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A RECURSION OPERATOR FOR THE GEODESIC FLOW ON N-DIMENSIONAL SPHERE

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Abstract. For a completely integrable system, the way of finding first integrals is not formulated in general. A new characterization for integrable systems using the particular tensor field is investigated which is called a recursion operator. A recursion operator T for a vector field Δ is a diagonizable (1, 1)-type tensor field, invariant under Δ and has vanishing Nijenhuis torsion. One of the important property of T is that T gives constants of the motion (the sequence of first integrals) for the vector field Δ. The purpose of this paper is to discuss a recursion operator T for the geodesic flow on S^n.

1. Introduction

For a completely integrable system, the way of finding the first integrals is not formulated in general.

Liouville proved that a system with n degrees of freedom is integrable by quadratures when there exist n independent first integrals in involution (cf. [1]).

In classical mechanics, *a completely integrable system* in the sense of Liouville are called simply *an integrable system*.

Integrable systems related to the recursion operator were characterized in many papers [2, 3, 6, 8] written since 1980.

There the integrable system is characterized by the recursion operator T in the Hamiltonian dynamical system on the cotangent bundle $T^*\mathcal{M}$ of a manifold $\mathcal{M}$.

The recurtion operator T is a diagonalizable $(1,1)$-tensor field which satisfies certain conditions. In particular, it can be written in the following form if we choose an action-angle variables $(J_k,\ \varphi^k)$

$$T = \sum_k \lambda^k(J_k) \left(\frac{\partial}{\partial J_k} \otimes \mathrm{d}J_k + \frac{\partial}{\partial \varphi^k} \otimes \mathrm{d}\varphi^k \right)$$

where $\lambda^k(J_k)$ are doubly degenerate eigenvalues. Functionally independent constants of the motion are obtained by taking the traces of powers of T, i.e.,

$$\mathrm{Tr}(T^k), \qquad k = 1, 2, \ldots, n.$$

There are some examples of constructing recursion operators from the viewpoint of physics - one-dimensional harmoic oscilator, Kepler problem, KdV equation, etc.

Here, we consider a recursion operator from the viewpoint of geometry, specifically for the geodesic flow on n-dimensional sphere S^n.

In this work, we construct recursion operators for the geodesic flow of the n-dimensional sphere S^n and consider their applications.

Besides the Introduction this paper consists of five additional sections.

Section 2 is devoted to notation and definitions which are used in this paper.

In Section 3 we consider the recursion operator. By the properties of the recursion operator, we are able to determine whether the given system is integrable.

Section 2 and Section 3 are based on [8].

In the sections that follow, we are dedicated to constitute a concrete example of a recursion operator and to present an application of this operator.

Section 4 is about the recursion operator for the geodesic flow on S^n that we had obtained in [5].

Section 5 is an application of the recursion operator. We obtain

- *A sequence of Abelian symmetries between Hamiltonian vector fields.*
- *A sequence of involutive Hamiltonian functions.*

And, finally Section 6 is the conclusion.

Remark 1. *A geometric quantization of the n-dimensional sphere is discussed in full detail in* [7]. *There the authors have obtained the quantum energy spectrum of the geodesic flow on S^n with correspondig multiplicities.*

2. Definitions

We introduce endomorphisms $\hat{T}$ and $\check{T}$ induced by a $(1,1)$-tensor T given in [8].

Definition 2. *Let T be a $(1,1)$-tensor field on a manifold $\mathcal{M}$ which we write in the form*

$$T = \sum_{i,j=1}^{n} T_i^{\ j} \mathrm{d}x^i \otimes \frac{\partial}{\partial x^j}\cdot$$

Then we define endomorphisms $\hat{T}$ and $\check{T}$ by the formulas

$$\hat{T} : T_p\mathcal{M} \ni X \mapsto \hat{T}X \in T_p\mathcal{M}, \qquad \hat{T}X = \sum_{i,j=1}^{n} T_i^{\ j} X^i \frac{\partial}{\partial x^j}$$

$$\check{T} : T_p^*\mathcal{M} \ni \alpha \mapsto \check{T}\alpha \in T_p^*\mathcal{M}, \qquad \check{T}\alpha = \sum_{i,j=1}^{n} \alpha_j T_i^{\ j} \mathrm{d}x^i$$

where the vector field X and the one-form α are of the form

$$X = \sum_{k=1}^{n} X^k \frac{\partial}{\partial x^k}, \qquad \alpha = \sum_{k=1}^{n} \alpha_k \mathrm{d}x^k.$$

Additionally, we introduce a separability of a dynamical vector field see ([8]).

Definition 3. *A dynamical vector field Δ is said to be separable on an open subset $\mathcal{O} \subseteq \mathcal{M}$ when there exists a basis $\{e_i\}$ of local vector fields on $\mathcal{O}$ such that*

$$\mathcal{L}_{e_i} \langle \Delta, \vartheta^j \rangle \neq 0 \Rightarrow i = j$$

where $\{\vartheta^j\}$ is the dual basis of $\{e_i\}$. If $\mathcal{O} = \mathcal{M}$, we say that Δ is separable.

3. Introduction of the Recursion Operator

Here we describe a new characterization of integrable systems. Specifically, we consider a diagonalizable tensor.

Theorem 4 (see [2] and [8]). *Let Δ be a vector field on a $2n$-dimensional manifold $\mathcal{M}$ and suppose Δ admits a diagonalizable $(1,1)$-tensor field T such that*

1. *T is invariant under Δ*
$$\mathcal{L}_\Delta T = 0.$$
2. *T has vanishing Nijenhuis torsion*
$$\mathcal{N}_T = 0.$$
3. *T has doubly degenerate eigenvalues λ^j with nowhere vanishing differentials*
$$\deg \lambda^j = 2, \qquad (\mathrm{d}\lambda^j)_p \neq 0, \qquad p \in \mathcal{M}, \qquad j = 1, \ldots, n.$$

Then, the vector field Δ is separable, completely integrable and Hamiltonian with respect to a certain symplectic structure.

This T is called a recursion operator of the vector field Δ. When a $(1,1)$-tensor T is a recursion operator, there are several important consequences.
E.g., there exist n vector fields Δ_k such that

$$\Delta_{k+1} = \hat{T}\Delta_k, \qquad k \geq 1$$

and n differential one-forms

$$\mathrm{d}\alpha = 0, \qquad \mathrm{d}\check{T}\alpha = 0, \qquad \mathcal{L}_T = 0 \Rightarrow \mathrm{d}\left(\check{T}^n\alpha\right) = 0, \qquad k \geq 1.$$

Next, under the flow generated by a vector field Δ, the invariance of T implies the invariance of $\hat{T}\Delta_n$, $\check{T}^n\alpha$ and that of its eigenvalues λ. Moreover, all Δ_k are Hamiltonian vector fields when the equation

$$i_\Delta\omega = -\mathrm{d}H$$

is true for the Hamiltonian function H and the symplectic structure ω.
Finally, the traces of T^k, i.e., $\mathrm{Tr}\left(T^k\right), k \geq 1$ are constants of motion of the system.

4. Construction of a Recursion Operator for the Geodesic Flow on S^n

We had obtained a recursion operator of the n-dimensional sphere S^n in [5].
The process is as follows

1. Considering the canonical Riemaniann metric on S^n.
2. Calculating the Hamiltonian function H from the metric.
3. Discribing the Hamiltonian system (H, Δ, ω) by the action-angle variables (J_k, φ^k).

Then, we get a recursion operator T.
And, for an application, the constants of motion are written as traces of the powers T

$$\{\mathrm{Tr}(T),\ \mathrm{Tr}(T^2),\ \ldots,\ \mathrm{Tr}(T^n)\}.$$

4.1. Canonical Riemaniann Metric on S^n

Using the spherical polar coordinate of the n-dimensional sphere of radius a, we consider its embedding given by the map ϕ, i.e.,

$$\phi(q^1,\dots,q^n) = \begin{pmatrix} a\cos q_1 \\ a\sin q_1\cos q_2 \\ \cdots\cdots \\ a\sin q_1\cdots\sin q_{n-2}\cos q_{n-1} \\ a\sin q_1\cdots\sin q_{n-2}\sin q_{n-1} \end{pmatrix}.$$

We see that

$$g_{ij} = \rho_i^2\delta_{ij}, \quad i,j = 1,\dots,n, \quad \rho_1 = a, \quad \rho_\ell = a\prod_{k=1}^{\ell-1}\sin q_k, \quad \ell = 2,\dots,n.$$

4.2. Calculating the Hamiltonian Function H

The corresponding Hamiltonian function H is calculated as

$$H(\boldsymbol{q},\boldsymbol{p}) = \frac{1}{2a^2}\sum_{k=1}^{n} P_k\cdot p_k^2, \qquad P_k = \begin{cases} 1, & k=1 \\ \displaystyle\prod_{i=1}^{k-1}\frac{1}{\sin^2 q_i}, & \text{otherwise.} \end{cases} \tag{1}$$

4.3. The Hamiltonian System (H,Δ,ω) in Action-Angle Variables

The Hamilton-Jacobi equation for (1) is

$$E = \frac{1}{2a^2}\sum_{k=1}^{n} P_k\left(\frac{\mathrm{d}S_k}{\mathrm{d}q_k}\right)^2$$

where S is the generating function

$$S = \sum_{i=1}^{n} S_i(q_i).$$

Let us change the variables via the formulas

$$Q_\ell := \left\{R_\ell - \left(\frac{\mathrm{d}S_\ell}{\mathrm{d}q_\ell}\right)^2\right\}\sin^2 q_\ell = \sum_{k=1}^{n-\ell} P_k\left(\frac{\mathrm{d}S_{\ell+k}}{\mathrm{d}q_{\ell+k}}\right)^2$$

where

$$R_\ell = \begin{cases} 2a^2E, & \ell = 1 \\ Q_{\ell-1}, & \text{otherwise} \end{cases}$$

and

$$P_k = \begin{cases} 1, & k = 1 \\ \displaystyle\prod_{i=\ell}^{\ell+k-2} \frac{1}{\sin^2 q_i}, & \text{otherwise.} \end{cases}$$

As Q_ℓ, E and a are constants, we can set

$$\alpha_1 := \sqrt{2a^2E}, \qquad \alpha_\ell := \sqrt{Q_{\ell-1}}$$

and therefore

$$p_\ell = \frac{\mathrm{d}S_\ell}{\mathrm{d}q_\ell} = \begin{cases} \sqrt{\alpha_\ell^2 - \dfrac{\alpha_{\ell+1}^2}{\sin^2 q_\ell}}, & \ell = 1, \ldots, n-1 \\ \alpha_\ell, & \ell = n. \end{cases} \tag{2}$$

Then, introducing the action variables $J_\ell(\boldsymbol{q}, \boldsymbol{p})$

$$J_\ell := \frac{1}{2\pi} \oint p_\ell \mathrm{d}q_\ell$$

we obtain

$$J_\ell = \begin{cases} \alpha_\ell - \alpha_{\ell+1}, & \ell = 1, \ldots, n-1 \\ \alpha_n, & \ell = n \end{cases}$$

and hence

$$\alpha_\ell = \sum_{k=\ell}^{n} J_k. \tag{3}$$

Therefore, from (1), (2) and (3), H is written as a function of J_i

$$H(\boldsymbol{J}) = \frac{1}{2a^2} \left(\sum_{i=1}^{n} J_i \right)^2.$$

The correspondig Hamiltonian vector field Δ is

$$\Delta = \{H, \cdot\,\} = \frac{1}{a^2} \sum_{i,\ell=1}^{n} J_i \frac{\partial}{\partial \varphi_\ell}$$

and the symplectic form ω is

$$\omega = \sum_{i=1}^{n} \mathrm{d}J_i \wedge \mathrm{d}\varphi^i. \tag{4}$$

From the above, the tensor field T is defined by the expression

$$T = \frac{1}{2} \sum_{i,\ell} \left\{ \left({}^t\mathcal{S}\right)^\ell{}_i \frac{\partial}{\partial J_i} \otimes \mathrm{d}J_\ell + \mathcal{S}^i{}_\ell \frac{\partial}{\partial \varphi^i} \otimes \mathrm{d}\varphi^\ell \right\} \tag{5}$$

where

$$\begin{aligned} \mathcal{S}^1{}_1 &= J_1 \\ \mathcal{S}^i{}_1 &= -\sum_k J_k + J_1 + 2J_i, \;\; i > 2 \\ \mathcal{S}^\ell{}_\ell &= \sum_k J_k - J_\ell, \qquad\qquad \ell > 2 \\ \mathcal{S}^i{}_\ell &= J_\ell, \qquad\qquad\qquad\quad \text{otherwise} \end{aligned} \tag{6}$$

which obviously fulfills the conditions for the recursion operator.

Thus, from the above we have obtained the following proposition

Proposition 5 ([5]). *For the introduced canonical Riemannian metric g on S^n, the geodesic flow of T^*S^n has a recursion operator T. The operator is written by means of the action-angle variables $(\boldsymbol{J}(\boldsymbol{q},\boldsymbol{p}), \boldsymbol{\varphi}(\boldsymbol{q},\boldsymbol{p}))$ as*

$$T = \frac{1}{2}\sum_{i,\ell}\left\{\left({}^tS\right)^\ell{}_i \frac{\partial}{\partial J_i} \otimes \mathrm{d}J_\ell + S^i{}_\ell \frac{\partial}{\partial \varphi_i} \otimes \mathrm{d}\varphi_\ell\right\}$$

where $(\boldsymbol{q},\boldsymbol{p})$ form a local coordinate system on T^*S^n.

Example 6. *When we consider the case of the three-dimensional sphere S^3, the recursion operator T is given by*

$$T = \frac{1}{2}\begin{pmatrix} J_1 & J_2 - J_3 & J_3 - J_2 & & & \\ J_2 & J_1 + J_3 & J_2 & & O & \\ J_3 & J_3 & J_1 + J_2 & & & \\ & & & J_1 & J_2 & J_3 \\ & O & & J_2 - J_3 & J_1 + J_3 & J_3 \\ & & & J_3 - J_2 & J_2 & J_1 + J_2 \end{pmatrix}.$$

5. Applications of the Recursion Operator for the Geodesic Flow on S^n

5.1. Constants of Motion

The constants of motion F_k of the geodesic flow on S^n are obtained via the traces of T^k, i.e.,

$$F_k = \mathrm{Tr}(T^k) = 2\sum_{i=1}^{n} \lambda_i^k, \qquad k = 1, \ldots, n$$

where λ_i are the eigenvalues of T.

Example 7. *For the three-dimensional sphere S^3 case, the constants of the motion, F_1, F_2 and F_3, are*

$$\begin{aligned} F_1 &= 3J_1 + J_2 + J_3 = \lambda_1 + \lambda_2 + \lambda_3 \\ F_2 &= 3(J_1^2 + J_2^2 + J_3^2) + 2(J_1 J_2 - J_2 J_3 + J_3 J_1) = \lambda_1^2 + \lambda_2^2 + \lambda_3^2 \\ F_3 &= \lambda_1^3 + \lambda_2^3 + \lambda_3^3. \end{aligned}$$

5.2. A Sequence of Abelian Symmetries

The symplectic form ω_1, which is induced by (5) and (6), can be written as follows

$$\omega_1 = \sum_{i,\ell} \mathcal{S}^i{}_\ell \mathrm{d}J_i \wedge \mathrm{d}\varphi^\ell.$$

In this way we get also

$$\mathrm{d}K_i = \sum_{k=1}^n ({}^tS)^i{}_k \mathrm{d}J_k, \qquad \omega_1 = \sum_{i=1}^n \mathrm{d}K_i \wedge \mathrm{d}\alpha_i, \qquad \alpha_i = \varphi_i.$$

The two-form ω_1 can be considered as the Lie derivative of the symplectic form ω given by equation (4) with respect to the vector field

$$\Gamma = \sum_{i=1}^n K_i \frac{\partial}{\partial J_i}$$

and therefore

$$\omega_1 = \mathcal{L}_\Gamma \omega.$$

Now, we can set the new vector fields Δ_{i+1} as follows

$$\Delta_{i+1} := [\Delta_i, \Gamma], \qquad i = 0, \dots, n-1$$

starting with

$$\Delta_0 = \Delta = \frac{J_1 + \cdots + J_n}{2a^2} \left(\frac{\partial}{\partial \varphi_1} + \cdots + \frac{\partial}{\partial \varphi_n} \right)$$

so that

$$\Delta_{i+1} = (-1)^{i+1} \frac{(i+1)\,!\; (J_1 + \cdots + J_n)^{i+2}}{2^{i+1} a^2} \left(\frac{\partial}{\partial \varphi_1} + \cdots + \frac{\partial}{\partial \varphi_n} \right).$$

The vector fields Δ_i generated by the commutators are Hamiltonian vector fields which commute

$$[\Delta_i,\ \Delta_\ell] = 0$$

and the corresponding Hamiltonian functions are

$$H_{i+1} = (-1)^{i+1} \frac{1}{(i+3) \cdot 2^{i+1} a^2} (J_1 + \cdots + J_n)^{i+3}.$$

6. Conclusion

The geodesic flow on n-dimensional sphere has a recursion operator T

$$T = \frac{1}{2} \sum_{i,\ell} \left\{ \left({}^t S\right)^{\ell}{}_{i} \frac{\partial}{\partial J_i} \otimes \mathrm{d}J_\ell + S^i{}_\ell \frac{\partial}{\partial \varphi_i} \otimes \mathrm{d}\varphi_\ell \right\}.$$

Using the properties of the T, we got a sequence of Abelian symmetric vector fields

$$\begin{aligned} \Delta_0 &= \Delta = \frac{J_1 + \cdots + J_n}{2a^2} \left(\frac{\partial}{\partial \varphi_1} + \cdots + \frac{\partial}{\partial \varphi_n} \right) \\ \Delta_{i+1} &= (-1)^{i+1} \frac{(i+1)\,!\,(J_1 + \cdots + J_n)^{i+2}}{2^{i+1} a^2} \left(\frac{\partial}{\partial \varphi_1} + \cdots + \frac{\partial}{\partial \varphi_n} \right) \end{aligned}$$

and a sequence of involutive Hamiltonian function

$$H_{i+1} = (-1)^{i+1} \frac{1}{(i+3) \cdot 2^{i+1} a^2} (J_1 + \cdots + J_n)^{i+3}.$$

Remark 8. *It is known that there exists another recursion operator T_1 which is generated by the original T in the case of Minkowski metric* [4]*. Similar consideration is also possible in the S^n case, but it is not easy to obtain T_1 for S^n because of the difficulty of solving PDEs which follow from the conditions $\mathcal{L}_\Delta T_1 = 0$ and $\mathcal{N}_{T_1} = 0$.*

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