Fifteenth International Conference on
Geometry, Integrability and Quantization
June 7–12, 2013, Varna, Bulgaria
Ivaïlo M. Mladenov, Andrei Ludu
and Akira Yoshioka, Editors
Avangard Prima, Sofia 2014, pp 162–172
doi: 10.7546/giq-15-2014-162-172

AN S^1-REDUCTION OF NON-FORMAL STAR PRODUCT

TOMOYO KANAZAWA and AKIRA YOSHIOKA

Department of Mathematics, Tokyo University of Science
Kagurazaka1-3, Shinjuku-ku, Tokyo 162-8601, JAPAN

Abstract. Starting from the Moyal product on eight-dimensional canonical Euclidean phase space $T^*\mathbb{R}^4$ with an S^1-symplectic action, we construct a non-formal star product, i.e., the deformation parameter is a real number, on the cotangent bundle of three-dimensional Euclidean space except the origin $T^*(\mathbb{R}^3\backslash\{0\})$ which is the reduced symplectic manifold by the S^1-action.

1. Introduction

In this paper we construct a non-formal star product on the phase space of the MIC-Kepler problem $T^*(\mathbb{R}^3\backslash\{0\})$. The phase space is equipped with a symplectic structure given as a sum of the canonical symplectic structure and the closed two form of the configuration space which represents the Dirac's magnetic monopole. The product is given by an S^1 reduction from the Moyal product on the cotangent bundle $T^*\mathbb{R}^4$.

Fedosov [2] gives a general formula for the reduction of star products for the case of formal deformation quantization. However, the situation is quite different when we treat the deformation parameter non formal. Most of techniques used in the formal star product case are not useful to the non-formal star product problems.

MIC-Kepler problem was proposed by McIntosh and Cisneros [9], which is a dynamical system describing a motion of an electron in the hydrogen atom under the influence of Kepler potential and the Dirac's magnetic monopole field.

The MIC-Kepler problem is formulated in terms with S^1 reduction by Iwai-Uwano [3, 4]. On the cotangent bundle $T^*(\mathbb{R}^4\backslash\{0\})$ the conformal Kepler problem is given and then the MIC-Kepler problem is obtained by the S^1-reduction. The classical system was investigated in [3] and the quantum version was studied in [9–11] where the eigenvalues and the multiplicities were calculated.

The quantized MIC-Kepler problem is considered in terms of the Moyal product on the cotangent bundle $T^*(\mathbb{R}^4\backslash\{0\})$ in Kanazawa-Yoshioka [5], Kanazawa [6,7]. In [5], the star eigen values and the multiplicities are calculated by means of the conformal Kepler problem and are found to be the same as the values given in [4]. Although these quantities are calculated by means of the Moyal product on the original phase space $T^*(\mathbb{R}^4\backslash\{0\})$, the relationship is still unclear between the Moyal star eigen functions on the original space and the S^1-reduced star product obtained in this paper. We will study the star eigenvalue problem with respect to the S^1-reduced star product in future.

2. MIC-Kepler Problem

McIntosh and Cisneros [9] studied the dynamical system describing the motion of a charged particle under the influence of Dirac's monopole field besides the Coulomb's potential. Iwai-Uwano [3] give the Hamiltonian description for the MIC-Kepler problem as follows.

We consider a closed two form Ω on $\dot{\mathbb{R}}^3 = \mathbb{R}^3\backslash\{0\}$

$$\Omega = \left(x_1\, \mathrm{d}x_2 \wedge \mathrm{d}x_3 + x_2\, \mathrm{d}x_3 \wedge \mathrm{d}x_1 + x_3\, \mathrm{d}x_1 \wedge \mathrm{d}x_2\right)/r^3$$

where $r = \sqrt{x_1^2 + x_2^2 + x_3^2}$. We consider the cotangent bundle $T^*\dot{\mathbb{R}}^3$ and a symplectic form

$$\sigma_\mu = \mathrm{d}\xi_1 \wedge \mathrm{d}x_1 + \mathrm{d}\xi_2 \wedge \mathrm{d}x_2 + \mathrm{d}\xi_3 \wedge \mathrm{d}x_3 + \Omega_\mu$$

where $(\boldsymbol{x}, \boldsymbol{\xi}) = (x_1, x_2, x_3, \xi_1, \xi_2, \xi_3) \in T^*\dot{\mathbb{R}}^3$ and the two-form $\Omega_\mu \equiv -\mu\,\Omega$ stands for Dirac's monopole field of strength $-\mu \in \mathbb{R}$. Then the MIC-Kepler problem is given as the triple $(\,T^*\dot{\mathbb{R}}^3,\, \sigma_\mu\,,\, H_\mu\,)$ where H_μ is the Hamiltonian function such that

$$H_\mu\,(\boldsymbol{x},\, \boldsymbol{\xi}) = \frac{1}{2}\left(\xi_1^2 + \xi_2^2 + \xi_3^2\right) + \frac{\mu^2}{2r^2} - \frac{k}{r}$$

and k is a positive constant.

3. Reduction

In this section, we recall the method of the S^1-reduction which reduces the conformal Kepler problem on $T^*\dot{\mathbb{R}}^4$ to the MIC-Kepler problem on $T^*\dot{\mathbb{R}}^3$ given in Iwai-Uwano [3].

3.1. S^1 Action

The space $\mathbb{R}^4$ is naturally identified with $\mathbb{C}^2$ which yields the diffeomorphism Φ of $T^*\mathbb{R}^4$ to $T^*\mathbb{C}^2 = \mathbb{C}^4$ such that

$$\Phi : T^*\mathbb{R}^4 \ni (y_1, y_2, y_3, y_4, \eta_1, \eta_2, \eta_3, \eta_4) \mapsto (z_1, z_2, \zeta_1, \zeta_2) \in \mathbb{C}^4$$

where

$$z_1 = y_1 + \mathrm{i}y_2, \qquad z_2 = y_3 + \mathrm{i}y_4, \qquad \zeta_1 = \eta_1 + \mathrm{i}\eta_2, \qquad \zeta_2 = \eta_3 + \mathrm{i}\eta_4. \quad (1)$$

With this identification the Euclidean inner product of $\mathbb{R}^4$ is written in $\mathbb{C}^2$ as

$$\langle z, z' \rangle = \Re\, z \cdot \bar{z}' = (z_1\bar{z}_1' + z_2\bar{z}_2' + \bar{z}_1 z_1' + \bar{z}_2 z_2')/2.$$

Also the canonical one-form θ on $T^*\mathbb{R}^4$ in these coordinates is

$$\theta(z, \zeta) = \langle \zeta, \mathrm{d}z \rangle.$$

Now we define an S^1 action on $\dot{\mathbb{R}}^4 = \mathbb{R}^4 \backslash \{0\} = \dot{\mathbb{C}}^2$ by $z \mapsto \mathrm{e}^{\mathrm{i}t} z$ which induces the action on the cotangent bundle $T^*\dot{\mathbb{R}}^4$

$$\varphi_t : (z, \zeta) \mapsto (\mathrm{e}^{\mathrm{i}t} z, \mathrm{e}^{\mathrm{i}t} \zeta).$$

We remark here t $(0 \leq t < 2\pi)$ replaces $t/2$ $(0 \leq t/2 < 2\pi)$ used in [3] as a parameter. The induced action φ_t preserves the canonical one form θ and then is an exact symplectic action. The induced vector field $v(z, \zeta)$ on $T^*\dot{\mathbb{R}}^4$ of the action is

$$v(z, \zeta) = (\mathrm{i}z, \mathrm{i}\zeta)$$

and

$$\psi(z, \zeta) = \iota_v \theta(z, \zeta) = \langle \zeta, \mathrm{i}z \rangle = \Im\, \zeta \cdot \bar{z} = (\zeta \cdot \bar{z} - \bar{\zeta} \cdot z)/2\mathrm{i}$$

is a moment map ψ of the action.

3.2. S^1 Reduction

Now we consider a level set of the moment map $\psi^{-1}(\mu)$ for $\mu \in \mathbb{R}$. We apply the Marsden-Weinstein reduction procedure to the S^1-bundle $\pi_\mu : \psi^{-1}(\mu) \to \psi^{-1}(\mu)/S^1$. Then we obtain the reduced symplectic manifold $(\psi^{-1}(\mu)/S^1, \sigma_\mu)$ such that $\iota_\mu^* \mathrm{d}\theta = \pi_\mu^* \sigma_\mu$, where $\iota_\mu : \psi^{-1}(\mu) \to T^*\dot{\mathbb{R}}^4$ is the inclusion map.

We will show that the reduced symplectic manifold $(\psi^{-1}(\mu)/S^1, \sigma_\mu)$ is realized as the phase space of the MIC-Kepler problem (cf. [3]).

The S^1 action induces the projection

$$\pi : \dot{\mathbb{R}}^4 \to \dot{\mathbb{R}}^3, \qquad \pi(z_1, z_2) = (x_1, x_2, x_3)$$

where

$$x_1 = \Re\, 2z_1\bar{z}_2, \qquad x_2 = \Im\, 2z_1\bar{z}_2, \qquad x_3 = |z_1|^2 - |z_2|^2.$$

We introduce a riemannian metric $g(z)$ on $\dot{\mathbb{R}}^4$ such that

$$g(z)(Z_1, Z_2) = 4|z|^2 \langle Z_1, Z_2 \rangle, \qquad Z_1, Z_2 \in T_z\dot{\mathbb{R}}^4$$

and a riemannian metric g_0 on $\dot{\mathbb{R}}^3$ as the canonical Euclidean inner product. By means of the riemannian metric g, we identify the cotangent bundle with the tangent bundle

$$\# : T^*\dot{\mathbb{R}}^4 \to T\dot{\mathbb{R}}^4, \qquad (z,\zeta) \mapsto (z,Z) = (z,\zeta/4|z|^2) \tag{2}$$

and by g_0, we identify $T^*\dot{\mathbb{R}}^3$ with $T\dot{\mathbb{R}}^3$, i.e.,

$$\#_0 : T^*\dot{\mathbb{R}}^3 \to T\dot{\mathbb{R}}^3.$$

Using the identification maps $\#, \#_0$ and the projection $\mathrm{d}\pi : T\dot{\mathbb{R}}^4 \to T\dot{\mathbb{R}}^3$ we define a map

$$\tilde{\pi} = (\#_0)^{-1} \circ \mathrm{d}\pi \circ \# : T^*\dot{\mathbb{R}}^4 \to T^*\dot{\mathbb{R}}^3. \tag{3}$$

0 Now we write the map $\tilde{\pi}(z,\zeta) = (x,\xi)$ explicitly. Let us introduce an orthonormal frame of $T_z\,\dot{\mathbb{R}}^4$ at every $z \in \dot{\mathbb{R}}^4$ with respect to the riemannian metric g such that

$$\Xi_1(z) = (z_2, z_1)/2|z|^2, \qquad \Xi_2(z) = (\mathrm{i}z_2, -\mathrm{i}z_1)/2|z|^2$$
$$\Xi_3(z) = (z_1, -z_2)/2|z|^2, \qquad \Xi_4(z) = (\mathrm{i}z_1, \mathrm{i}z_2)/2|z|^2.$$

The frame is equivariant under the S^1 action

$$\Xi_j(\mathrm{e}^{\mathrm{i}t}z) = \mathrm{e}^{\mathrm{i}t}\Xi_j(z), \qquad j = 1, \cdots, 4$$

and satisfies

$$\partial/\partial_{x_1} = \pi_{*z}\Xi_1(z), \qquad \partial/\partial_{x_2} = \pi_{*z}\Xi_2(z)$$
$$\partial/\partial_{x_3} = \pi_{*z}\Xi_3(z), \qquad 0 = \pi_{*z}\Xi_4(z).$$

Using the frame we write an element $(z,Z) \in T\dot{\mathbb{R}}^4$ as a linear combination of Ξ_j

$$Z = \alpha_1\Xi_1(z) + \alpha_2\Xi_2(z) + \alpha_3\Xi_3(z) + \alpha_4\Xi_4(z) \tag{4}$$

where

$$\alpha_j = g(Z, \Xi_j(z)) = 4|z|^2\langle Z, \Xi_j(z)\rangle, \qquad j = 1, \cdots, 4. \tag{5}$$

Then substituting (2) into (5) shows

Proposition 1. The components of the map defined by (3) $\tilde{\pi}(z,\zeta) = (x,\xi)$ are written explicitly as

$$x_1(z) = \Re\, 2z_1\bar{z}_2, \qquad x_2(z) = \Im\, 2z_1\bar{z}_2, \qquad x_3(z) = |z_1|^2 - |z_2|^2$$

and

$$\xi_1(z,\zeta) = \langle\zeta, \Xi_1(z)\rangle, \qquad \xi_2(z,\zeta) = \langle\zeta, \Xi_2(z)\rangle, \qquad \xi_3(z,\zeta) = \langle\zeta, \Xi_3(z)\rangle.$$

Moreover the moment map ψ is written as

$$\psi(z,\zeta) = \alpha_4(\#(z,\zeta)) = \langle\zeta, \Xi_4(z)\rangle.$$

Now we will write the reduced phase space $(\psi^{-1}(\mu)/S^1, \sigma_\mu)$ using these notations. We set

$$M_0 = \psi^{-1}(\mu).$$

By means of the decomposition (4), the identifications $\# : T^*\dot{\mathbb{R}}^4 \to T\dot{\mathbb{R}}^4$ and $\#_0 : T^*\dot{\mathbb{R}}^3 \to T\dot{\mathbb{R}}^3$ give a diffeomorphism τ (cf. Fedosov [2]) such as

Lemma 1.

$$\tau : T^*\dot{\mathbb{R}}^4 \to M_0 \times \mathbb{R}, \qquad \tau(z, \zeta) = ((z, \zeta_0), s)$$

where $(z, \zeta_0) \in M_0$ such that

$$\zeta_0 = 4|z|^2 \left\{ \langle \zeta, \Xi_1(z) \rangle \Xi_1(z) + \langle \zeta, \Xi_2(z) \rangle \Xi_2(z) + \langle \zeta, \Xi_3(z) \rangle \Xi_3(z) \right\} + 2\mu\, \Xi_4(z)$$

and

$$s = \psi(z, \zeta) - \mu.$$

Through this isomorphism the S^1 action on $T^*\dot{\mathbb{R}}^4$ is transformed on $M_0 \times \mathbb{R}$ as

$$\varphi_t : ((z, \zeta_0), s) \mapsto ((\mathrm{e}^{\mathrm{i}t} z, \mathrm{e}^{\mathrm{i}t} \zeta_0), s)$$

and the projection $\tilde{\pi}$ in Porposition 1 yields a projection map

$$\tilde{\pi} : M_0 \to T^*\dot{\mathbb{R}}^3, \qquad \tilde{\pi}(z, \zeta_0) = (x, \xi)$$

such that

$$\xi_1 = \langle \zeta_0, \Xi_1(z) \rangle, \qquad \xi_2 = \langle \zeta_0, \Xi_2(z) \rangle, \qquad \xi_3 = \langle \zeta_0, \Xi_3(z) \rangle.$$

The canonical one form is written on $M_0 \times \mathbb{R}$ as

$$(\tau^{-1})^* \theta = \tilde{\pi}^* \tilde{\theta} + (\mu + s)\gamma(z)$$

where $\tilde{\theta}$ is the canonical one form on $T^*\dot{\mathbb{R}}^3$ and

$$\gamma(z) = \langle \mathrm{d}z, \mathrm{i}z \rangle / |z|^2 = \Im\, \bar{z} \cdot \mathrm{d}z / |z|^2.$$

By a direct calculation we have

Lemma 2. $\mathrm{d}\gamma(z) = \pi^* \Omega$, where

$$\Omega = (\, x_1\, \mathrm{d}x_2 \wedge \mathrm{d}x_3 \, + \, x_2\, \mathrm{d}x_3 \wedge \mathrm{d}x_1 \, + \, x_3\, \mathrm{d}x_1 \wedge \mathrm{d}x_2) / 2r^3$$

is the closed two-form on $\dot{\mathbb{R}}^3$ given in the introduction.

Thus we have

Theorem 1. ([3]) The reduced phase space $(\psi^{-1}(\mu)/S^1, \sigma_\mu)$ is isomorphic to the symplectic manifold $(\, T^*\dot{\mathbb{R}}^3,\, \sigma_\mu\,)$ such that $\iota_\mu^* \mathrm{d}\theta = \pi_\mu^* \sigma_\mu$, where

$$\sigma_\mu = \mathrm{d}\tilde{\theta} + \Omega_\mu = \mathrm{d}\xi_1 \wedge \mathrm{d}x_1 \, + \, \mathrm{d}\xi_2 \wedge \mathrm{d}x_2 \, + \, \mathrm{d}\xi_3 \wedge \mathrm{d}x_3 \, + \, \Omega_\mu.$$

4. Star Product on $(T^*\dot{\mathbb{R}}^3, \sigma_\mu)$

We will construct a star product (cf. [1]) for $(T^*\dot{\mathbb{R}}^3, \sigma_\mu)$, namely, an associative product $*_\mu$ for polynomials $\tilde{f}(x,\xi), \tilde{g}(x,\xi)$ such that

$$\tilde{f}(x,\xi) *_\mu \tilde{g}(x,\xi) = \tilde{f}(x,\xi)\tilde{g}(x,\xi) + \tfrac{\mathrm{i}\hbar}{2}\left\{\tilde{f}(x,\xi), \tilde{g}(x,\xi)\right\}_\mu + \cdots + \hbar^n C_n(\tilde{f}(x,\xi), \tilde{g}(x,\xi)) + \cdots$$

where $\hbar$ is a positive constant, $\{\cdot,\cdot\}_\mu$ is the Poisson bracket given by the symplectic form σ_μ and C_n, $n = 2, 3, \ldots$ are bidifferential operators.
We will obtain $*_\mu$ by means of the S^1 reduction of $T^*\dot{\mathbb{R}}^4$.

4.1. The Moyal Product on $T^*\dot{\mathbb{R}}^4$

On $T^*\dot{\mathbb{R}}^4$, the canonical coordinate (y,η) defines the Moyal product $*_0$ which is expressed in terms with (z,ζ) of the identification (1) such that

$$\begin{aligned} f *_0 g(z,\zeta) &= f(z,\zeta)\exp(\hbar D) g(z,\zeta) \\ &= f(z,\zeta)\left(\sum_{n\geq 0} \tfrac{1}{n!}(\mathrm{i}\hbar)^n D^n\right) g(z,\zeta) \\ &= f(z,\zeta)g(z,\zeta) + \mathrm{i}\hbar\,\{f(z,\zeta), g(z,\zeta)\} \\ &\quad + \cdots + \tfrac{1}{n!}(\mathrm{i}\hbar)^n f(z,\zeta)(D^n)g(z,\zeta) + \cdots \end{aligned} \tag{6}$$

where $\{f(z,\zeta), g(z,\zeta)\}$ is the Poisson bracket of the canonical symplectic form θ on $T^*\dot{\mathbb{R}}^4$ and D^n is the nth power of the Poisson biderivation

$$D = \overleftarrow{\partial}_z \wedge \overrightarrow{\partial}_{\bar{\zeta}} + \overleftarrow{\partial}_{\bar{z}} \wedge \overrightarrow{\partial}_{\zeta}, \qquad f(z,\zeta) D g(z,\zeta) = \{f(z,\zeta), g(z,\zeta)\}\,.$$

The generators $z_j, \bar{z}_j, \zeta_k, \bar{\zeta}_k$ $(j,k = 1,2)$ satisfy the commutation relations of the Moyal product

$$[z_j, \bar{\zeta}_k] = [\bar{z}_j, \zeta_k] = 2\mathrm{i}\hbar\delta_{jk}$$

and other commutators are equal to zero.

4.2. Algebra

We consider the components of the projection given in Proposition 1

$$\begin{aligned} &x_1 = x_1(z), && x_2 = x_2(z), && x_3 = x_3(z) \\ &\xi_1 = \xi_1(z,\zeta), && \xi_2 = \xi_2(z,\zeta), && \xi_3 = \xi_3(z,\zeta). \end{aligned}$$

These are functions defined on $T^*\dot{\mathbb{R}}^4$. We see easily by direct calculation

Proposition 2. The Moyal product $*_0$ for the functions $x_j(z),\ \xi_j(z,\zeta),\ j=1,2,3$ are as follows

1. $x_j(z) *_0 x_k(z) = x_j(z)x_k(z)$
2. $\xi_j(z,\zeta) *_0 x_k(z) = \xi_j(z,\zeta)x_k(z) - \frac{i\hbar}{2}\delta_{jk}$
   $x_k(z) *_0 \xi_j(z,\zeta) = \xi_j(z,\zeta)x_k(z) + \frac{i\hbar}{2}\delta_{jk}$
3. $\xi_j(z,\zeta) *_0 \xi_k(z,\zeta) = \xi_j(z,\zeta)\xi_k(z,\zeta) - \epsilon_{jkl}\frac{i\hbar}{4}\frac{x_l(z)}{|z|^6}\psi(z,\zeta) + \frac{\hbar^2}{4}\frac{x_j(z)x_k(z)}{|z|^8}$.

Also for the moment map $\psi(z,\zeta)$ we have $\psi(z,\zeta) *_0 x_j(z) = \psi(z,\zeta)x_j(z)$, $\psi(z,\zeta) *_0 \xi_j(z,\zeta) = \psi(z,\zeta)\xi_j(z,\zeta)$.

Proposition 2 gives

Proposition 3. The commutation relations of the functions $x_j(z),\ \xi_j(z,\zeta),\ j = 1,2,3$ are

$$[x_j(z), x_k(z)]_{*_0} = 0, \qquad [\xi_j(z,\zeta), x_k(z)]_{*_0} = -i\hbar\delta_{jk}$$
$$[\xi_j(z,\zeta), \xi_k(z,\zeta)]_{*_0} = -\epsilon_{jkl}\frac{i\hbar}{2}\frac{x_l(z)}{|z|^6}\psi(z,\zeta)$$

and ψ commutes with $x_j(z), \xi_j(z,\zeta),\ (j=1,2,3)$ where $[f,g]_{*_0} = f *_0 g - g *_0 f$ is the commutator with respect to the multiplication $*_0$.

In what follows, we will give an algebra of functions on $T^*\dot{\mathbb{R}}^4$ under the Moyal product $*_0$. For a smooth function $a(x) = a(x_1,x_2,x_3)$ on $T^*\dot{\mathbb{R}}^3$, we substitute functions $x_j = x_j(z),\ (j=1,2,3)$ to obtain a smooth function $a(x(z))$ on $T^*\dot{\mathbb{R}}^4$. We consider an element of the form

$$f = \sum_{j,k,l,m\geq 0} a_{jklm}(x(z)) *_0\ \psi(z,\zeta)^j_* *_0 \xi_1{}^k_* *_0\ \xi_2{}^l_* *_0\ \xi_3{}^m_* \tag{7}$$

where the summation is a finite sum

$$\xi_1{}^k_* = \underbrace{\xi_1(z,\zeta) *_0 \cdots *_0 \xi_1(z,\zeta)}_{k}, \quad \text{etc.,} \quad \text{and} \quad a_{jklm}(x) \in C^\infty(T^*\dot{\mathbb{R}}^3).$$

Let us consider a subset $\mathcal{A} \subset C^\infty(T^*\dot{\mathbb{R}}^4)$ such that

$$\mathcal{A} = \{f \in C^\infty(T^*\dot{\mathbb{R}}^4);\ f \ \text{ is of the form (7)}\}.$$

Then using the commutation relation we can show that for any $f, g \in \mathcal{A}$, the multipilcation $f *_0 g$ also belongs to $\mathcal{A}$, namely and we have

Proposition 4. The set $\mathcal{A}$ becomes an associative algebra under the multiplication $*_0$.

Remark 2. *By calculating the Moyal product $*_0$ and relations in* Proposition 2*, we see easily that an element of $\mathcal{A}$ can be written in the form*

$$f_c = \sum_{j,k,l,m\geq 0} a_{jklm}(x(z))\, \psi(z,\zeta)^j\, \xi_1^k(z,\zeta)\, \xi_2^l(z,\zeta)\, \xi_3^m(z,\zeta) \tag{8}$$

where the multiplication is the usual multiplication of functions. Hence it is obvious that $\mathcal{A}$ is just the set

$$\mathcal{P} = \{f_c \in C^\infty(T^*\dot{\mathbb{R}}^4)\,;\ f_c \text{ is of the form } (8)\}.$$

4.3. Expression by Complete Symmetrization

In order to obtain the reduced algebra by the S^1-reduction, we consider the expression of elements of $\mathcal{A}$ by means of the complete symmetrization. For an element of $\mathcal{A}$ such as $Q = a(x(z)) *_0 \psi(z,\zeta)^j_* *_0 {\xi_1}^k_* *_0 {\xi_2}^l_* *_0 {\xi_3}^m_*$, we take the complete symmetrization in the following way. We regard this element as a monomial of the generators $a(x(z)), \psi(z,\zeta), \xi_p(z,\zeta)$, $p = 1,2,3$, of total degree $N = j+k+l+m+1$, namely, regard as an element of the form $A_1 *_0 \cdots *_0 A_N$ where $A_q, (q = 1, \cdots, N)$ is one of $a(x(z)), \psi(z,\zeta), \xi_p(z,\zeta)$, $(p = 1,2,3)$. The complete symmetrization of Q is an element given by

$$\tfrac{1}{N!} \sum_{\sigma \in \mathfrak{S}_N} A_{\sigma(1)} *_0 \cdots *_0 A_{\sigma(N)}$$

where $\mathfrak{S}_N$ is the N permutation group.

Now we have the following.

Proposition 5. Any element f of the algebra $(\mathcal{A}, *_0)$ is expressed uniquely by means of the symmetrization as

$$f = \sum_{j,k,l,m \geq 0} a_{jklm}(x(z)) \cdot \psi(z,\zeta)^j \cdot \xi_1^k(z,\zeta) \cdot \xi_2^l(z,\zeta) \cdot \xi_3^m(z,\zeta) \tag{9}$$

where each term $a_{jklm}(x(z)) \cdot \psi(z,\zeta)^j \cdot \xi_1^k \cdot \xi_2^l \cdot \xi_3^m$ is the complete symmetrization of the term $a_{jklm}(x(z)) *_0 \psi(z,\zeta)^j_* *_0 {\xi_1}^k_* *_0 {\xi_2}^l_* *_0 {\xi_3}^m_*$.

4.4. Ideal and Quotient Algebra

We consider an ideal of $\mathcal{A}$ and the quotient algebra to obtain certain star product algebra on $T^*\dot{\mathbb{R}}^3$ which is regarded as the reduced algebra by the S^1-action. We put $\mathcal{J}_\mu$ the two-sided ideal of $(\mathcal{A}, *_0)$ generated by $\psi(z,\zeta) - \mu$ where μ is a fixed real constant. Now we define an associative algebra $(\tilde{\mathcal{A}}, *_0') = (\mathcal{A}, *_0)/\mathcal{J}_\mu$ where $*_0'$ is the induced product. We naturally have an algebra homomorphism

$$\phi : (\mathcal{A}, *_0) \to (\tilde{\mathcal{A}}, *_0').$$

For an element of $\mathcal{A}$ such as $Q = a(x(z)) *_0 \psi(z,\zeta)^j_* *_0 {\xi_1}^k_* *_0 {\xi_2}^l_* *_0 {\xi_3}^m_*$, and using the associativity of $*_0$ we have that $\phi(Q) = a(x(z))\, \mu^j *_0' {\xi_1}^k_* *_0' {\xi_2}^l_* *_0' {\xi_3}^m_*$.

Also $f \in \mathcal{A}$ expressed in the symmetrized form (9) can be written

$$\phi(f) = \sum_{j,k,l,m \geq 0} a_{jklm}(x(z))\ \mu^j \cdot \xi_1^k(z,\zeta)\ \cdot\ \xi_2^l(z,\zeta)\ \cdot\ \xi_3^m(z,\zeta)$$

where $a_{jklm}(x(z))\ \mu^j \cdot \xi_1^k(z,\zeta)\ \cdot\ \xi_2^l(z,\zeta)\ \cdot\ \xi_3^m(z,\zeta)$ is the complete symmetrization taken in $(\tilde{\mathcal{A}}, *_0')$.

4.5. Star Product on $T^*\dot{\mathbb{R}}^3$

Now we define a star product on $T^*\dot{\mathbb{R}}^3$. In the previous section, we define a product algebra $(\mathcal{A}, *_0)$ on $T^*\dot{\mathbb{R}}^4$. We will show this product naturally induces a star product on the symplectic manifold $(T^*\dot{\mathbb{R}}^3, \sigma_\mu)$.

Polynomials. We consider a smooth function on $T^*\dot{\mathbb{R}}^3$ of polynomial form of ξ_1, ξ_2, ξ_3 such that

$$\tilde{f} = \sum_{k,l,m \geq 0} a_{klm}(x_1, x_2, x_3)\, \xi_1^k\, \xi_2^l\, \xi_3^m, \quad a_{klm}(x_1, x_2, x_3) \in C^\infty(T^*\dot{\mathbb{R}}^3). \tag{10}$$

We denote by $\tilde{\mathcal{P}}$ the set of such polynomials $\tilde{f}$ of the form (10).

4.6. Star Product

Now we give a star product on $\tilde{\mathcal{P}}$. To an element $\tilde{f} \in \tilde{\mathcal{P}}$ we assign an element $\Psi(\tilde{f}) \in \tilde{\mathcal{A}}$ such that

$$\Psi(\tilde{f}) = \sum_{k,l,m \geq 0} a_{klm}(x(z))\ \cdot \xi_1(z,\zeta)^k\ \cdot \xi_2(z,\zeta)^l\ \cdot \xi_3(z,\zeta)^m$$

where each term $a_{klm}(x(z))\ \cdot \xi_1(z,\zeta)^k \cdot \xi_2(z,\zeta)^l\ \cdot \xi_3(z,\zeta)^m$ is the complete symmetrization of the term $a_{klm}(x(z)) *_0'\ \xi_1^k(z,\zeta) *_0'\ \xi_2^l(z,\zeta) *_0'\ \xi_3^m(z,\zeta) \in \tilde{\mathcal{A}}$.
For example we see

$$\Psi(\xi_1\xi_2) = \xi_1(z,\zeta) \cdot \xi_2(z,\zeta) = (\xi_1(z,\zeta) *_0' \xi_2(z,\zeta) + \xi_2(z,\zeta) *_0' \xi_1(z,\zeta))/2.$$

As to other generators we see similarly.
We remark here that the map Ψ induces a map of $\tilde{\mathcal{P}}$ to $\tilde{\mathcal{A}} = \mathcal{A}/\mathcal{J}_\mu$. It is easy to see the induced map $\Psi : \tilde{\mathcal{P}} \to \tilde{\mathcal{A}}$ is a linear isomorphism.
Now, we introduce an associative product $*_\mu$ on $\tilde{\mathcal{P}}$ by

Definition 1.

$$\tilde{f} *_\mu \tilde{g} = \Psi^{-1}(\Psi\tilde{f} *_0' \Psi\tilde{g}), \qquad \tilde{f}, \tilde{g} \in \tilde{\mathcal{P}}.$$

Here we remark the following. Since $\phi : \mathcal{A} \to \tilde{\mathcal{A}} = \mathcal{A}/\mathcal{J}_\mu$ is a surjective algebra homomorphism, for $\Psi\tilde{f}, \Psi\tilde{g} \in \tilde{\mathcal{A}} = \mathcal{A}/\mathcal{J}_\mu$ there exist $f, g \in \mathcal{A}$ such that $\Psi\tilde{f} = \phi(f)$, $\Psi\tilde{g} = \phi(g)$, and hence we have $\Psi\tilde{f} *'_0 \Psi\tilde{g} = \phi(f *_0 g)$. Thus, we can calculate the star product $*_\mu$ by the formula

$$\tilde{f} *_\mu \tilde{g} = \Psi^{-1}\phi(f *_0 g). \tag{11}$$

Then we have

Theorem 3. *The product $*_\mu$ is a star product, namely*

$$\tilde{f} *_\mu \tilde{g} = \tilde{f}\,\tilde{g} + \tfrac{i\hbar}{2}\{\tilde{f}, \tilde{g}\}_\mu + \cdots + \hbar^n C_n(\tilde{f}, \tilde{g}) + \cdots$$

where $\{\tilde{f}, \tilde{g}\}_\mu$ is the Poisson bracket of the symplectic structure σ_μ and C_n is a bidifferential operator on $T^*\dot{\mathbb{R}}^3$ for every $n = 2, 3, \ldots$.

Proof: For elements of $\tilde{\mathcal{P}}$

$$\tilde{f} = \sum_{k,l,m \geq 0} a_{klm}(x)\, \xi_1^k\, \xi_2^l\, \xi_3^m, \qquad \tilde{g} = \sum_{\alpha,\beta,\gamma \geq 0} b_{\alpha\beta\gamma}(x)\, \xi_1^\alpha\, \xi_2^\beta\, \xi_3^\gamma$$

we can take $f, g \in \mathcal{A}$ satisfying $\Psi(\tilde{f}) = \phi(f)$, $\Psi(\tilde{g}) = \phi(g)$ such as

$$\begin{aligned} f &= \sum_{k,l,m \geq 0} a_{klm}(x(z)) \cdot \xi_1^k(z,\zeta) \cdot \xi_2^l(z,\zeta) \cdot \xi_3^m(z,\zeta) \\ g &= \sum_{\alpha,\beta,\gamma \geq 0} b_{\alpha\beta\gamma}(x(z)) \cdot \xi_1^\alpha(z,\zeta) \cdot \xi_2^\beta(z,\zeta) \cdot \xi_3^\gamma(z,\zeta). \end{aligned}$$

We calculate $\phi(f *_0 g)$ of (11) in the following way. As we see in Remark 2 we can rewrite the elements f, g in the form (8). Then we calculate the Moyal product $f *_0 g$ by means of the formula (6) which is given in the polynimal form (8). By a direct calculation, we rewrite this into the form (7) (see for example, [8]) and further changing the orders of elements by means of the formulae in Proposition 3 we rewrite into the complete symmetrized form (9). Then we can take the quotient by the ideal $\mathcal{J}_\mu$ to obtain $\phi(f *_0 g)$. Remark that these calculations are given by differential operators or bidifferential operators. Thus the obtained product $\tilde{f} *_\mu \tilde{g} = \Psi^{-1}\phi(f *_0 g)$ has the expansion

$$\tilde{f} *_\mu \tilde{g} = fg + i\hbar C_1(f,g) + (i\hbar)^2 C_2(f,g) + \cdots + (i\hbar)^n C_n(f,g) + \cdots$$

where $C_n, n = 1, 2, \cdots$ are bidifferential operators. We use the formulae in Proposition 3 for the calculation of $[f,g]_{*_\mu} = f *_\mu g - g *_\mu f$ and then we easily see that C_1 is equal to the Poisson bracket of the symplectic structure σ_μ. Then we obtain the proof. ■

References

[1] Bayen F., Flato M. and Fronsdal C., Lichnerowicz A. and Sternheimer D., *Deformation Theory and Quantization I & II*, Ann. Phys. **111** (1978) 61–151.

[2] Fedosov V., *Deformation Quantization and Index Theory*, Akademie, Berlin 1996.

[3] Iwai T. and Uwano Y., *The Four-dimensional Conformal Kepler Problem Reduces to the Three-dimensional Kepler Problem with a Centrifugal Potential and Dirac's Monopole Field. Classical Theory*, J. Math. Phys. **27** (1986) 1523–1529.

[4] Iwai T. and Uwano Y., *The Quantised MIC-Kepler Problem and Its Symmetry Group for Negative Energies*, J. Phys. A: Math. & Gen. **21** (1988) 4083–4104.

[5] Kanazawa T. and Yoshioka A., *Star Product and Its Application to the MIC-Kepler Problem*, J. Geom. Symmetry Phys. **25** (2012) 57–75.

[6] Kanazawa T., *A Specific Illustration of Section Derived From $*$-Unitary Evolution Function*, In: New Developments in Geometric Mechanics, T. Iwai (Ed), RIMS, Kyoto 2012, pp 172–191.

[7] Kanazawa T., *Green's Function, Wavefunction and Wigner Function of The MIC-Kepler Problem*, J. Geom. Symmetry Phys. **30** (2013) 63–73.

[8] Masuda T. and Omori H., *Algebras of Quantum Groups as Quantized Poisson Manifolds*, In: Geometry and Its Applications, T. Nagano (Ed), World Scientific, Singapore 1993, pp 109–120.

[9] McIntosh H. and Cisneros A., *Degeneracy in the Presence of a Magnetic Monopole*, J. Math. Phys. **11** (1970) 896–916.

[10] Mladenov I. and Tsanov V., *Geometric Quantisation of the MIC-Kepler Problem*, J. Phys. A: Math. & Gen. **20** (1987) 5865-5871.

[11] Mladenov I., *Scattering of Charged Partcles off Dyons*, J. Physics A: Math. & Gen. **21** (1988) L1-L4.