-1	0	$\mathfrak{se}(1,1)$
$\mathfrak{g}^\alpha_{3.4}$	VI_α	$\alpha>0$ $\alpha\neq 1$	1	-1	0	
$\mathfrak{g}^0_{3.5}$	VII_0	0	1	1	0	$\mathfrak{se}(2)$
$\mathfrak{g}^\alpha_{3.5}$	VII_α	$\alpha>0$	1	1	0	
$\mathfrak{g}_{3.6}$	$VIII$	0	1	1	-1	$\mathfrak{sl}(2,\mathbb{R})$, $\mathfrak{so}(2,1)$
$\mathfrak{g}_{3.7}$	IX	0	1	1	1	$\mathfrak{su}(2)$, $\mathfrak{so}(3)$

Accordingly, any quadratic Hamilton Poisson system $(\mathfrak{g}^*_-, H_Q)$ is equivalent to a system on one of the corresponding eleven types of Lie-Poisson spaces.

Note 1. *We find it convenient to use a basis for* $\mathfrak{g}_{2.1} \oplus \mathfrak{g}_1$ *different from the one listed in* Table 1. *More precisely, we use the basis* $E_1' = \frac{1}{2}(E_1 - E_2)$, $E_2' = -\frac{1}{2}E_3$, $E_3' = \frac{1}{2}(E_1 + E_2)$; *the only nonzero commutator is then* $[E_1', E_2'] = E_1'$.

A standard computation yields the group of linear Poisson (or dually Lie algebra) automorphisms for each Lie-Poisson space (see, e.g. [14]). The Casimir functions for real algebras of dimension up to five were obtained by Patera *et al* in [24]. Of the above eleven types, only $\mathfrak{g}_{2.1} \oplus \mathfrak{g}_1$, $\mathfrak{g}_{3.1}$, $\mathfrak{g}_{3.4}^0$, $\mathfrak{g}_{3.5}^0$, $\mathfrak{g}_{3.6}$ and $\mathfrak{g}_{3.7}$ admit global Casimir functions. Table 3 (appended) lists the Poisson matrix, the linear Poisson automorphisms, and the Casimir functions for each of the aforementioned Lie-Poisson spaces (with respect to the appropriate dual basis).

2. Classification of Systems

For each Lie-Poisson space $\mathfrak{g}_-^*$ (admitting a global Casimir function), we shall classify the Hamilton-Poisson systems $(\mathfrak{g}_-^*, H_Q)$ on $\mathfrak{g}_-^*$. Although elementary, some of the computations involved are quite lengthy.

Note 2. *The case of trivial dynamics (i.e., $H_0(p) = 0$) will not be covered explicitly.*

Theorem 1 (cf [5]). *Let $(\mathfrak{g}_-^*, H_Q)$ be a (homogeneous, positive semidefinite) quadratic Hamilton-Poisson system.*

1. *If $\mathfrak{g}_-^* \cong (\mathfrak{g}_{2.1} \oplus \mathfrak{g}_1)_-^*$, then $(\mathfrak{g}_-^*, H_Q)$ is equivalent to exactly one of the following systems on $(\mathfrak{g}_{2.1} \oplus \mathfrak{g}_1)_-^*$*
$$H_1(p) = p_1^2, \qquad H_2(p) = p_2^2, \qquad H_3(p) = p_1^2 + p_2^2$$
$$H_4(p) = (p_1 + p_3)^2, \qquad H_5(p) = p_2^2 + (p_1 + p_3)^2.$$
2. *If $\mathfrak{g}_-^* \cong (\mathfrak{g}_{3.1})_-^*$, then $(\mathfrak{g}_-^*, H_Q)$ is equivalent to exactly one of the following systems on $(\mathfrak{g}_{3.1})_-^*$*
$$H_1(p) = p_3^2, \qquad H_2(p) = p_2^2 + p_3^2.$$
3. *If $\mathfrak{g}_-^* \cong (\mathfrak{g}_{3.4}^0)_-^*$, then $(\mathfrak{g}_-^*, H_Q)$ is equivalent to exactly one of the following systems on $(\mathfrak{g}_{3.4}^0)_-^*$*
$$H_1(p) = p_1^2, \qquad H_2(p) = p_3^2, \qquad H_3(p) = p_1^2 + p_3^2$$
$$H_4(p) = (p_1 + p_2)^2, \qquad H_5(p) = (p_1 + p_2)^2 + p_3^2.$$
4. *If $\mathfrak{g}_-^* \cong (\mathfrak{g}_{3.5}^0)_-^*$, then $(\mathfrak{g}_-^*, H_Q)$ is equivalent to exactly one of the following systems on $(\mathfrak{g}_{3.5}^0)_-^*$*
$$H_1(p) = p_2^2, \qquad H_2(p) = p_3^2, \qquad H_3(p) = p_2^2 + p_3^2.$$
5. *If $\mathfrak{g}_-^* \cong (\mathfrak{g}_{3.6})_-^*$, then $(\mathfrak{g}_-^*, H_Q)$ is equivalent to exactly one of the following systems on $(\mathfrak{g}_{3.6})_-^*$*
$$H_1(p) = p_1^2, \qquad H_2(p) = p_3^2, \qquad H_3(p) = p_1^2 + p_3^2$$
$$H_4(p) = (p_2 + p_3)^2, \qquad H_5(p) = p_2^2 + (p_1 + p_3)^2.$$

6. *If* $\mathfrak{g}_-^* \cong (\mathfrak{g}_{3.7})_-^*$, *then* $(\mathfrak{g}_-^*, H_Q)$ *is equivalent to exactly one of the following systems on* $(\mathfrak{g}_{3.7})_-^*$

$$H_1(p) = p_1^2, \qquad H_2(p) = p_1^2 + \tfrac{1}{2}p_2^2.$$

Proof: We give full details only for item 2. The proofs for items 1, 3, and 4 are similar (although a little more involved) and hence omitted. In most cases, application of equivalences of type $\mathfrak{E}1)$, $\mathfrak{E}2)$, or $\mathfrak{E}3)$ is enough to arrive at the result; items 5 and 6 are the exceptions (i.e., additional linear isomorphisms that are not dilations nor linear Poisson automorphisms are required to arrive at the normal forms). Direct application of $\mathfrak{E}1)$ for item 5 is not fruitful and a modified approach is required in this case (we give an outline). The proof for item 6 (and a full proof for item 5) will appear elsewhere.

2) Let $H_Q(p) = p^\top Q\, p$ be a system on $(\mathfrak{g}_{3.1})_-^*$, where

$$Q = \begin{bmatrix} a_1 & b_1 & b_2 \\ b_1 & a_2 & b_3 \\ b_2 & b_3 & a_3 \end{bmatrix}.$$

Suppose $a_3 \neq 0$. Then

$$\psi = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{b_2}{a_3} & -\frac{b_3}{a_3} & 1 \end{bmatrix}$$

is a linear Poisson automorphism such that

$$\psi^\top Q\, \psi = \begin{bmatrix} a_1 - \frac{b_2^2}{a_3} & b_1 - \frac{b_2 b_3}{a_3} & 0 \\ b_1 - \frac{b_2 b_3}{a_3} & a_2 - \frac{b_3^2}{a_3} & 0 \\ 0 & 0 & a_3 \end{bmatrix} = \begin{bmatrix} a_1' & b_1' & 0 \\ b_1' & a_2' & 0 \\ 0 & 0 & a_3 \end{bmatrix}.$$

If $a_2' = 0$, then H is equivalent to H_1. Suppose $a_2' \neq 0$. Then

$$\psi' = \begin{bmatrix} \frac{1}{\sqrt{a_3}\sqrt{a_2'}} & 0 & 0 \\ -\frac{b_1'}{\sqrt{a_3}\left(a_2'\right)^{3/2}} & \frac{1}{\sqrt{a_2'}} & 0 \\ 0 & 0 & \frac{1}{\sqrt{a_3}} \end{bmatrix}$$

is a linear Poisson automorphism such that

$$\psi'^\top\, \psi^\top\, Q\, \psi\, \psi' = \begin{bmatrix} \frac{a_1' a_2' - \left(b_1'\right)^2}{a_3\left(a_2'\right)^2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Thus H is equivalent to H_2 as $(H \circ \psi \circ \psi')(p) = H_2(p) + c_1 p_1^2$ for some $c_1 \geq 0$, where $\psi \circ \psi'$ is a linear Poisson automorphism and $C(p) = p_1^2$ is a Casimir function.

Now suppose $a_3 = 0$. If $a_2 = 0$, then H is equivalent to the trivial system $H_0(p) = 0$. Suppose $a_2 \neq 0$. Then

$$\psi = \begin{bmatrix} -\frac{1}{\sqrt{a_2}} & 0 & 0 \\ \frac{b_1}{a_2^{3/2}} & 0 & \frac{1}{\sqrt{a_2}} \\ 0 & 1 & 0 \end{bmatrix}$$

is a linear Poisson automorphism such that

$$\psi^\top Q\, \psi = \begin{bmatrix} \frac{a_1 a_2 - b_1^2}{a_2^2} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Hence H is equivalent to H_1.

It remains to be shown that H_1 and H_2 are not equivalent. Assume that they are equivalent, i.e., suppose there exists a linear isomorphism ψ such that $\psi \cdot \vec{H}_1 = \vec{H}_2 \circ \psi$. Let ψ have matrix $(\psi_{ij})_{1 \leq i,j \leq 3}$. Then

$$\begin{bmatrix} -2\psi_{12} p_1 p_3 \\ -2\psi_{22} p_1 p_3 \\ -2\psi_{32} p_1 p_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -2\left(\psi_{11} p_1 + \psi_{12} p_2 + \psi_{13} p_3\right)\left(\psi_{31} p_1 + \psi_{32} p_2 + \psi_{33} p_3\right) \\ 2\left(\psi_{11} p_1 + \psi_{12} p_2 + \psi_{13} p_3\right)\left(\psi_{21} p_1 + \psi_{22} p_2 + \psi_{23} p_3\right) \end{bmatrix}.$$

A simple argument shows that ψ is not an isomorphism, hence a contradiction.

5) Let $H_Q(p) = p^\top Q\, p$ be a system on $(\mathfrak{g}_{3.6})^*_-$. Symmetric matrices are diagonalizable by orthogonal matrices (see, e.g. [1]); hence there exists $\theta \in \mathbb{R}$ such that

$$Q' = \rho_3(\theta)^\top Q\, \rho_3(\theta) = \begin{bmatrix} a_1 & 0 & b_2 \\ 0 & a_2 & b_3 \\ b_2 & b_3 & a_3 \end{bmatrix} \quad \text{where} \quad \rho_3(\theta) = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

for some $a_i, b_i \in \mathbb{R}$. (Note that $\rho_3(\theta)$ is a linear Poisson automorphism.) If $a_1 = 0$ or $a_2 = 0$, then

$$Q' = \begin{bmatrix} 0 & 0 & 0 \\ 0 & a_2 & b_3 \\ 0 & b_3 & a_3 \end{bmatrix} \quad \text{or} \quad \rho_3(\tfrac{\pi}{2})^\top Q'\, \rho_3(\tfrac{\pi}{2}) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & a_1 & b_2 \\ 0 & b_2 & a_3 \end{bmatrix}$$

respectively. These cases will be dealt with below.

Assume $a_1, a_2 \neq 0$ and let $K = \operatorname{diag}(1, 1, -1)$ be the matrix of the quadratic Casimir function. There exists $x \geq 0$ such that the matrix $Q' + xK$ has a Cholesky decomposition $Q' + xK = R^\top R$, where

$$R = \begin{bmatrix} r_1 & 0 & r_3 \\ 0 & r_2 & r_4 \\ 0 & 0 & r_5 \end{bmatrix}$$

with $r_1, r_2 \neq 0$ and $r_5 = 0$. It can be shown that there exists $\psi^\top \in \mathrm{SO}(2,1)$ and $s > 0$ such that $s\,\psi^\top R^\top$ equals

$$\begin{bmatrix} x & 0 & 0 \\ y & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 0 \\ x & 1 & 0 \\ y & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 \\ x & 1 & 0 \\ k & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 0 \\ x & 0 & 0 \\ y & 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 0 \\ x & 1 & 0 \\ y & 1 & 0 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} k & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

for some $x, y, \in \mathbb{R}$, and $k \in \{-1, 1\}$ (cf [12]). Accordingly, H_Q is equivalent to a system $H_{\mathcal{R}_i}(p) = p^\top R_i\, p$ with quadratic form

$$R_1 = \begin{bmatrix} a_1 & b_1 & 0 \\ b_1 & a_2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad R_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & a_2 & b_3 \\ 0 & b_3 & a_3 \end{bmatrix}$$

$$R_3 = \begin{bmatrix} 1 & x & k \\ x & 1 + x^2 & kx \\ k & kx & 1 \end{bmatrix} \qquad R_4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

(Here $k = \pm 1$, $x, a_1, a_2, a_3, b_1, b_3 \in \mathbb{R}$, and each matrix R_i is positive semidefinite.) By using $\mathfrak{E}1$), $\mathfrak{E}2$), and $\mathfrak{E}3$), and in some cases finding an explicit linear isomorphism, the result then follows. ■

We now consider a classification in the context of all three-dimensional Lie-Poisson spaces. First we determine which of the systems in Theorem 1 are equivalent (a summary appears in Table 2). The main classification result then follows.

Proposition 1. *In each of the following cases, any two systems are equivalent:*

1. $((\mathfrak{g}_{2.1} \oplus \mathfrak{g}_1)^*_-, (p_1 + p_3)^2)$, $((\mathfrak{g}_{3.1})^*_-, p_3^2)$, $(\mathfrak{g}^0_{3.4})^*_-, p_1^2)$, $((\mathfrak{g}^0_{3.5})^*_-, p_2^2)$.
2. $((\mathfrak{g}_{3.1})^*_-, p_2^2 + p_3^2)$, $((\mathfrak{g}^0_{3.5})^*_-, p_3^2)$, $((\mathfrak{g}_{3.6})^*_-, p_3^2)$, $((\mathfrak{g}_{3.7})^*_-, p_1^2)$.
3. $((\mathfrak{g}^0_{3.4})^*_-, p_1^2 + p_3^2)$, $((\mathfrak{g}^0_{3.5})^*_-, p_2^2 + p_3^2)$, $((\mathfrak{g}_{3.6})^*_-, p_1^2 + p_3^2)$, $((\mathfrak{g}_{3.7})^*_-, p_1^2 + \frac{1}{2}p_2^2)$.
4. $((\mathfrak{g}_{2.1} \oplus \mathfrak{g}_1)^*_-, p_1^2)$, $((\mathfrak{g}^0_{3.4})^*_-, (p_1 + p_2)^2)$.
5. $((\mathfrak{g}^0_{3.4})^*_-, p_3^2)$, $((\mathfrak{g}_{3.6})^*_-, p_1^2)$.
6. $((\mathfrak{g}^0_{3.4})^*_-, (p_1 + p_2)^2 + p_3^2)$, $((\mathfrak{g}_{3.6})^*_-, p_2^2 + (p_1 + p_3)^2)$.

Proof: We prove only item 1 as the other items follow similarly. We claim that each of the systems is equivalent to $((\mathfrak{g}_{3.1})^*_-, p_3^2)$. Indeed,

$$\psi_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \qquad \psi_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{bmatrix}, \qquad \psi_3 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{bmatrix}$$

are linear isomorphisms with codomain $(\mathfrak{g}_{3.1})^*_-$ such that $\psi_i \cdot \vec{H}_i \circ \psi_i^{-1} = \vec{H}$. Here $\vec{H}$ is the vector field associated with $((\mathfrak{g}_{3.1})^*_-, p_3^2)$; $\vec{H}_1$, $\vec{H}_2$, and $\vec{H}_3$ are the vector fields associated with $((\mathfrak{g}_{2.1} \oplus \mathfrak{g}_1)^*_-, (p_1 + p_3)^2)$, $(\mathfrak{g}_{3.4}^0)^*_-, p_1^2)$, and $((\mathfrak{g}_{3.5}^0)^*_-, p_2^2)$, respectively. ■

Table 2. Equivalence of systems (systems in the same column are equivalent).

$\mathfrak{g}_{2.1} \oplus \mathfrak{g}_1$	p_1^2	$(p_1+p_3)^2$				
$\mathfrak{g}_{3.1}$		p_3^2		$p_2^2 + p_3^2$		
$\mathfrak{g}_{3.4}^0$	$(p_1+p_2)^2$	p_1^2	p_3^2		$(p_1+p_2)^2 + p_3^2$	$p_1^2 + p_3^2$
$\mathfrak{g}_{3.5}^0$		p_2^2		p_3^2		$p_2^2 + p_3^2$
$\mathfrak{g}_{3.6}$			p_1^2	p_3^2	$p_2^2 + (p_1+p_3)^2$	$p_1^2 + p_3^2$
$\mathfrak{g}_{3.7}$				p_1^2		$p_1^2 + \frac{1}{2}p_2^2$

In terms of the geometry of the integral curves, there are three types of quadratic Hamilton-Poisson systems. We say that a system $(\mathfrak{g}^*_-, H)$ is *linear*, if for each integral curve of $\vec{H}$ there exists a line containing its trace. Likewise, $(\mathfrak{g}^*_-, H)$ is called *planar* if it is not linear and for each integral curve of $\vec{H}$ there exists a plane containing its trace. Otherwise, $(\mathfrak{g}^*_-, H)$ is called *non-planar*. (The properties of being linear, planar, and non-planar are each invariant under equivalence, i.e., if two systems are equivalent, then they must belong to the same class.)

Theorem 2. *Let $(\mathfrak{g}^*_-, H_Q)$ be a (homogeneous, positive semidefinite) quadratic Hamilton-Poisson system.*

1. *If $(\mathfrak{g}^*_-, H)$ is linear, then it is equivalent to exactly one of the systems*

$$((\mathfrak{g}_{2.1} \oplus \mathfrak{g}_1)^*_-, p_2^2), \qquad ((\mathfrak{g}_{3.4}^0)^*_-, (p_1+p_2)^2), \qquad ((\mathfrak{g}_{3.5}^0)^*_-, p_2^2).$$

2. *If $(\mathfrak{g}^*_-, H)$ is planar, then it is equivalent to exactly one of the systems*

$$((\mathfrak{g}_{2.1} \oplus \mathfrak{g}_1)^*_-, p_1^2 + p_2^2), \qquad ((\mathfrak{g}_{2.1} \oplus \mathfrak{g}_1)^*_-, p_2^2 + (p_1+p_3)^2)$$
$$((\mathfrak{g}_{3.4}^0)^*_-, p_3^2), \qquad ((\mathfrak{g}_{3.5}^0)^*_-, p_3^2), \qquad ((\mathfrak{g}_{3.6})^*_-, (p_2+p_3)^2).$$

3. *If* $(\mathfrak{g}_-^*, H)$ *is non-planar, then it is equivalent to exactly one of the systems*

$$((\mathfrak{g}_{3.4}^0)_-^*, (p_1+p_2)^2+p_3^2), \qquad ((\mathfrak{g}_{3.5}^0)_-^*, p_2^2+p_3^2).$$

Proof sketch. By the Theorem 1 and Proposition 1, the system $(\mathfrak{g}_-^*, H_\mathcal{Q})$ is indeed equivalent to one of the given normal forms. Computationally taxing and tedious calculations (facilitated by MATHEMATICA) show that no two normal forms are equivalent. However, for the majority of pairs this has already been established in Theorem 1.

For most of the linear or planar systems, simple inspection of the equations of motion prove that they are linear or planar (as the evolution along certain coordinates is constant). For example, for $((\mathfrak{g}_{3.6})_-^*, (p_2+p_3)^2)$ the integral curves are clearly contained in a plane $\{(x, y, \pm\sqrt{h_0} - y)\,;\, x, y \in \mathbb{R}\}$, were $h_0 = (p_2(0)+p_3(0))^2$. In order to show that certain systems are not linear (resp. not planar) one may simply show that intersection of the level sets $H^{-1}(h_0)$ and $C^{-1}(c_0)$ (corresponding to the constants of motion H and C) is not contained in a line (resp. plane) for some initial value. ■

Remark 1. *Tudoran* [29] *showed that a number of quadratic Hamilton-Poisson systems are equivalent to the free rigid body dynamics*

$$\begin{aligned} \dot{p}_1 &= (\lambda_3-\lambda_2)p_2p_3, \qquad \lambda_1,\lambda_2,\lambda_3\in\mathbb{R} \\ \dot{p}_1 &= (\lambda_1-\lambda_3)p_2p_3 \\ \dot{p}_1 &= (\lambda_2-\lambda_1)p_2p_3. \end{aligned}$$

The above system may be realized as $((\mathfrak{g}_{3.7})_-^*, \lambda_1p_1^2+\lambda_2p_2^2+\lambda_3p_3^2)$. (*We may assume* $\lambda_1,\lambda_2,\lambda_3 > 0$ *by adding a multiple of the Casimir.*) *Note however, that by the above theorem, any (non-trivial) system* $((\mathfrak{g}_{3.7})_-^*, \lambda_1p_1^2+\lambda_2p_2^2+\lambda_3p_3^2)$ *is equivalent to* $((\mathfrak{g}_{3.5}^0)_-^*, p_3^2)$ *or* $((\mathfrak{g}_{3.5}^0)_-^*, p_2^2+p_3^2)$.

Remark 2. *We point out some interesting features inferred from the above theorem and preceding proposition.*

- *Any system on* $(\mathfrak{g}_{3.1})_-^*$ *or* $(\mathfrak{g}_{3.7})_-^*$ *is equivalent to one on* $(\mathfrak{g}_{3.5}^0)_-^*$.
- *Any system on* $(\mathfrak{g}_{2.1}\oplus\mathfrak{g}_1)_-^*$ *or* $(\mathfrak{g}_{3.1})_-^*$ *is a planar* (*or linear*) *one.* (*This follows immediately from the fact that* $C(p)=p_3$ *and* $C(p)=p_1$, *respectively, are Casimir functions for these spaces.*)
- *Every system on* $(\mathfrak{g}_{3.1})_-^*$, $(\mathfrak{g}_{3.4}^0)_-^*$, $(\mathfrak{g}_{3.5}^0)_-^*$, *or* $(\mathfrak{g}_{3.7})_-^*$ *may be realized on more than one Lie-Poisson space.* (*For* $(\mathfrak{g}_{3.6})_-^*$, *the only exceptions are those systems equivalent to* $((\mathfrak{g}_{3.6})_-^*, (p_2+p_3)^2)$.)

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Appendix

Table 3. Poisson matrices, linear Poisson automorphisms, and Casimir functions for Lie-Poisson spaces

Algebra	Π	$\mathrm{Aut}\,(\mathfrak{g}^*_-)$	Casimir
$\mathfrak{g}_{2.1} \oplus \mathfrak{g}_1$	$\begin{bmatrix} 0 & -p_1 & 0 \\ p_1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} x & 0 & 0 \\ y & 1 & u \\ 0 & 0 & v \end{bmatrix}$	p_3
$\mathfrak{g}_{3.1}$	$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -p_1 \\ 0 & p_1 & 0 \end{bmatrix}$	$\begin{bmatrix} yw - zv & 0 & 0 \\ x & y & z \\ u & v & w \end{bmatrix}$	p_1
$\mathfrak{g}^0_{3.4}$	$\begin{bmatrix} 0 & 0 & -p_2 \\ 0 & 0 & -p_1 \\ p_2 & p_1 & 0 \end{bmatrix}$	$\begin{bmatrix} x & \sigma y & 0 \\ y & \sigma x & 0 \\ u & v & \sigma \end{bmatrix}, \ \sigma = \pm 1$	$p_1^2 - p_2^2$
$\mathfrak{g}^0_{3.5}$	$\begin{bmatrix} 0 & 0 & p_2 \\ 0 & 0 & -p_1 \\ -p_2 & p_1 & 0 \end{bmatrix}$	$\begin{bmatrix} x & -\sigma y & 0 \\ y & \sigma x & 0 \\ u & v & \sigma \end{bmatrix}, \ \sigma = \pm 1$	$p_1^2 + p_2^2$
$\mathfrak{g}_{3.6}$	$\begin{bmatrix} 0 & p_3 & p_2 \\ -p_3 & 0 & -p_1 \\ -p_2 & p_1 & 0 \end{bmatrix}$	$M^\top J M = J$ $J = \mathrm{diag}(1, 1, -1)$ $\det M = 1$	$p_1^2 + p_2^2 - p_3^2$
$\mathfrak{g}_{3.7}$	$\begin{bmatrix} 0 & -p_3 & p_2 \\ p_3 & 0 & -p_1 \\ -p_2 & p_1 & 0 \end{bmatrix}$	$M^\top M = I$ $I = \mathrm{diag}(1, 1, 1)$ $\det M = 1$	$p_1^2 + p_2^2 + p_3^2$

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