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SYMMETRY REDUCTION OF ASYMMETRIC HEAVENLY EQUATION AND 2+1-DIMENSIONAL BI-HAMILTONIAN SYSTEM

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Abstract. Asymmetric heavenly equation, presented in a two-component form, is known to be 3+1-dimensional bi-Hamiltonian system. We show that symmetry reduction of this equation yields a new two component 2+1-dimensional integrable bi-Hamiltonian system. We prove that this new 2+1-dimensional system admits bi-Hamiltonian structure, so that it is integrable according to Magri's theorem.

1. Introduction

Asymmetric heavenly equation was obtained as one of the canonical equations in the classification of nonlinear second order partial differential equations that possess partner symmetries [1]. The asymmetric heavenly equation in 3+1-dimension is given by

$$u_{tx}u_{ty} - u_{tt}u_{xy} + au_{tz} + bu_{xz} + cu_{xx} = 0 \tag{1}$$

where u is the unknown function that depends on the four independent variables t, x, y, z and the subscripts denote partial derivatives of u, e.g., $u_{tx} = \partial^2 u/\partial t\partial x$, $u_{xx} = \partial^2 u/\partial x^2$..., while a, b, c are constants. By choosing $u_t = v$ as the second unknown, we have converted the asymmetric heavenly equation to the two-component evolution system [2]

$$u_t = v, \qquad v_t = \frac{1}{u_{xy}}\big(v_x v_y + av_z + bu_{xz} + cu_{xx}\big) \equiv Q. \tag{2}$$

The physical significance of the singe scalar equation (1) follows from the fact that it is equivalent to complex Einstein field equations for (anti-)self-dual gravitational fields [3], with u being the metric potential.

In [2] we found all point symmetries of asymmetric heavenly equation. In general, a symmetry reduction of this equation has no Hamiltonian structure. Here we show that one particular symmetry reduction yields a two-component 2+1-dimensional new bi-Hamiltonian integrable system. For this system we present Hamiltonian structure, recursion operator, Lax pair, Lie point symmetries and integrals of motion.

In Section 2, we present new 2+1-dimensional integrable system in two-component form obtained by a symmetry reduction of asymmetric heavenly system.

In Section 3, we obtain the first Hamiltonian structure of this system of equations. We start with a degenerate Lagrangian and construct its Dirac bracket [4] to find a Hamiltonian operator.

In Section 4, we construct a recursion operator in a matrix form using the results presented in [2]. The recursion operator and operator of the symmetry condition form a Lax pair of the Olver-Ibragimov-Shabat type for the two-component system [5].

In Section 5, we give explicitly the second Hamiltonian structure which shows that the asymmetric heavenly equation is an integrable bi-Hamiltonian system.

In Section 6, we present all point symmetries of the reduced system. Using the inverse Noether theorem for Hamiltonian symmetries, we determine the corresponding integrals of motion.

2. Symmetry Reduction of Asymmetric Heavenly Equation

Basic generators of one-parameter subgroups of a total Lie group of point symmetries for the asymmetric heavenly system (2) has the form [2]

$$\begin{aligned}
X_1 &= y\partial_y + u\partial_u + v\partial_v \\
X_2 &= (\frac{akx}{b^2} - kt + aF'(s))\partial_t + bF'(s)\partial_x + ky\partial_y \\
X_d &= ((bt - ax)d_{yz} - vd_{yy})\,\partial_t + cd_y\partial_x - bd_z\partial_y + bd_y\partial_z \\
&\quad +(-\frac{1}{2}d_{yy}v^2 + (\frac{1}{2}a^2x^2 - abtx + \frac{1}{2}b^2t^2)d_{zz} - actd_z)\partial_u \\
&\quad + \left(-bd_{yz}v + (b^2t - abx)d_{zz} - acd_z\right)\partial_v \\
X_f &= f_y\partial_t + (bt - ax)f_z\partial_u + bf_z\partial_v \\
X_g &= g(y,z)\partial_u.
\end{aligned} \tag{3}$$

We use a particular choice of $d = y$ for X_d and we obtain

$$X_y = c\partial_x + b\partial_z. \tag{4}$$

The invariants of X_y are determined by the characteristic system as

$$X = bx - cz, \qquad Y = y, \qquad T = t, \qquad U = u, \qquad V = v. \tag{5}$$

The symmetry reduction implies the ansatz: $u = U(X,Y,T)$ and $v = V(X,Y,T)$. The total derivatives in terms of new variables become

$$D_x = bD_X, \qquad D_y = D_Y, \qquad D_z = -cD_X, \qquad D_t = D_T \tag{6}$$

where D stand for partial derivative ∂, i.e., $D_x u = \partial_x u = u_x$. Substituting this into the original system (2) and renaming $U \to u$, $V \to v$ and $T \to t$ we obtain new 2+1-dimensional reduced system in two component form as

$$u_t = v, \qquad v_t = \frac{v_x}{u_{xy}}\left(v_y - \frac{ac}{b}\right) \equiv Q, \qquad b \neq 0. \tag{7}$$

3. Dirac's Constraints Analysis, Symplectic and Hamiltonian Structure of Reduced System

In general, in order to prove that 2+1-dimensional reduced system (7) is an integrable bi-Hamiltonian system, we should start with a degenerate Lagrangian and follow the same procedure as we did for asymmetric heavenly system [2]. Therefore we apply (6) to the Lagrangian given in [2] and we obtain reduced Lagrangian density for the system (7) as follows

$$L = b\left(\frac{v^2}{2} - vu_t\right)u_{xy} - \frac{ac}{2}u_t u_x. \tag{8}$$

In order to get a Hamiltonian formulation, we need to apply Dirac's constraints [4] analysis. Thus, we define the canonical momenta

$$\Pi_i = \frac{\partial L}{\partial u_t^i} \tag{9}$$

which satisfy the canonical Poisson brackets

$$[\Pi_i(\xi), u^k\eta] = \delta_i^k \delta(\xi - \eta) \tag{10}$$

where ξ, η are generic names for independent variables, each of which stands for the collection of our original independent variables x, y. In other words, $\delta(\xi - \eta) = \delta(x - x')\delta(y - y'^)$ for $\xi = \{x, y\}$ and $\eta = \{x', y'\}$ and using (9) we get π_u and π_v

$$\pi_u = \frac{\partial L}{\partial u_t} = -bvu_{xy} - \frac{ac}{2}u_x, \qquad \pi_v = \frac{\partial L}{\partial v_t} = 0 \tag{11}$$

that cannot be solved for velocities u_t and v_t, and therefore the Lagrangian (8) is degenerate. Following Dirac's theory of constraints [4], we treat the definitions (11) as the second class constraints

$$\phi_u = \pi_u + bvu_{xy} + \frac{ac}{2}u_x = 0, \qquad \phi_v = \pi_v = 0 \tag{12}$$

and calculate the Poisson bracket of the constraints

$$K_{ij} = [\phi_i(x,y), \phi_j(x',y')], \qquad i,j = 1,2 \tag{13}$$

where $\phi_1 = \phi_u$ and $\phi_2 = \phi_v$. Organizing them in the form of a matrix, we find

$$K = \begin{pmatrix} -((bv_y - ac)D_x + bv_x D_y + bv_{xy}) & bu_{xy} \\ -bu_{xy} & 0 \end{pmatrix}. \tag{14}$$

Below we show that this explicitly skew-symmetric operator is a symplectic operator in the sense of Fuchssteiner and Fokas [6]. Here the corresponding symplectic two-form is the volume integral

$$\Omega = \int_V \omega \mathrm{d}x\mathrm{d}y\mathrm{d}z \tag{15}$$

of the density

$$\omega = \frac{1}{2}\mathrm{d}u^i \wedge K_{ij}\,\mathrm{d}u^j = bu_{xy}\,\mathrm{d}u\wedge\mathrm{d}v - \frac{b}{2}v_x\,\mathrm{d}u\wedge\mathrm{d}u_y - \frac{1}{2}(bv_y - ac)\,\mathrm{d}u\wedge\mathrm{d}u_x \tag{16}$$

where $u^1 = u$ and $u^2 = v$. In ω under the sign of volume integral (15), we can neglect all the terms that are either total derivatives or total divergences due to suitable boundary condition on the boundary surface of the volume. For the exterior differential of this two-form we obtain

$$\mathrm{d}\omega = b\,\mathrm{d}u_{xy} \wedge \mathrm{d}u \wedge \mathrm{d}v - \frac{b}{2}\,\mathrm{d}v_x \wedge \mathrm{d}u \wedge \mathrm{d}u_y - \frac{b}{2}\,\mathrm{d}v_y \wedge \mathrm{d}u \wedge \mathrm{d}u_x$$

or

$$\mathrm{d}\omega = \frac{b}{2}\,D_x[\mathrm{d}u_y \wedge \mathrm{d}u \wedge \mathrm{d}v] + \frac{b}{2}\,D_y[\mathrm{d}u_x \wedge \mathrm{d}u \wedge \mathrm{d}v] \Leftrightarrow 0. \tag{17}$$

Here the application of total derivatives to differential forms is performed in the context of the variational complex as explained in the Olver's book, [7, Section 5.4]. In (17) $\mathrm{d}\omega$ is a total divergence, which is equivalent to zero as we have explained above, so that two-form Ω is closed and therefore it is a symplectic two-form and so K, defined by (14), is a symplectic operator[6]. Hence its inverse is a Hamiltonian operator

$$J_0 = K^{-1} = \begin{pmatrix} 0 & -\dfrac{1}{bu_{xy}} \\ & \\ \dfrac{1}{bu_{xy}} & J_0^{22} \end{pmatrix} \tag{18}$$

where

$$J_0^{22} = \frac{ac}{b^2 u_{xy}} D_x \frac{1}{u_{xy}} - \frac{1}{2b}\left(\frac{v_y}{u_{xy}^2} D_x + D_x \frac{v_y}{u_{xy}^2} + \frac{v_x}{u_{xy}^2} D_y + D_y \frac{v_x}{u_{xy}^2}\right) \tag{19}$$

and D_t, D_x, D_y denote operators of total derivatives with respect to t, x, y, respectively. The closeness of the symplectic two-form (16) is equivalent to satisfaction of the Jacobi identity for the Hamiltonian operator J_0 [6].

The Hamiltonian density, corresponding to J_0, is defined as

$$H_1 = \pi_u u_t + \pi_v v_t - L$$

which leads to

$$H_1 = -\frac{b}{2} v^2 u_{xy}. \tag{20}$$

One can obtain the flow specified in equation (2) by applying J_0 to variational derivatives of Hamiltonian density H_1

$$\begin{pmatrix} u_t \\ v_t \end{pmatrix} = J_0 \begin{pmatrix} \delta_u H_1 \\ \delta_v H_1 \end{pmatrix} = \begin{pmatrix} v \\ \frac{v_x}{u_{xy}} \left(v_y - \frac{ac}{b} \right) \end{pmatrix} \tag{21}$$

where δ_u and δ_v are Euler-Lagrange operators [7] with respect to u and v applied to the Hamiltonian density H_1 (they correspond to variational derivatives of the Hamiltonian functional $\int\limits_V H_1 \mathrm{d}V$).

4. Recursion Operator and Lax Pair for Reduced System

Lie equations for symmetries of reduced system (7) have the form

$$\begin{pmatrix} u_\tau \\ v\tau \end{pmatrix} = \begin{pmatrix} \varphi \\ \psi \end{pmatrix} \equiv \Phi \tag{22}$$

where φ and $\psi = \varphi_t$ are components of the symmetry characteristic and τ is a symmetry group parameter. From the Fréchet derivative of the flow (7), we find the equation that determines its symmetries $\hat{A}(\Phi) = 0$, where the operator $\hat{A}$ is

$$\hat{A} = \begin{pmatrix} D_t & -1 \\ \frac{Q}{u_{xy}} D_x D_y & D_t - \frac{1}{u_{xy}} (v_y + \frac{ac}{b}) D_x - \frac{v_x}{u_{xy}} D_y \end{pmatrix}. \tag{23}$$

The recursion operator is defined as a linear operator which maps any symmetry of a given equation again into a symmetry of the same equation. As a consequence, this operator commutes with the operator $\hat{A}$ of the symmetry condition $\hat{A}(\Phi) = 0$ on solution of the latter equation and equation (7). It is obtained by a symmetry reduction from the recursion operator for the four dimensional asymmetric heavenly system corresponding to (1) that was given in [2], and reads

$$R = \begin{pmatrix} D_x^{-1} \left(v_x D_y - \frac{ac}{b} D_x \right) & -D_x^{-1} u_{xy} \\ Q D_y & -v_y \end{pmatrix} \tag{24}$$

where D_x^{-1} is the inverse of D_x. The commutator of the recursion operator R and the operator $\hat{A}$ of the symmetry condition has the form

$$[R,\hat{A}] = \begin{pmatrix} -D_x^{-1}(v_t-Q)_x D_y & D_x^{-1}(u_t-v)_{xy} \\ \frac{1}{u_{xy}}\big(-cb^2(u_t-v)_{xx} + bQ(u_t-v)_{xy} \\ \quad -b^2c(u_t-v)_{xx} - bv_y(v_t-Q)_x \\ \quad -bv_x(v_t-Q)_y + ac(v_t-Q)_x\big)D_y & (v_t-Q)_y \end{pmatrix} \tag{25}$$

and as a consequence, the operators R and $\hat{A}$ form a Lax pair of the Olver-Ibragimov-Shabat [5] type for the asymmetric heavenly system (7), so that R and $\hat{A}$ commute on solutions of this system.

5. Second Hamiltonian Structure and Hamiltonian Function

By using theorem of Magri [8, 9], one can generate the second Hamiltonian operator, by applying the recursion operator (24) to the first Hamiltonian operator $J_1 = RJ_0$ with the result

$$J_1 = RJ_0 = \begin{pmatrix} -D_x^{-1} & \frac{v_y}{u_{xy}} \\ -\frac{v_y}{u_{xy}} & J_1^{22} \end{pmatrix} \tag{26}$$

where J_1^{22} in an explicitly skew-symmetric form is defined as

$$J_1^{22} = \frac{1}{2b}\left(v_y^2 D_x \frac{1}{u_{xy}^2} + \frac{1}{u_{xy}^2}D_x v_y^2\right) - \frac{ac}{2b^2}\left(\frac{v_y}{u_{xy}}D_x\frac{1}{u_{xy}} + \frac{1}{u_{xy}}D_x\frac{v_y}{u_{xy}}\right)$$
$$+\frac{1}{2b}\left(\frac{v_x}{u_{xy}}D_y\frac{v_y}{u_{xy}} + \frac{v_y}{u_{xy}}D_y\frac{v_x}{u_{xy}}\right) - \frac{1}{2b}\left(QD_y\frac{1}{u_{xy}} + \frac{1}{u_{xy}}D_yQ\right). \tag{27}$$

This operator is manifestly skew-symmetric. The proof of the Jacobi identity is straightforward and lengthy. The calculation are simplified by using Olver's criterion, namely Theorem 7.8 in his book [7], formulated in terms of functional multi-vectors. Moreover, J_0 and J_1 are compatible Hamiltonian operators, that is, every linear combination $\alpha J_0 + \beta J_1$ with constant coefficients α and β satisfies the Jacobi identity. We again note that operator (26) could be obtained by a symmetry reduction from the second Hamiltonian structure in [2]. Thus, we obtain the second Hamiltonian form of the reduced system (7)

$$\begin{pmatrix} u_t \\ v_t \end{pmatrix} = J_1 \begin{pmatrix} \delta_u H_0 \\ \delta_v H_0 \end{pmatrix} \tag{28}$$

with the Hamiltonian density

$$H_0 = b(c_0 - y)vu_{xy} \tag{29}$$

where c_0 is a constant, so that reduced system (7) is a bi-Hamiltonian system, that is, it can be written in the two Hamiltonian forms

$$\begin{pmatrix} u_t \\ v_t \end{pmatrix} = J_0 \begin{pmatrix} \delta_u H_1 \\ \delta_v H_1 \end{pmatrix} = J_1 \begin{pmatrix} \delta_u H_0 \\ \delta_v H_0 \end{pmatrix}. \tag{30}$$

By repeated applications of the recursion operator to the first Hamiltonian operator J_0 according to Magri's theorem we could generate an in finite sequence of Hamiltonian operators as

$$J_n = R^n J_0, \qquad n = 1, 2, 3, \dots \tag{31}$$

which proves that reduced system considered in two component form is a multi-Hamiltonian system in above sense.

6. Symmetries and Integrals of Motions

Hamiltonian operators provide a natural link between commuting symmetries in evolutionary form [7] and conserved quantities (integral of motions) that are in involution with respect to Poisson brackets. Our two-component reduced system (7) is also a member of an infinite hierarchy of symmetries. Point symmetries generators of (7) read

$$\begin{aligned} X_1 &= f(x)\partial_x \\ X_2 &= (-tg_v - tvh_v - uh_v - w_v + th)\partial_t + e(y)\partial_y \\ &\quad +(-tvg_v + tv^2h_v - uvh_v - vw_v + tg + uh + w)\partial_u + g\partial_v \end{aligned} \tag{32}$$

where $f(x), g(y,v), h(y,v), w(x,y,v)$ and $e(y)$ are arbitrary functions. These point symmetries are generated by integrals of motion, that is, they are variational symmetries and the integrals are given by the Hamiltonian form of Noether's theorem

$$\begin{pmatrix} \delta_u H \\ \delta_v H \end{pmatrix} = K \begin{pmatrix} \hat{\eta}^u \\ \hat{\eta}^v \end{pmatrix} = \begin{pmatrix} -((bv_y - ac)D_x + bv_x D_y + bv_{xy}) & bu_{xy} \\ -bu_{xy} & 0 \end{pmatrix} \begin{pmatrix} \hat{\eta}^u \\ \hat{\eta}^v \end{pmatrix}. \tag{33}$$

We determine the integral of motion H (for a variational symmetry), corresponding to known symmetry characteristics $\hat{\eta}^u$, $\hat{\eta}^v$ via the relation (33). For the first symmetry X_1, the corresponding symmetry characteristic is: $\hat{\eta}^u = -u_x f(x)$, $\hat{\eta}^v = -v_x f(x)$ and we obtain

$$H^1 = (bvu_xu_{xy} + \frac{ac}{2}u_x^2)f(x) \tag{34}$$

which is the conserved density corresponding to the first point symmetry X_1. For the second point symmetry X_2, we can find an integral of motion only for a special choice of arbitrary functions. For example, if we choose $h = 0, w = 0, g = b$ and $e = 1$ we obtain

$$X_2 = \partial_y + bt\partial_u + b\partial_v, \qquad \hat{\eta}^u = bt - u_y, \qquad \hat{\eta}^v = b - v_y. \tag{35}$$

Using (35) into (33) we get

$$H^2 = bvu_{xy}(u_y - bt) + \frac{1}{2}(b^2 - ac)uu_{xy}. \tag{36}$$

It may be possible to find different integrals of motion for different choices of arbitrary functions.

7. Conclusion

We have shown that a certain symmetry reduction of the 3+1-dimensional asymmetric heavenly equation, taken in a two-component form yields a two component 2 + 1-dimensional multi-Hamiltonian integrable system. For this system, we have presented explicitly two Hamiltonian operators, a recursion operator for symmetries, a complete set of point symmetries and corresponding integrals of the motion. The first impression of the major part of this work could be that it is an easy and even trivial task to obtain a three-dimensional multi-Hamiltonian system by a symmetry reduction of the original four-dimensional asymmetric heavenly system. All main objects $J_0, J_1, K, \hat{A}, R, H_0, H_1$ and L could be obtained by the symmetry reduction. However, a slight change in a symmetry chosen for the reduction, ruins all these properties and creates a difficulty in discovering bi-Hamiltonian structure of the reduced system. If we choose more general symmetries for the reduction, for example from the optimal system of one-dimensional subalgebras from [2], then we shall be unable to discover even a single Hamiltonian structure of reduced systems. The problem of conservation of multi-Hamiltonian structure under symmetry reductions seems to be an important and interesting subject for a future research.

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