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MOTION OF CHARGED PARTICLES IN THE EQUATORIAL PLANE OF A MAGNETIC DIPOLE FIELD

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Abstract. Newton–Lorentz equations describing the motion of charged particles in the equatorial plane of a magnetic dipole field are considered. The parametric equations of the trajectories of the particles are obtained explicitly in terms of Jacobi elliptic functions and elliptic integrals.

1. Introduction

In this article we consider the system of Newton–Lorentz equations describing the planar motion of a charged particle in the equatorial plane of a magnetic dipole field. This system belongs to the class of dynamical systems of two degrees of freedom whose integrability in the Liouville–Arnold sense (see, e.g., [1, Section 5]) has been studied recently by the present authors in [2].

The magnitude of the magnetic dipole field depends only on the distance from the origin and hence, see [3], the corresponding Newton–Lorentz system is integrable by quadratures since it possesses two functionally independent integrals of motion, one of which is the speed of the particle. Here, our aim is using the techniques developed in [2] to express the parametric equations of the trajectories of the particles explicitly in analytic form.

2. Motion of Charged Particles in an Electromagnetic Field

2.1. Lorentz Force and Newton–Lorentz Equation

The electromagnetic force $\mathbf{F}$ acting on an electrically charged particle with electric charge q is given by the Lorentz force law $\mathbf{F} = q\mathbf{E} + q\,(\mathrm{d}\mathbf{x}/\mathrm{d}t \times \mathbf{B})$, (see [4, §5.1.2]) where $\mathbf{x}$ is the position vector of the particle, which depends on the time t, $\mathbf{E}$ and $\mathbf{B}$ are the electric and magnetic fields, respectively.

The motion of such a particle of mass m is described by the classical Newton–Lorentz equation $m\,\mathrm{d}^2\mathbf{x}/\mathrm{d}t^2 = \mathbf{F}$ that is

$$m\frac{\mathrm{d}^2\mathbf{x}}{\mathrm{d}t^2} = q\mathbf{E} + q\left(\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} \times \mathbf{B}\right)\cdot \tag{1}$$

2.2. Magnetic Dipole Field

Consider a magnetic dipole - a "small" circular loop of wire with area A lying in the XOY plane ("equatorial" plane of the dipole), centered at the origin O, and carrying a current I running counterclockwise as viewed from the positive OZ axis (see Fig. 1).

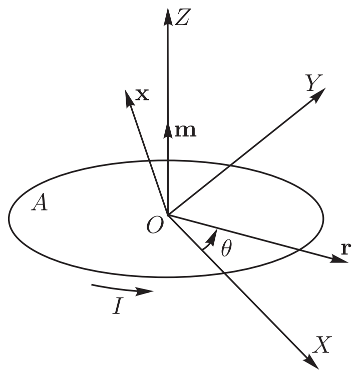

Figure 1. A magnetic dipole field.

The magnetic field of such a dipole is stationary and has the form

$$\mathbf{B}\,(\mathbf{x}) = \frac{\mu_0}{4\pi}\frac{1}{|\mathbf{x}|^3}\left[3\,(\mathbf{m}\cdot\hat{\mathbf{x}})\,\hat{\mathbf{x}} - \mathbf{m}\right] \tag{2}$$

(see, e.g, [4, pp. 244–246]) where $\hat{\mathbf{x}} = \mathbf{x}/\,|\mathbf{x}|$, μ_0 is the magnetic permeability in vacuum and $\mathbf{m} = IA\mathbf{k}$ is the field source's magnetic dipole moment (here, $\mathbf{k}$ is the unit vector along the coordinate axis OZ).

2.3. Motion of Particles in the Equatorial Plane of a Magnetic Dipole Field

Suppose that a charged particle moves without leaving the equatorial plane of a magnetic dipole, i.e., the XOY plane (see Fig. 1). Denote by **r** its position vector and let $r = |\mathbf{r}| = \sqrt{x^2 + y^2}$. Then, on account of equations (1) and (2), the equation of motion of the particle written in scalar representation reads

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + \frac{\sigma}{r^3}\frac{\mathrm{d}y}{\mathrm{d}t} = 0, \qquad \frac{\mathrm{d}^2 y}{\mathrm{d}t^2} - \frac{\sigma}{r^3}\frac{\mathrm{d}x}{\mathrm{d}t} = 0, \qquad \sigma = \frac{\mu_0 I A}{4\pi m} > 0 \tag{3}$$

which is an integrable dynamical system of the Frenet-Serret type (cf. [2]) since it has the following two functionally independent first integrals

$$\left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2 = \frac{1}{r^4}\left(\nu r^2 - \varepsilon r + \sigma\right)\left(\nu r^2 + \varepsilon r - \sigma\right), \qquad \frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{1}{r^2}\left(\varepsilon - \frac{\sigma}{r}\right) \tag{4}$$

where θ is the polar angle (see Fig. 1), $\nu \geq 0$ and ε are constants of integration (integrals of motion). Actually, $\sqrt{\nu}$ is the speed of the particle.

3. Analytic Solution of the Equations of Motion

3.1. General Solution in Terms of the Weierstrass Elliptic Function $\wp$

In terms of a new variable τ, which is such that

$$\frac{\mathrm{d}t}{\mathrm{d}\tau} = r^2 \tag{5}$$

the first integrals (4) take the form

$$\left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)^2 = \nu^2 r^4 - \varepsilon^2 r^2 + 2\sigma\varepsilon r - \sigma^2 \tag{6}$$

and

$$\frac{\mathrm{d}\theta}{\mathrm{d}\tau} = \varepsilon - \frac{\sigma}{r}\cdot \tag{7}$$

Thus, to find the solutions of the considered system (3) one can first find the general solution $r(\tau)$ of equation (6), which involves only the variable r, and then using this result and integrating equation (7) to obtain the polar angle

$$\theta(\tau) = \int \left(\varepsilon - \frac{\sigma}{r(\tau)}\right)\mathrm{d}\tau. \tag{8}$$

Rewriting equation (6) in the form

$$\left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)^2 = P\left[r\right], \qquad P\left[r\right] = a_0 r^4 + 4a_1 r^3 + 6a_2 r^2 + 4a_3 r + a_4$$

where

$$a_0 = \nu^2, \qquad a_1 = 0, \qquad a_2 = -\frac{\varepsilon^2}{6}, \qquad a_3 = \frac{\varepsilon\sigma}{2}, \qquad a_4 = -\sigma^2 \tag{9}$$

it becomes obvious (see [5, §20.6]) that its general real-valued solution can be written in the form

$$r\,(\tau) = \mathring{r} + \frac{1}{4}\frac{P_1[\mathring{r}]}{\wp\,(\tau; g_2, g_3) - \frac{1}{24}P_2[\mathring{r}]} \tag{10}$$

where $\mathring{r}$ is a real root of the polynomial $P[r]$, $\wp\,(\tau; g_2, g_3)$ is the Weierstrass elliptic function, and

$$g_2 = a_0a_4 - 4a_1a_3 + 3a_2^2, \qquad g_3 = a_0a_2a_4 + 2a_1a_2a_3 - a_2^3 - a_0a_3^2 - a_1^2a_4 \tag{11}$$

$$P_1 = \frac{\mathrm{d}P}{\mathrm{d}r}, \qquad P_2 = \frac{\mathrm{d}^2P}{\mathrm{d}r^2}\cdot \tag{12}$$

Now, taking into account equations (9) and (12) we can express equations (10) and (11) as follows

$$r\,(\tau) = \mathring{r} + 6\frac{\mathring{r}\left(2\mathring{r}^2\nu^2 - \varepsilon^2\right) + \varepsilon\sigma}{12\wp\,(\tau; g_2, g_3) + \varepsilon^2 - 6\mathring{r}^2\nu^2} \tag{13}$$

and

$$g_2 = \frac{1}{12}\varepsilon^4 - \nu^2\sigma^2, \qquad g_3 = \frac{1}{216}\varepsilon^2\left(\varepsilon^4 - 18\nu^2\sigma^2\right). \tag{14}$$

3.2. Expression of the Solution in Terms of the Jacobian Elliptic Functions

Let us consider first the discriminant $\Delta = g_2^3 - 27g_3^2$ of the polynomials $P[r]$ and $R[r] = 4r^3 - g_2r - g_3$ (see, e.g., [6, pp. 40-44]). According to equations (14) it has the form

$$\Delta = \frac{1}{16}\upsilon^4\sigma^4(\varepsilon^2 - 4\nu\sigma)(\varepsilon^2 + 4\nu\sigma).$$

Evidently, $\Delta > 0$ if and only if $\varepsilon^2 > 4\nu\sigma$ since $\nu > 0$ and $\sigma > 0$ as was indicated in Subsection 1.3, otherwise $\Delta < 0$. It is well known that if $\Delta \neq 0$, then the Weierstrass elliptic function $\wp\,(\tau; g_2, g_3)$ can be expressed in terms of Jacobi elliptic functions (cf. [7, pp. 649–652]) as follows

i) if $\Delta > 0$, which means $\varepsilon^2 > 4\nu\sigma$ in view of the above considerations, then

$$\wp\,(\tau; g_2, g_3) = e_3 + \frac{e_1 - e_3}{\mathrm{sn}^2\left(\sqrt{e_1 - e_3}\,\tau, k\right)} \tag{15}$$

where $\mathrm{sn}(\cdot,\cdot)$ is the sine Jacobian elliptic function with elliptic modulus

$$k = \sqrt{\frac{e_2 - e_3}{e_1 - e_3}} \tag{16}$$

and $e_1 > e_2 > e_3$ are the roots of the cubic polynomial $R[r]$, which are real in this case. Using equations (14) they can be written in the form

$$e_1 = \frac{\varepsilon^2}{24} + \frac{\sqrt{\varepsilon^4 - 16\nu^2\sigma^2}}{8}, \qquad e_2 = \frac{\varepsilon^2}{24} - \frac{\sqrt{\varepsilon^4 - 16\nu^2\sigma^2}}{8}, \qquad e_3 = -\frac{\varepsilon^2}{12}\cdot \tag{17}$$

ii) if $\Delta < 0$, which means $\varepsilon^2 < 4\nu\sigma$ as mentioned above, then the polynomial $R[r]$ has one real root

$$e_2 = -\frac{\varepsilon^2}{12} \tag{18}$$

as well as a couple of complex conjugated roots and

$$\wp\,(\tau; g_2, g_3) = e_2 + H_2 \frac{1 + \operatorname{cn}\left(2\sqrt{H_2}\,\tau, k\right)}{1 - \operatorname{cn}\left(2\sqrt{H_2}\,\tau, k\right)} \tag{19}$$

where $\operatorname{cn}(\cdot, \cdot)$ is the cosine Jacobi elliptic function, while

$$k = \sqrt{\frac{1}{2} - \frac{3e_2}{4H_2}}, \qquad H_2 = \sqrt{3e_2^2 - \frac{g_2}{4}}\cdot \tag{20}$$

In both cases k denotes the elliptic modulus of the corresponding elliptic functions. In Case i), substituting equation (15) in equation (13) and using equations (16) and (17) one obtains

$$r\,(\tau) = \mathring{r} - \frac{4\left(\varepsilon^2\mathring{r} - \varepsilon\sigma - 2\nu^2\mathring{r}^3\right)\operatorname{sn}^2(\lambda\tau, k)}{\sqrt{\varepsilon^4 - 16\nu^2\sigma^2} + \varepsilon^2 - 4\nu^2\mathring{r}^2\operatorname{sn}^2(\lambda\tau, k)} \tag{21}$$

where

$$\lambda = \frac{\sqrt{\sqrt{\varepsilon^4 - 16\nu^2\sigma^2} + \varepsilon^2}}{2\sqrt{2}}, \qquad k = \frac{\sqrt{\varepsilon^4 - \varepsilon^2\sqrt{\varepsilon^4 - 16\nu^2\sigma^2} - 8\nu^2\sigma^2}}{2\sqrt{2}\nu\sigma}\cdot \tag{22}$$

In Case ii), substituting equation (19) in equation (13) and using equations (14), (18) and (20) one obtains

$$r\,(\tau) = \mathring{r} + \frac{\left(\varepsilon^2\mathring{r} - \varepsilon\sigma - 2\nu^2\mathring{r}^3\right)\left(\operatorname{cn}(\lambda\tau, k) - 1\right)}{\nu\left(\sigma - \nu\mathring{r}^2\right) + \nu\left(\sigma + \nu\mathring{r}^2\right)\operatorname{cn}(\lambda\tau, k)} \tag{23}$$

where

$$\lambda = \sqrt{2\nu\sigma}, \qquad k = \sqrt{\frac{1}{2} + \frac{\varepsilon^2}{8\nu\sigma}}\cdot \tag{24}$$

Let us remark that in the limiting cases $k = 0$ and $k = 1$ in which $\Delta = 0$ the expressions in the left-hand sides of equations (21) and (23) reduce to trigonometric or hyperbolic functions.

It is noteworthy that the roots of the polynomial $P[r]$, of which at least one needs to be used in the expressions (10), (21) and (23), can be straightforwardly written in the form

$$\alpha = \frac{\varepsilon - \sqrt{\varepsilon^2 - 4\nu\sigma}}{2\nu}, \qquad \beta = \frac{\varepsilon + \sqrt{\varepsilon^2 - 4\nu\sigma}}{2\nu}$$
$$\gamma = -\frac{\varepsilon + \sqrt{\varepsilon^2 + 4\nu\sigma}}{2\nu}, \qquad \delta = -\frac{\varepsilon - \sqrt{\varepsilon^2 + 4\nu\sigma}}{2\nu}.$$

In Case i), all these roots are real since $\varepsilon^2 > 4\nu\sigma$ under the corresponding assumptions, while in Case ii) only the roots γ and δ are real since $\varepsilon^2 < 4\nu\sigma$ in this case. Graphs of the polynomial $P[r]$ corresponding to two different sets of parameters ν, ε, and σ are depicted in Fig. 2.

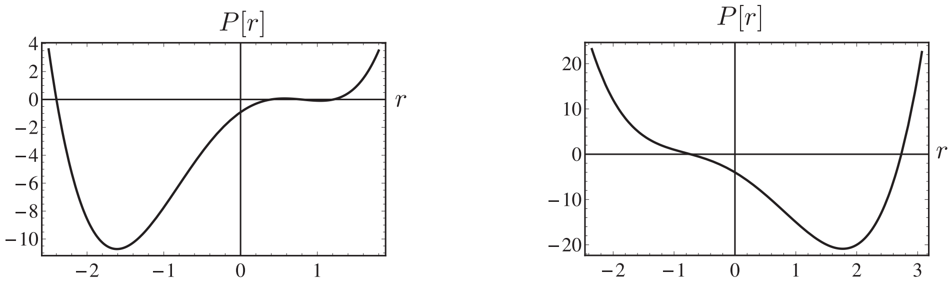

Figure 2. Graphs of the polynomial $P[r]$ with $\nu = 1$, $\varepsilon = 2$, $\sigma = 0.956$, $\alpha = 0.790238$, $\beta = 1.20976$, $\gamma = -2.39857$, $\delta = 0.398571$ (left) and $\nu = 1$, $\varepsilon = -2$, $\sigma = 2$, $\gamma = -0.732051$, $\delta = 2.73205$ (right).

4. Explicit Analytic Representation of the Trajectories

4.1. Analytic Representation of the Trajectories in Case i)

Making use of the result for the radius $r(\tau)$ given by equation (21) one can calculate the polar angle in formula (8) in the following explicit form

$$\theta(\tau) = b_1\tau + b_2\Pi(b_3, \operatorname{am}(\lambda, k), k) \tag{25}$$

where λ and k are given in equations (22), $\Pi(\cdot,\cdot,\cdot)$ denotes the incomplete elliptic integral of the third kind and the real parameters b_1, b_2 and b_3 are given by the formulae

$$b_1 = \frac{\nu^2\mathring{r}^2\sigma}{\nu^2\mathring{r}^3 - \varepsilon^2\mathring{r} + \varepsilon\sigma} + \varepsilon, \qquad b_2 = \frac{\sigma\left(\varepsilon^2\mathring{r} - \varepsilon\sigma - 2\nu^2\mathring{r}^3\right)}{\lambda\mathring{r}\left(\nu^2\mathring{r}^3 - \varepsilon^2\mathring{r} + \varepsilon\sigma\right)}$$
$$b_3 = \frac{4\left(\varepsilon^2\mathring{r} - \varepsilon\sigma - \nu^2\mathring{r}^3\right)}{\mathring{r}\left(\sqrt{\varepsilon^4 - 16\nu^2\sigma^2} + \varepsilon^2\right)}.$$

Thus, equations (21) and (25) provide an explicit analytic parametrization of the trajectories in Case i). Let us recall that in this case all the roots of the polynomial $P[r]$ are real and can be utilized for the purpose.

4.2. Analytic Representation of the Trajectories in Case ii)

In this case, using equation (23) for the function $r(\tau)$ and the expressions for the roots γ and δ given above we obtain

$$r\,(\tau) = \frac{2\sigma}{\varepsilon \pm \sqrt{\varepsilon^2 + 4\nu\sigma}\,\mathrm{cn}(\lambda\tau, k)} \tag{26}$$

where the upper sign corresponds to $\mathring{r} = \delta$ while the lower one corresponds to $\mathring{r} = \gamma$. Here, λ and k are given by equations (24).

Now, we can easily rewrite equation (8) for the respective polar angle in the explicit form

$$\theta(\tau) = \frac{\varepsilon}{2}\tau \mp \mathrm{sign}\left[\mathrm{sn}(\lambda\tau, k))\right] \arccos\left[\mathrm{dn}(\lambda\tau, k)\right] \tag{27}$$

where again the upper sign corresponds to $\mathring{r} = \delta$ and the lower one – to $\mathring{r} = \gamma$.

Thus, equations (26) and (27) provide an explicit analytic parametrization of the trajectories in Case ii).

4.3. Examples

Finally, in Figs. 3, 4 and 5 we give several examples of trajectories obtained from parametric representations (21), (25) and (26), (27).

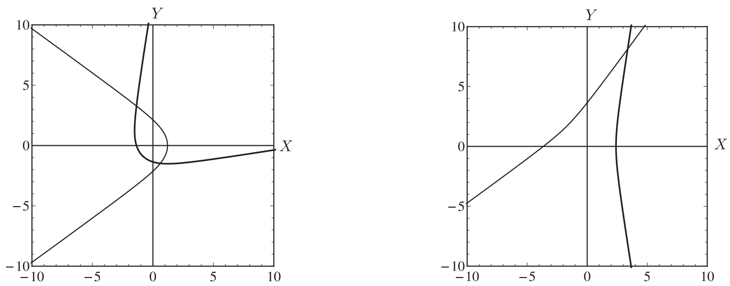

Figure 3. Unbounded trajectories derived from parametric equations (21), (25) with $\nu = 1$, $\varepsilon = 2$, $\sigma = 0.956$, $\mathring{r} = 0.790238$ (left, thin line), $\mathring{r} = -2.39857$ (left, thick line) and with $\nu = 1$, $\varepsilon = -2$, $\sigma = 0.956$, $\mathring{r} = -1.20976$ (right, thin line), $\mathring{r} = 2.39857$ (right, thick line)

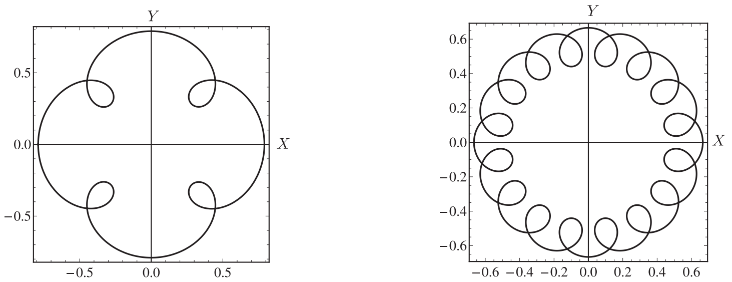

Figure 4. Closed trajectories obtained from equations (21), (25) with $\nu = 1$, $\varepsilon = 2$, $\sigma = 0.956$, $\mathring{r} = 0.790238$ (left) and $\nu = 1$, $\varepsilon = -3$, $\sigma = 1.554$, $\mathring{r} = -0.665734$ (right).

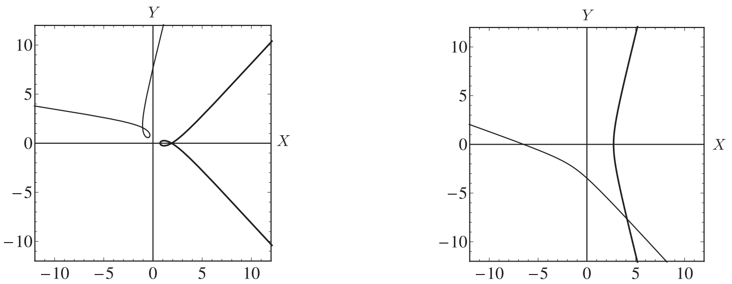

Figure 5. Unbounded trajectories obtained using the parametric equations (26), (27) with $\nu = 1$, $\varepsilon = 2$, $\sigma = 2$, $\mathring{r} = -2.73205$ (left, thin line), $\mathring{r} = 0.73205$ (left, thick line) and $\nu = 1$, $\varepsilon = -2$, $\sigma = 2$, $\mathring{r} = -0.73205$ (right, thin line), $\mathring{r} = 2.73205$ (right, thick line). The graph of the polynomial $P[r]$ in the latter case is depicted in Fig. 2 (right).

5. Concluding Remarks

Let us recall that the system of Newton–Lorentz equations considered here describes the planar motion of a charged particle in the equatorial plane of the magnetic dipole field. This system falls into the class of two degrees of freedom dynamical systems of the so called Frenet–Serret type. It is integrable by quadratures since it possesses two functionally independent integrals of motion.

In the present contribution, using the aforementioned property of the Newton–Lorentz system we have presented in explicit form, via equations (21), (25) and (26), (27), the parametric equations of its trajectories in terms of Jacobi elliptic functions and elliptic integrals depending on the auxiliary variable τ. The time dependence can be recovered by integrating equation (5).

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