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SOLUTION OF THE DIRAC EQUATION FOR THE Q-DEFORMED MANNING-ROSEN POTENTIAL

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Abstract. Approximate solution for the Dirac equation with the q-deformed Manning-Rosen potential, under the condition of spin and pseudospin symmetry are obtained. Also the energy spectrum and wave functions are obtained by the Nikiforov-Uvarov (NU) method. The special cases $q = 1$, Hulthén potential $(b \to 0)$ and the nonrelativistic limit are studied for the q-deformed Manning-Rosen potential, and then results are compared with the other works.

1. Introduction

A particle in a strong potential field should be described with the Klein-Gordon (KG) and Dirac equations. The solutions of the Dirac or KG equations having the spin, and pseudospin symmetry have been extensively studied in the last years. The spin symmetry arises if the magnitude of the spherical scalar potential $S(r)$ and vector potential $V(r)$ are nearly equal in nuclei (i.e., $\Delta(r) = V(r) - S(r) = C_s = \mathrm{const}$). However, the pseudospin symmetry occurs when $S(r) \sim -V(r)$ (i.e., $\Sigma(r) = V(r) + S(r) = C_{ps} = \mathrm{const}$) [7].

In recent years, many authors have worked on solving these equations with physical potentials including Morse potential [1], Hulthén potential [5, 10, 18], Woods-Saxon potential [4, 20], reflectionless-type potential [6, 15], Rosen-Morse potential [15, 19], Manning-Rosen potential [9, 15], five-parameter exponent-type potential [17], etc.

Various methods are used to obtain the solutions of the wave equations for this type of exponential potentials, like the supersymmetric quantum mechanics and shape invariant [8, 12], the standard methods [14], the asymptotic iteration method (AIM) [3] and the Nikiforov-Uvarov (NU) method [13], etc. Recently, the NU method has received increasing interest for solving Schrödinger equation [11], KG [18] and Dirac [16] equations.

Based on the observations and experimental data in nuclear physics with above mentioned potential, becomes clear that the results obtained via Dirac equation do not coincide with experimental results. On the other hand, the deformed potentials are deformed versions of the usual potentials which are obtained by introducing a deformation parameter q, that have more accordance with observational data. Therefore, our aim, in the present work, is to investigate analytical bound state solutions of the Dirac equation with q-deformed Manning-Rosen potential, using the NU method. Also, we will show that, when we take $q = 1$, as one expected, we obtain the result of usual Manning-Rosen potential [15]. A q-deformed Manning-Rosen potential is expressed as

$$V_q(r) = V(x) = \frac{C\exp(-\alpha x) + \frac{1}{\sqrt{q}} D\exp(-2\alpha x)}{\sqrt{q}(1 - \frac{1}{\sqrt{q}}\exp(-\alpha x))^2} \tag{1}$$

where C and D are two dimensionless parameters, α is the range of the potential and q defines the deformation parameter of the potential. The above q-deformation in equation (1), is obtained according to the prescription by Arai [2]. In fact this deformation of the potential function can be obtained by following coordinate transformation

$$r \to x + \frac{1}{\alpha}\ln\sqrt{q}. \tag{2}$$

In the following sections, at first the Nikiforov-Uvarov (NU) method will be reviewed briefly. In Section 3 we solve the q-deformed Manning-Rosen potential [8, 14]. We obtain the energy eigenvalue and the corresponding eigenfunctions for any spin-orbit quantum number κ. Then we study the solutions of the Dirac equation with spin and pseudospin symmetry. Here we use the NU method for our aim. Finally, we summary our results in Section 4.

2. The Nikiforov-Uovarov Method

In this section we recall briefly the Nikiforov-Uovarov (NU) method. Using the NU method we can solve the second order differential equation. The master equation in this method is

$$\sigma^2(z)\ddot{\psi}_n(z) + \sigma(z)\tilde{\tau}(z)\dot{\psi}_n(z) + \tilde{\sigma}(z)\psi_n(z) = 0 \tag{3}$$

where $\sigma(z)$ and $\tilde{\sigma}(z)$ are polynomials, at most of second-degree and $\tilde{\tau}(z)$ is a polynomial, at most of first-degree. We choose $\psi_n(z)$ in the form

$$\psi_n(z) = \phi_n(z)y_n(z). \tag{4}$$

By substituting equation (4) into equation (3) we have

$$\sigma(z)\ddot{y}_n(z)+\tau(z)\dot{y}_n(z)+\lambda y_n(z)=0 \tag{5}$$

where

$$\tau(z)=\tilde{\tau}(z)+2\pi(z) \tag{6}$$

$$\pi(z)=\frac{1}{2}[\dot{\sigma}(z)-\tilde{\tau}(z)]\pm\sqrt{\frac{1}{4}(\dot{\sigma}(z)-\tilde{\tau}(z))^2-\tilde{\sigma}(z)+k\sigma(z)} \tag{7}$$

and λ is a constant parameter given by the formula

$$\lambda=\lambda_n=-n\dot{\tau}(z)-\frac{n(n-1)}{2}\ddot{\sigma}(z), \qquad n=0,1,2,... \tag{8}$$

There is also a relationship between λ and k, i.e.,

$$k=\lambda-\dot{\pi}(z). \tag{9}$$

The $y_n(z)$ part of the anzatz (4) is the hypergeometric-type function which can be written as

$$y_n(z)=\frac{B_n}{\rho(z)}\frac{\mathrm{d}^n}{\mathrm{d}z^n}[\sigma^n(z)\rho(z)] \tag{10}$$

where B_n is the normalization constant, and $\rho(z)$ is the weight function that should satisfy the condition

$$\dot{\sigma}(z)\rho(z)+\sigma(z)\dot{\rho}(z)=\tau(z)\rho(z). \tag{11}$$

Finally, $\phi_n(z)$ can be calculated as the solution of the differential equation

$$\dot{\phi}(z)=(\frac{\pi(z)}{\sigma(z)})\phi(z). \tag{12}$$

In appendix A, we solved a general example by this method.

3. Dirac Equation with Q-Deformed Manning-Rosen Potential

3.1. The Dirac Equation

The Dirac equation with scalar and vector potentials, $S(r)$ and $V(r)$ is

$$\left(\vec{\alpha}.\vec{P}+\beta(m+s(r))+v(r)-E\right)\psi(\vec{r})=0, \qquad \hbar=c=1 \tag{13}$$

where m and E are the mass and the energy of Dirac particle. Also $\vec{P}$ is the linear momentum operator, $\vec{\alpha}$ and β are the Dirac 4×4 matrices

$$\vec{\alpha}=\begin{pmatrix}0 & \sigma_i\\ \sigma_i & 0\end{pmatrix}, \qquad \beta=\begin{pmatrix}I & 0\\ 0 & -I\end{pmatrix} \tag{14}$$

where σ_i, $i=1,2,3$, are the Pauli's 2×2 matrices and I is the 2×2 unit matrix. For spherical nuclei, the nucleon angular momentum J and spin matrix operator $\hat{\kappa}=-\beta(\vec{\sigma}.\vec{L}+1)$ commute with the Dirac Hamiltonian. The eigenvalues of $\hat{\kappa}$ are

$\kappa = \pm(j + \frac{1}{2})$ with $(-)$ for aligned spin ($s_{\frac{1}{2}}; p_{\frac{3}{2}}$, etc.) and $(+)$ for unaligned spin ($p_{\frac{1}{2}}; d_{\frac{3}{2}}$ etc.). Hence, we use further the quantum number κ since it is sufficient to label the orbitals. For a given $\kappa = \pm1, \pm2, ..., j =\mid \kappa + \frac{1}{2} \mid - \frac{1}{2}$ and $\tilde{l} =\mid \kappa - \frac{1}{2} \mid - \frac{1}{2}$ the wave functions in Pauli-Dirac representation are given by

$$\psi_{n\kappa}(\vec{r}) = \begin{pmatrix} \frac{1}{r} F_{n\kappa}(r)\, y^{l}_{jm}(\hat{r}) \\ \frac{1}{r} G_{n\kappa}(r)\, y^{\tilde{l}}_{jm}(\hat{r}) \end{pmatrix} \tag{15}$$

in which $F_{n\kappa}(r)$ and $G_{n\kappa}(r)$ are the radial wave functions and $y^{l}_{jm}(\hat{r})$, $y^{\tilde{l}}_{jm}(\hat{r})$ are the spherical harmonic functions. Here n, l and j are the single-nucleon radial, orbital and total angular momentum quantum numbers, respectively, and m is the projection of the angular momentum on the z axis. Pseudo-orbital angular momentum was defined as $\tilde{l}$. Substituting equation (15) into equation (13), we have

$$\left(\frac{\mathrm{d}}{\mathrm{d}r} + \frac{\kappa}{r}\right) F_{n\kappa}(r) = (m + E_{n\kappa} - \Delta(r))G_{n\kappa}(r) \tag{16}$$

$$\left(\frac{\mathrm{d}}{\mathrm{d}r} - \frac{\kappa}{r}\right) G_{n\kappa}(r) = (m - E_{n\kappa} + \Sigma(r))F_{n\kappa}(r). \tag{17}$$

By eliminating $G_{n\kappa}(r)$ in equation (16) and putting it in equation (17), we get a differential equation of second order for the upper radial spinor component

$$\begin{aligned}\left(\frac{\mathrm{d}^2}{\mathrm{d}r^2} - \frac{\kappa(\kappa+1)}{r^2} + \frac{\frac{\mathrm{d}\Delta(r)}{\mathrm{d}r}}{m + E_{n\kappa} - \Delta(r)}\left(\frac{\mathrm{d}}{\mathrm{d}r} + \frac{\kappa}{r}\right)\right) F_{n\kappa}(r) \\ = ((m + E_{n\kappa} - \Delta(r))(m - E_{n\kappa} + \Sigma(r)))F_{n\kappa}(r)\end{aligned} \tag{18}$$

and also by eliminating $F_{n\kappa}(r)$ between equation (16) and equation (17), we have

$$\begin{aligned}\left(\frac{\mathrm{d}^2}{\mathrm{d}r^2} - \frac{\kappa(\kappa-1)}{r^2} + \frac{\frac{\mathrm{d}\Sigma(r)}{\mathrm{d}r}}{m - E_{n\kappa} + \Sigma(r)}\left(\frac{\mathrm{d}}{\mathrm{d}r} - \frac{\kappa}{r}\right)\right) G_{n\kappa}(r) \\ = \Big((m + E_{n\kappa} - \Delta(r)\Big)\Big(m - E_{n\kappa} + \Sigma(r))\Big)G_{n\kappa}(r) = 0\end{aligned} \tag{19}$$

where $\kappa(\kappa+1) = l(l+1)$ and $\kappa(\kappa-1) = \tilde{l}(\tilde{l}+1)$.

3.1.1. Dirac Equation with Spin Symmetry

In this subsection we would like to study the Dirac equation with spin symmetry. The condition of spin symmetry is $\frac{\mathrm{d}\Delta(r)}{\mathrm{d}r} = 0$ or $\Delta(r) = C_s = \text{const}$, so by substituting $\Sigma(r) = V_q(r)$ into equation (4), we have

$$\left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{\kappa(\kappa+1)}{(x+\frac{1}{\alpha}\ln\sqrt{q})^2} + \alpha^2 A_1^2 + \alpha^2 B_1 \times \frac{C\exp(-\alpha x) + \frac{1}{\sqrt{q}}\exp(-2\alpha x)}{\sqrt{q}(1-\frac{1}{\sqrt{q}}\exp(-\alpha x))^2}\right) F_{n\kappa}(x) = 0 \quad (20)$$

where

$$A_1^2 = \frac{1}{\alpha^2}(m - E_{n\kappa})(m + E_{n\kappa} - C_s), \qquad B_1 = \frac{1}{\alpha^2}(m + E_{n\kappa} - C_s) \quad (21)$$

with $\kappa = l$ for $\kappa < 0$ and $\kappa = -(l+1)$ for $\kappa > 0$. The eigenenergies E_{nk}, depend only on n and l, i.e., $E_{nk} = E(n; l(l+1))$. For $l \neq 0$, the states with $j = l \pm \frac{1}{2}$ are degenerate. The lower component of Dirac equation become

$$G_{n\kappa}(r) = \frac{1}{m + E_{n\kappa} - C_s}\left(\frac{\mathrm{d}}{\mathrm{d}r} + \frac{\kappa}{r}\right) F_{n\kappa}(r). \quad (22)$$

The differential equation (20), can not be solved exactly using the NU method for $\kappa \neq 1$, due to the presence of the centrifugal term $\kappa(\kappa+1)r^{-2}$. So, we should do an approximation. Here we use the following approximation of (20)

$$\frac{1}{(x+\frac{1}{\alpha}\ln\sqrt{q})^2} \approx \frac{\alpha^2\exp(-\alpha x)}{(1-\frac{1}{\sqrt{q}}\exp(-\alpha x))^2}. \quad (23)$$

By introducing $z = \exp(-\alpha x)$, we can rewrite equation (20) as

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}z^2} + \frac{(1-\frac{z}{\sqrt{q}})}{z(1-\frac{z}{\sqrt{q}})}\frac{\mathrm{d}}{\mathrm{d}z} + \frac{1}{z^2(1-\frac{z}{\sqrt{q}})^2}\left(-\frac{1}{q}(A_1^2 + B_1 D)z^2 + \frac{1}{\sqrt{q}}(2A_1^2 - B_1 C - \sqrt{q}\kappa(\kappa+1))z - A_1^2\right)\right) F_{n\kappa}(z) = 0. \quad (24)$$

It is necessary to compare equation (24) with equation (3). Subsequently, the following values for the parameters in equation (3) are obtained as

$$\tilde{\tau}(z) = 1 - \frac{z}{\sqrt{q}}, \qquad \sigma(z) = z(1 - \frac{z}{\sqrt{q}}), \qquad \tilde{\sigma}(z) = -q_2 z^2 + q_1 z - q_0 \quad (25)$$

where

$$q_2 = \frac{1}{q}(A_1^2 + B_1 D), \quad q_1 = \frac{1}{\sqrt{q}}(2A_1^2 - B_1 C - \sqrt{q}\kappa(\kappa+1)), \quad q_0 = A_1^2. \quad (26)$$

Using the introduced parameters in Appendix A and the results in Table 1, we obtain the following expression

$$\pi(z) = A_1 - \frac{1}{\sqrt{q}}\left(\frac{1}{2} + A_1 + \sqrt{B_1C + B_1D + \sqrt{q}\kappa(\kappa+1) + \frac{1}{4}}\right)z \quad (27)$$

where

$$\kappa = -\frac{1}{\sqrt{q}}\left(B_1C + \sqrt{q}\kappa(\kappa+1) + 2A_1\sqrt{B_1C + B_1D + \sqrt{q}\kappa(\kappa+1) + \frac{1}{4}}\right). \quad (28)$$

We can also write

$$\tau(z) = 1+2A_1 - \frac{2}{\sqrt{q}}\left(1 + A_1 + \sqrt{B_1C + B_1D + \sqrt{q}\kappa(\kappa+1) + \frac{1}{4}}\right)z \quad (29)$$

where $\dot{\tau}(z) = -\frac{2}{\sqrt{q}}[1 + A_1 + \sqrt{B_1C + B_1D + \sqrt{q}\kappa(\kappa+1) + \frac{1}{4}}\,] < 0$.

To find the energy eigenvalues of our Dirac equation we can use the equations (8), (9). So we have

$$2A_1\left(n + \frac{1}{2} + \Lambda\right) + 2(n+\frac{1}{2})\Lambda + \sqrt{q}\kappa(\kappa+1) + B_1C + (n+\frac{1}{2})^2 + \frac{1}{4} = 0 \quad (30)$$

where $\Lambda = \sqrt{B_1C + B_1D + \sqrt{q}\kappa(\kappa+1) + \frac{1}{4}}$.

Now we will obtain the non-relativistic limit of the energy by these replacements $E_{n\kappa} - m \to E_{n\kappa}$, $E_{n\kappa} + m \to 2m$, $\kappa = l$. By substituting A_1 and B_1 from equation (21) into the above equation, we get the energy in the form

$$E_{nl} = -\frac{\alpha^2}{2m}\left(\frac{(n+\frac{1}{2})^2 + \frac{1}{4} + \sqrt{q}l(l+1) + \frac{2mC}{\alpha^2} + (n+\frac{1}{2})\hat{\Lambda}}{2n+1+\hat{\Lambda}}\right)^2 \quad (31)$$

where $\hat{\Lambda} = \sqrt{1 + 4\sqrt{q}l(l+1) + \frac{8m}{\alpha^2}(C+D)}$ and $C_s = 0$.

By choosing $C = -\frac{\alpha^2}{2m}A$, $D = \frac{\alpha^2}{2m}(A + b(b-1))$ we have

$$E_{nl} = -\frac{\alpha^2}{2m}\left(\frac{(n+1)^2 - A + \sqrt{q}l(l+1) + (2n+1)\beta}{2(n+\beta+1)}\right)^2 \quad (32)$$

where $\beta = \frac{-1+\hat{\Lambda}_1}{2}$, $\hat{\Lambda}_1 = \sqrt{(1-2b)^2 + 4\sqrt{q}l(l+1)}$. Now by considering the limiting case $q = 1$, we obtain the following expression for the non-relativistic energy

$$E_{nl} = -\frac{\alpha^2}{2m}\left(\frac{(n+1)^2 - A + l(l+1) + (2n+1)\hat{\beta}}{2(n+\hat{\beta}+1)}\right)^2 \quad (33)$$

Table 1. The specific values for the necessary parametric constants for q-deformed Manning-Rosen potential with spin symmetry.

parameter	value
c_1	1
c_2	$\frac{1}{\sqrt{q}}$
c_3	$\frac{1}{\sqrt{q}}$
c_4	0
c_5	$-\frac{1}{2\sqrt{q}}$
c_6	$\frac{1}{4q}[1+4(A_1^2+B_1D)]$
c_7	$\frac{1}{\sqrt{q}}[-2A_1^2+B_1C+\sqrt{q}\kappa(\kappa+1)]$
c_8	A_1^2
c_9	$\frac{1}{4q}[1+4\sqrt{q}\kappa(\kappa+1)+4B_1(C+D)]$
c_{10}	$1+2A_1$
c_{11}	$\frac{2}{\sqrt{q}}[1+A_1+\Lambda_1]$
c_{12}	A_1
c_{13}	$-\frac{1}{\sqrt{q}}(\frac{1}{2}+A_1+\Lambda_1)$

where $\hat{\beta} = \frac{-1+\hat{\Lambda}_2}{2}$, $\hat{\Lambda}_2 = \sqrt{(1-2b)^2+4l(l+1)}$, The result for E_{nl} in equation (33) is completely in agreement with the energy of the Schrödinger equation with Manning-Rosen potential. Now we obtain the energy spectrum of this problem with Hulthén potential, easily by replacing $b \to 0$, because in this limit the Manning-Rosen potential transform to the Hulthén potential

$$E_{nl} = -\frac{\alpha^2}{2m}\left(\frac{(n+1)^2 - A + \sqrt{q}l(l+1) + (2n+1)\beta}{2(n+\beta+1)}\right)^2. \tag{34}$$

Then by taking $q = 1$, we have

$$E_{nl} = -\frac{\alpha^2}{2m}\left(\frac{-A+(n+l+1)^2}{2(n+l+1)}\right)^2. \tag{35}$$

In this part, referring to Appendix A, we calculate the wave functions as

$$\phi_n(z) = z^{A_1}(1-\frac{z}{\sqrt{q}})^{\frac{1}{2}+\Lambda} \tag{36}$$

and the weight function takes the form

$$\rho(z) = z^{2A_1}(1-\frac{z}{\sqrt{q}})^{2\Lambda}. \tag{37}$$

Furthermore, we find the function, y_n , which is just the Hypergeometric function

$$y_n(z) \approx P_n^{(2A_1,2\Lambda)}(1-\frac{2z}{\sqrt{q}}). \tag{38}$$

Using the relation $F_{n\kappa}(z)=\phi_n(z)y_n(z)$, we get the radial upper spinor as

$$\begin{aligned}F_{n\kappa}(r) =&N_{n\kappa}q^{\frac{A_1}{2}}\exp(-\alpha A_1 r)(1-\exp(-\alpha r))^{\frac{1}{2}+\Lambda}\\ &\times {}_2F_1(-n,n+2A_1+2\Lambda+1,2A_1+1;\exp(-\alpha r))\end{aligned} \tag{39}$$

where

$${}_2F_1(-n,n+\eta+\mu+1,\eta+1;\frac{1-s}{2}) = \frac{n!\Gamma(\eta+1)}{\Gamma(n+\eta+1)}P_n^{(\eta,\mu)}(s) \tag{40}$$

$z(r)=\sqrt{q}e^{-\alpha r}$ and $N_{n\kappa}$ is the normalization constant. This equation satisfies the requirements, i.e., $F_{n\kappa}(r)=0$ as $r=0$ and $F_{n\kappa}(r)=0$ as $r\to\infty$. Therefore, the wave functions, $F_{n\kappa}(r)$ in equation (39) is valid physical solution in the closed interval $z\in[0,\sqrt{q}]$ or $r\in(0,\infty)$. We obtain the lower spinor component in equation (22), by using equation (39)

$$\begin{aligned}G_{n\kappa}(r) =&\frac{1}{m+E_{n\kappa}-C_s}\left(\left(\frac{\kappa}{r}-\alpha+\frac{\alpha(\frac{1}{2}+\Lambda)\exp(-\alpha r)}{(1-\exp(-\alpha r))}\right)F_{n\kappa}(r)\right.\\ &+\frac{q^{\frac{1}{2}(1+A_1)}\alpha n N_{n\kappa}(n+2A_1+2\Lambda+1)}{(m+E_{n\kappa}-C_s)(2A_1+1)}(1-\exp(-\alpha r))^{\frac{1}{2}+\Lambda}\\ &\left.\times\exp(-2\alpha r)\,{}_2F_1(-n+1,n+2(1+A_1+\Lambda),2(A_1+1);\exp(-\alpha r))\right).\end{aligned} \tag{41}$$

3.1.2. Dirac Equation with Pseudospin Symmetry

The condition of pseudospin symmetry is $\frac{\mathrm{d}\Sigma(r)}{\mathrm{d}r}=0$ or $\Sigma(r)=C_{ps}=\text{const.}$ Now by inserting $\Delta(r)=V_q(r)$ in equation (19) we have

$$\begin{aligned}&\left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2}+\frac{\kappa(\kappa-1)}{(x+\frac{1}{\alpha}\ln\sqrt{q})^2}+A_2^2\alpha^2+B_2\alpha^2\right.\\ &\left.\times\frac{C\exp(-\alpha x)+\frac{D}{\sqrt{q}}\exp(-2\alpha x)}{\sqrt{q}(1-\frac{1}{\sqrt{q}}\exp(-\alpha x))^2}\right)G_{n\kappa}(x)=0\end{aligned} \tag{42}$$

where

$$A_2^2=\frac{1}{\alpha^2}(m+E_{n\kappa})(m-E_{n\kappa}+C_{ps}),\qquad B_2=\frac{1}{\alpha^2}(m-E_{n\kappa}+C_{ps}) \tag{43}$$

and $\kappa=-\tilde{l}$ for $\kappa<0$ and $\kappa=\tilde{l}+1$ for $\kappa>0$. The energy eigenvalues, E_{nk}, depend only on n and l, i.e., $E_{nk}=E(n;\ l(\ l+1))$. For $l\neq0$, the states with $j=\ l\pm\frac{1}{2}$ are degenerated.

By using transformations $z = \exp(-\alpha x)$ and equation (23), we have

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}z^2} + \frac{(1-\frac{z}{\sqrt{q}})}{z(1-\frac{z}{\sqrt{q}})}\frac{\mathrm{d}}{\mathrm{d}z} + \frac{1}{z^2(1-\frac{z}{\sqrt{q}})^2}\left(-\frac{1}{q}(A_2^2 - B_2 D)z^2 \right.\right.$$
$$\left.\left. + \frac{1}{\sqrt{q}}(2A_2^2 + B_2 C - \sqrt{q}\kappa(\kappa+1))z - A_2^2\right)\right) G_{n\kappa}(z) = 0. \quad (44)$$

Now, we follow the same procedures in the previous subsection and with attention to Table 2, we can obtain energy equation

$$2(n+\frac{1}{2}+\Lambda)A_2 + 2(n+\frac{1}{2})\Lambda - B_2 C + \sqrt{q}\kappa(\kappa-1) + (n+\frac{1}{2})^2 + \frac{1}{4} = 0 \quad (45)$$

where $\Lambda = \sqrt{\sqrt{q}\kappa(\kappa-1) - B_2 C - B_2 D + \frac{1}{4}}$. Also the wave function is

$$\begin{aligned} G_{n\kappa}(r) =& \acute{N_{n\kappa}} q^{\frac{A_2}{2}} \exp(-\alpha A_2 r)(1-\exp(-\alpha r))^{\frac{1}{2}+\Lambda} \\ & \times {}_2F_1(-n, n+2A_2+2\Lambda+1, 2A_2+1; \exp(-\alpha r)) \end{aligned} \quad (46)$$

where $\acute{N_{n\kappa}}$ is the normalization constant.

4. Conclusion

In the present paper, we have studied the approximate solutions of the Dirac equation for the q-deformed Manning-Rosen potential for any arbitrary spin-orbit quantum number κ under conditions of the spin and pseudospin symmetries by the NU method, using the approximation scheme to deal with the centrifugal term. Under the condition of spin symmetry and pseudospin symmetry, i.e., $\Delta(r) = C_s$, $\Sigma(r) = C_{ps}$, respectively, we have solved the energy equation and we have also obtained the wave functions in terms of the hypergeometric functions. We have shown that in the limiting case $q = 1$, our results for the energy eigenvalues are agree with previous obtained values of usual Manning-Rozen potential [15]. If we choose $\Delta(r) = \text{const} = C_s = 0$ and consider the non-relativistic limit, i.e., $E_{n\kappa} - m = E_{nl}$, $E_{n\kappa} + m \sim 2m$, we obtain the expression (31) for the spectrum of our problem. By considering the special values $C = -\frac{\alpha^2}{2m}A$, $D = \frac{\alpha^2}{2m}[A + b(b-1)]$ and $q = 1$, we have found that the spectrum of energy is consistent with the result of [9]. So this is the correctly shrödinger energy spectrum as non-relativistic limit of Dirac equation. We have also investigated the special case $b = 0$ $(C = -D)$ for the q-deformed Manning-Rosen potential, which corresponds to the case of the Hulthén potential.

Table 2. The specific values for the necessary parametric constants and functions for q-deformed Manning-Rosen potential with pseudospin simmetry.

parameter and functions	value
c_1	1
c_2	$\frac{1}{\sqrt{q}}$
c_3	$\frac{1}{\sqrt{q}}$
c_4	0
c_5	$-\frac{1}{2\sqrt{q}}$
c_6	$\frac{1}{4q}[1+4(A_2^2-B_2D)]$
c_7	$\frac{1}{\sqrt{q}}[-2A_2^2-B_2C+\sqrt{q}\kappa(\kappa-1)]$
c_8	A_2^2
c_9	$\frac{1}{4q}[1+4\sqrt{q}\kappa(\kappa-1)-4B_2(C+D)]$
c_{10}	$1+2A_2$
c_{11}	$\frac{2}{\sqrt{q}}[1+A_2+\Lambda_2]$
c_{12}	A_2
c_{13}	$-\frac{1}{\sqrt{q}}(\frac{1}{2}+A_2+\Lambda_2)$
q_2	$\frac{1}{q}(A_2^2-B_2D)$
q_1	$\frac{1}{\sqrt{q}}(2A_2^2+B_2C-\sqrt{q}\kappa(\kappa-1))$
q_0	A_2^2
$\pi(z)$	$A_2-\frac{1}{\sqrt{q}}(\frac{1}{2}+A_2+\Lambda_2)z$
k	$-\frac{1}{\sqrt{q}}[-B_2C+\sqrt{q}\kappa(\kappa-1)+2A_2\Lambda_2]$
$\tau(z)$	$1+2A_2-\frac{2}{\sqrt{q}}[1+A_2+\Lambda_2]z$
$\phi_n(z)$	$z^{A_2}(1-\frac{z}{\sqrt{q}})^{\frac{1}{2}+\Lambda_2}$
$\rho(z)$	$z^{2A_2}(1-\frac{z}{\sqrt{q}})^{2\Lambda_2}$
$y_n(z)$	$P_n^{(2A_2,2\Lambda_2)}(1-\frac{2z}{\sqrt{q}})$

Appendix A: Parametric Generalization of the NU Method

In this Appendix we consider parametric generalization of the NU method. We begin with the following equation

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}z^2}+\frac{(c_1-c_2z)}{z(1-c_3z)}\frac{\mathrm{d}}{\mathrm{d}z}+\frac{1}{z^2(1-c_3z)^2}\big(-q_2z^2+q_1z-q_0\big)\right)\psi(z)=0. \quad (47)$$

By comparing the above equation with equation (1), we have

$$\tilde{\tau}(z)=c_1-c_2z, \quad \sigma(z)=z(1-c_3z), \quad \tilde{\sigma}(z)=-q_2z^2+q_1z-q_0. \quad (48)$$

The polynomials functions given by

$$\pi(z) = c_4 + c_5 z - ((\sqrt{c_9} + c_3\sqrt{c_8})z - \sqrt{c_8}) \tag{49}$$

$$k = -(c_7 + 2c_3c_8) - 2\sqrt{c_8c_9} \tag{50}$$

$$\tau(z) = c_1 + 2c_4 - (c_2 - 2c_5)z - 2((\sqrt{c_9} + c_3\sqrt{c_8})z - \sqrt{c_8}) \tag{51}$$

where the parameters $c_i, i = 1...9$ are to be determined during the solution procedure

$$c_4 = \frac{1}{2}(1 - c_1), \qquad c_5 = \frac{1}{2}(c_2 - 2c_3), \qquad c_6 = q_2 + c_5^2$$
$$c_7 = 2c_4c_5 - q_1, \qquad c_8 = c_4^2 + q_0, \qquad c_9 = c_3(c_3c_8 + c_7) + c_6$$

And, it gives the energy equation and the wave functions as

$$n((n-1)c_3 + c_2 - 2c_5) - c_5 + (2n+1)(\sqrt{c_9} + c_3\sqrt{c_8}) + c_7 + 2c_3c_8 + 2\sqrt{c_8c_9} = 0$$

$$\begin{aligned} \phi_n(z) &= z^{c_{12}}(1 - c_3z)^{-c_{12} - \frac{c_{13}}{c_3}} \\ \rho(z) &= z^{c_{10}-1}(1 - c_3z)^{\frac{c_{11}}{c_3} - c_{10} - 1} \\ y_n(z) &= P_n^{(c_{10}-1, \frac{c_{11}}{c_3} - c_{10} - 1)}(1 - 2c_3z) \end{aligned} \tag{52}$$

where $P_n^{(c_{10}-1, \frac{c_{11}}{c_3} - c_{10} - 1)}(1-2c_3z)$ are the Jacobi polynomials, and $c_{10} - 1 > -1$, $\dfrac{c_{11}}{c_3} - c_{10} - 1 > -1$. So the function Ψ take following form

$$\psi_n(z) = N_n z^{c_{12}}(1 - c_3z)^{-c_{12} - \frac{c_{13}}{c_3}} P_n^{(c_{10}-1, \frac{c_{11}}{c_3} - c_{10} - 1)}(1 - 2c_3z) \tag{53}$$

where

$$c_{10} = c_1 + 2c_4 + 2\sqrt{c_8}, \qquad c_{11} = c_2 - 2c_5 + 2(\sqrt{c_9} + c_3\sqrt{c_8})$$
$$c_{12} = c_4 + \sqrt{c_8}, \qquad c_{13} = c_5 - (\sqrt{c_9} + c_3\sqrt{c_8}).$$

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