



## $f$ -BIHARMONIC MAPS BETWEEN RIEMANNIAN MANIFOLDS

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**Abstract.** We show that if  $\psi$  is an  $f$ -biharmonic map from a compact Riemannian manifold into a Riemannian manifold with non-positive curvature satisfying a condition, then  $\psi$  is an  $f$ -harmonic map. We prove that if the  $f$ -tension field  $\tau_f(\psi)$  of a map  $\psi$  of Riemannian manifolds is a Jacobi field and  $\phi$  is a totally geodesic map of Riemannian manifolds, then  $\tau_f(\phi \circ \psi)$  is a Jacobi field. We finally investigate the stress  $f$ -bienergy tensor, and relate the divergence of the stress  $f$ -bienergy of a map  $\psi$  of Riemannian manifolds with the Jacobi field of the  $\tau_f(\psi)$  of the map.

### 1. Introduction

Harmonic maps between Riemannian manifolds were first established by Eells and Sampson in 1964. Afterwards, there are two reports and one survey paper by Eells and Lemaire [15–17] about the developments of harmonic maps up to 1988. Chiang, Ratto, Sun and Wolak also studied harmonic and biharmonic maps in [4–9].  $f$ -harmonic maps which generalize harmonic maps, were first introduced by Lichnerowicz [25] in 1970, and were studied by Course [12, 13] recently. The  $f$ -harmonic maps relate to the equation of the motion of a continuous system of spins with inhomogeneous neighbor Heisenberg interaction in mathematical physics. Moreover,  $F$ -harmonic maps between Riemannian manifolds were first introduced by Ara [1, 2] in 1999, which could be considered as the special cases of  $f$ -harmonic maps.

Let  $f : (M_1, g) \rightarrow (0, \infty)$  be a smooth function. By definition the  **$f$ -biharmonic** maps between Riemannian manifolds are the critical points of  **$f$ -bienergy**

$$E_2^f(\psi) = \frac{1}{2} \int_{M_1} f |\tau_f(\psi)|^2 dv$$

where  $dv$  the volume form determined by the metric  $g$ . The  $f$ -biharmonic maps between Riemannian manifolds which generalized biharmonic maps by Jiang [20, 21] in 1986, were first studied by Ouakkas, Nasri and Djaa [27] in 2010.

In section two, we describe the motivation, and review  $f$ -harmonic maps and their relationship with  $F$ -harmonic maps. In Theorem 3.1, we show that if  $\psi$  is an  $f$ -biharmonic map from a compact Riemannian manifold into a Riemannian manifold with non-positive curvature satisfying a condition, then  $\psi$  is an  $f$ -harmonic map. It is well-known from [18] that if  $\psi$  is a harmonic map of Riemannian manifolds and  $\phi$  is a totally geodesic map of Riemannian manifolds, then  $\phi \circ \psi$  is harmonic. However, if  $\psi$  is  $f$ -biharmonic and  $\phi$  is totally geodesic, then  $\phi \circ \psi$  is not necessarily  $f$ -biharmonic. Instead, we prove in Theorem 3.3 that if the  $f$ -tension field  $\tau_f(\psi)$  of a smooth map  $\psi$  of Riemannian manifolds is a Jacobi field and  $\phi$  is totally geodesic, then  $\tau_f(\phi \circ \psi)$  is a Jacobi field. It implies Corollary 3.4 [8] that if  $\psi$  is a biharmonic map between Riemannian manifolds and  $\phi$  is totally geodesic, then  $\phi \circ \psi$  is a biharmonic map. We finally investigate the stress  $f$ -bienergy tensors. If  $\psi$  is an  $f$ -biharmonic of Riemannian manifolds, then it usually does not satisfy the conservation law for the stress  $f$ -bienergy tensor  $S_2^f(\psi)$ . However, we obtain in Theorem 4.2 that if  $\psi : (M_1, g) \rightarrow (M_2, h)$  be a smooth map between two Riemannian manifolds, then

$$\operatorname{div} S_2^f(Y) = \pm \langle J_{\tau_f(\psi)}(Y), d\psi(Y) \rangle \quad \text{for all } Y \in \Gamma(TM_1) \quad (1)$$

where  $\operatorname{div} S_2^f$  is the divergence of  $S_2^f$  and  $J_{\tau_f(\psi)}$  is the Jacobi field of  $\tau_f(\psi)$  (there is a - or + sign convention in the formula). Hence, if  $\tau_f(\psi)$  is a Jacobi field, then  $\psi$  satisfies the conservation law for  $S_2^f$ . It implies Corollary 4.4 [22] that if  $\psi$  is a biharmonic map between Riemannian manifolds, then  $\psi$  satisfies the conservation law for the stress bi-energy tensor  $S_2(\psi)$ . We also discuss a few applications concerning the vanishing of the stress  $f$ -bienergy tensor.

## 2. Preliminaries

### 2.1. Motivation

In mathematical physics, the equation of the motion of a continuous system of spins with inhomogeneous neighborhood Heisenberg interaction is

$$\frac{\partial \psi}{\partial t} = f(x)(\psi \times \Delta \psi) + \nabla f \cdot (\psi \times \nabla \psi) \quad (2)$$

where  $\Omega \subset \mathbb{R}^m$  is a smooth domain in the Euclidean space,  $f$  is a real-valued function defined on  $\Omega$ ,  $\psi(x, t) \in S^2$ , and  $\times$  is the cross product in  $\mathbb{R}^3$  and  $\Delta$  is the **Laplace operator** in  $\mathbb{R}^m$ . Such a model is called the inhomogeneous Heisenberg ferromagnet [10, 11, 14]. Physically, the function  $f$  is called the coupling function,

and is the continuum of the coupling constant between the neighboring spins. It is known from [18] that the tension field of a map  $\psi$  into  $S^2$  is  $\tau(\psi) = \Delta\psi + |\nabla\psi|^2\psi$ . Observe that the right hand side of (2) can be expressed as

$$\psi \times (f\tau(\psi) + \nabla f \cdot \nabla\psi). \quad (3)$$

Hence,  $\psi$  is a smooth stationary solution (i.e.,  $\frac{\partial\psi}{\partial t} = 0$ ) of (2) if and only if

$$f\tau(\psi) + \nabla f \cdot \nabla\psi = 0 \quad (4)$$

i.e.,  $\psi$  is an  $f$ -harmonic map. Consequently, there is a one-to-one correspondence between the set of the stationary solutions of the inhomogeneous Heisenberg spin system (2) on the domain  $\Omega$  and the set of  $f$ -harmonic maps from  $\Omega$  into  $S^2$ . The inhomogeneous Heisenberg spin system (2) is also called inhomogeneous Landau-Lifshitz system (cf. [19, 23, 24]).

## 2.2. $f$ -harmonic Maps

Let  $f : (M_1, g) \rightarrow (0, \infty)$  be a smooth function. Many aspects of the  $f$ -harmonic maps which generalize harmonic maps, were studied in [12, 13, 19, 24] recently. Let  $\psi : (M_1, g) \rightarrow (M_2, h)$  be a smooth map from an  $m$ -dimensional Riemannian manifold  $(M_1, g)$  into an  $n$ -dimensional Riemannian manifold  $(M_2, h)$ . A map  $\psi : (M_1, g) \rightarrow (M_2, h)$  is  **$f$ -harmonic** if and only if  $\psi$  is a critical point of the  $f$ -energy

$$E_f(\psi) = \frac{1}{2} \int_{M_1} f |d\psi|^2 dv.$$

In terms of the Euler-Lagrange equation,  $\psi$  is  $f$ -harmonic if and only if the  **$f$ -tension field**

$$\tau_f(\psi) = f\tau(\psi) + d\psi(\text{grad } f) = 0 \quad (5)$$

where  $\tau(\psi) = \text{Tr}_g Dd\psi$  is the tension field of  $\psi$ . In particular, when  $f = 1$ ,  $\tau_f(\psi) = \tau(\psi)$ .

Let  $F : [0, \infty) \rightarrow [0, \infty)$  be a  $C^2$  function such that  $F' > 0$  on  $(0, \infty)$ .  $F$ -harmonic maps between Riemannian manifolds were introduced in [1, 2]. For a smooth map  $\psi : (M_1, g) \rightarrow (M_2, h)$  of Riemannian manifolds, the  $F$ -energy of  $\psi$  is defined by

$$E_F(\psi) = \int_{M_1} F\left(\frac{|d\psi|^2}{2}\right) dv. \quad (6)$$

When  $F(t) = t$ ,  $\frac{(2t)^{p/2}}{p}$  ( $p \geq 4$ ),  $(1 + 2t)^\alpha$  ( $\alpha > 1$ ,  $\dim M_1 = 2$ ), and  $e^t$ , they are the energy, the  $p$ -energy, the  $\alpha$ -energy of Sacks-Uhlenbeck [28], and the exponential energy, respectively. A map  $\psi$  is  $F$ -harmonic if and only if  $\psi$  is a critical point of the  $F$ -energy functional. In terms of the Euler-Lagrange equation,

$\psi : M_1 \rightarrow M_2$  is an  $F$ -harmonic map if and only if the  **$F$ -tension field**

$$\tau_F(\psi) = F' \left( \frac{|\mathrm{d}\psi|^2}{2} \right) \tau(\psi) + \psi_* \left( \mathrm{grad} \left( F' \left( \frac{|\mathrm{d}\psi|^2}{2} \right) \right) \right) = 0. \quad (7)$$

**Proposition 1.** 1) If  $\psi : (M_1, g) \rightarrow (M_2, h)$  an  $F$ -harmonic map without critical points (i.e.,  $|\mathrm{d}\psi_x| \neq 0$  for all  $x \in M_1$ ), then it is an  $f$ -harmonic map with  $f = F' \left( \frac{|\mathrm{d}\psi|^2}{2} \right)$ . In particular, a  $p$ -harmonic map without critical points is an  $f$ -harmonic map with  $f = |\mathrm{d}\psi|^{p-2}$ .

2) [15, 25]. A map  $\psi : (M_1^m, g) \rightarrow (M_2^n, h)$  is  $f$ -harmonic if and only if  $\psi : (M_1^m, f^{\frac{2}{m-2}}g) \rightarrow (M_2^n, h)$  is a harmonic map.

**Proof:** 1) It follows from (5) and (7) immediately (cf. Corollary 1.1 in [26]).

2) See [15]. ■

### 3. $f$ -biharmonic maps

Let  $f : (M_1, g) \rightarrow (0, \infty)$  be a smooth function.  $f$ -biharmonic maps between Riemannian manifolds which generalized biharmonic maps [20, 21], were first studied by Ouakkas, Nasri and Djaa [27] recently. An  $f$ -biharmonic map  $\psi : (M_1, g) \rightarrow (M_2, h)$  between Riemannian manifolds is the critical point of the  $f$ -bienergy functional

$$(E_2)_f(\psi) = \frac{1}{2} \int_{M_1} \|\tau_f(\psi)\|^2 \mathrm{d}v \quad (8)$$

where the  $f$ -tension field  $\tau_f(\psi) = f\tau(\psi) + \mathrm{d}\psi(\mathrm{grad} f)$ . In terms of Euler-Lagrange equation,  $\psi$  is  $f$ -biharmonic if and only if the  **$f$ -bitension field** of  $\psi$

$$(\tau_2)_f(\psi) = \pm \Delta_2^f \tau_f(\psi) \pm fR'(\tau_f(\psi), \mathrm{d}\psi)\mathrm{d}\psi = 0 \quad (9)$$

where

$$\Delta_2^f \tau_f(\psi) = \bar{D}f\bar{D}\tau_f(\psi) - f\bar{D}_D\tau_f(\psi) = \sum_{i=1}^m (\bar{D}_{e_i}f\bar{D}_{e_i}\tau_f(\psi) - f\bar{D}_{D_{e_i}e_i}\tau_f(\psi)).$$

Here,  $D, \bar{D}$  are the connections on  $TM_1, \psi^{-1}TM_2$ , respectively,  $\{e_i\}_{1 \leq i \leq m}$  is a local orthonormal frame at any point in  $M_1$ , and  $R'$  is the Riemannian curvature of  $M_2$ . There is a + or - sign convention in (9), and we take + sign in the context for simplicity. In particular, if  $f = 1$ , then  $(\tau_2)_f(\psi) = \tau_2(\psi)$ , the bitension field of  $\psi$ .

**Theorem 2.** If  $\psi : (M_1, g) \rightarrow (M_2, h)$  is a  $f$ -biharmonic map from a compact Riemannian manifold  $M_1$  into a Riemannian manifold  $M_2$  with non-positive curvature satisfying

$$\langle f\bar{D}_{e_i}\bar{D}_{e_i}\tau_f(\psi) - \bar{D}_{e_i}f\bar{D}_{e_i}\tau_f(\psi), \tau_f(\psi) \rangle \geq 0 \quad (10)$$

then  $\psi$  is  $f$ -harmonic.

**Proof:** Since  $\psi : M_1 \rightarrow M_2$  is  $f$ -biharmonic, it follows from (9) that

$$(\tau_2)_f(\psi) = \bar{D}f\bar{D}\tau_f(\psi) - f\bar{D}_D\tau_f(\psi) + fR'(\tau_f(\psi), d\psi)d\psi = 0. \quad (11)$$

Suppose that the compact supports of the maps  $\frac{\partial\psi_t}{\partial t}$  and  $\bar{D}_{e_i}\frac{\partial\psi_t}{\partial t}$  ( $\{\psi_t\} \in C^\infty(M_1 \times (-\epsilon, \epsilon), M_2)$ ) is a one parameter family of maps with  $\psi_0 = \psi$  are contained in the interior of  $M$ . We compute

$$\begin{aligned} \frac{1}{2}f\Delta\|\tau_f(\psi)\|^2 &= f\langle\bar{D}_{e_i}\tau_f(\psi), \bar{D}_{e_i}\tau_f(\psi)\rangle + f\langle\bar{D}^*\bar{D}\tau_f(\psi), \tau_f(\psi)\rangle \\ &= f\langle\bar{D}_{e_i}\tau_f(\psi), \bar{D}_{e_i}\tau_f(\psi)\rangle + f\langle\bar{D}_{e_i}\bar{D}_{e_i}\tau_f(\psi) \\ &\quad - \bar{D}_{D_{e_i e_i}}\tau_f(\psi), \tau_f(\psi)\rangle \\ &= f\langle\bar{D}_{e_i}\tau_f(\psi), \bar{D}_{e_i}\tau_f(\psi)\rangle + \langle f\bar{D}_{e_i}\bar{D}_{e_i}\tau_f(\psi) \\ &\quad - \bar{D}_{e_i}f\bar{D}_{e_i}\tau_f(\psi) + \bar{D}_{e_i}f\bar{D}_{e_i}\tau_f(\psi) - f\bar{D}_{D_{e_i e_i}}\tau_f(\psi), \tau_f(\psi)\rangle \\ &= f\langle\bar{D}_{e_i}\tau_f(\psi), \bar{D}_{e_i}\tau_f(\psi)\rangle + \langle f\bar{D}_{e_i}\bar{D}_{e_i}\tau_f(\psi) \\ &\quad - \bar{D}_{e_i}f\bar{D}_{e_i}\tau_f(\psi), \tau_f(\psi)\rangle - \langle f(R'(d\psi, d\psi)\tau_f(\psi), \tau_f(\psi)) \rangle \geq 0 \end{aligned} \quad (12)$$

(where  $\bar{D}^*\bar{D} = \bar{D}\bar{D} - \bar{D}_D$  [20]) by (10), (11),  $f > 0$  and  $R' \leq 0$ . It implies that

$$\frac{1}{2}\Delta\|\tau_f(\psi)\|^2 \geq 0.$$

By applying the Bochner's technique, we know that  $\|\tau_f(\psi)\|^2$  is constant and that

$$\bar{D}_{e_i}\tau_f(\psi) = 0 \quad \text{for all } i = 1, 2, \dots, m.$$

It follows from Eells-Lemaire [15] results that  $\tau_f(\psi)=0$ , i.e.,  $\psi$  is  $f$ -harmonic on  $M_1$ . ■

**Corollary 3** [20]. *If  $\psi : (M_1, g) \rightarrow (M_2, h)$  is a biharmonic map from a compact Riemannian  $M_1$  manifold into a Riemannian manifold  $M_2$  with non-positive curvature, then  $\psi$  is harmonic.*

**Proof:** When  $f = 1$  and  $\psi : M_1 \rightarrow M_2$  is a biharmonic map from a compact Riemannian  $M_1$  manifold into a Riemannian manifold  $M_2$  with non-positive curvature, (11) becomes

$$\tau_2(\psi) = \bar{D}^*\bar{D}\tau(\psi) + R'(\tau(\psi), d\psi)d\psi = 0.$$

The identity (13) reduces to

$$\begin{aligned} \frac{1}{2}\Delta\|\tau(\psi)\|^2 &= \langle\bar{D}_{e_i}\tau(\psi), \bar{D}_{e_i}\tau(\psi)\rangle + \langle\bar{D}^*\bar{D}\tau(\psi), \tau(\psi)\rangle \\ &= \langle\bar{D}_{e_i}\tau(\psi), \bar{D}_{e_i}\tau(\psi)\rangle - \langle R'(d\psi, d\psi)\tau(\psi), \tau(\psi)\rangle \geq 0 \end{aligned}$$

since  $\psi$  is biharmonic, and  $M_2$  is a Riemannian manifold with non-positive curvature  $R'$ . Note that (10) is automatically satisfied. It follows from the similar arguments as Theorem 3.1 that  $\psi$  is harmonic.  $\blacksquare$

It is well-known from [18] that if  $\psi : (M_1, g) \rightarrow (M_2, h)$  is a harmonic map of two Riemannian manifolds and  $\phi : (M_2, h) \rightarrow (M_3, k)$  is totally geodesic of two Riemannian manifolds, then  $\phi \circ \psi : (M_1, g) \rightarrow (M_3, k)$  is harmonic. However, if  $\psi : (M_1, g) \rightarrow (M_2, h)$  is an  $f$ -biharmonic map, and  $\phi : (M_2, h) \rightarrow (M_3, k)$  is totally geodesic, then  $\phi \circ \psi : (M_1, g) \rightarrow (M_3, k)$  is not necessarily an  $f$ -biharmonic map. We obtain the following theorem instead.

**Theorem 4.** *If  $\tau_f(\psi)$  is a Jacobi field for a smooth map  $\psi : (M_1, g) \rightarrow (M_2, h)$  of two Riemannian manifolds, and  $\phi : (M_2, h) \rightarrow (M_3, k)$  is a totally geodesic map of two Riemannian manifolds, then  $\tau_f(\phi \circ \psi)$  is a Jacobi field.*

**Proof:** Let  $D, D', \bar{D}, \bar{D}', \hat{D}, \hat{D}'$  and  $\hat{D}''$  are the respective connections on  $TM_1, TM_2, \psi^{-1}TM_2, \phi^{-1}TM_3, (\phi \circ \psi)^{-1}TM_3, T^*M_1 \otimes \psi^{-1}TM_2, T^*M_2 \otimes \phi^{-1}TM_3$  and  $T^*M_1 \otimes (\phi \circ \psi)^{-1}TM_3$ . Then we have

$$\bar{D}''_X d(\phi \circ \psi)(Y) = (\hat{D}'_{d\psi(X)} d\phi) d\psi(Y) + d\phi \circ \bar{D}_X d\psi(Y) \quad (13)$$

for all  $X, Y \in \Gamma(TM_1)$ . We have also

$$R^{M_3}(d\phi(X'), d\phi(Y'))d\phi(Z') = R^{\phi^{-1}TM_3}(X', Y')d\phi(Z') \quad (14)$$

for all  $X', Y', Z' \in \Gamma(TM_2)$ .

It is well-known from [18] that the tension field of the composition  $\phi \circ \psi$  is given by

$$\tau(\phi \circ \psi) = d\phi(\tau(\psi)) + \text{Tr}_g Dd\phi(d\psi, d\psi) = d\phi(\tau(\psi))$$

since  $\phi$  is totally geodesic. Then the  $f$ -tension field of the composition of  $\phi \circ \psi$  is

$$\tau_f(\psi \circ \phi) = d\phi(\tau_f(\psi)) + f \text{Tr}_g Dd\phi(d\psi, d\psi) = d\phi(\tau_f(\psi))$$

since  $\phi$  is totally geodesic. Recall that  $\{e_i\}_{i=1}^m$  is a local orthonormal frame at any point in  $M_1$ , and let  $\bar{D}^* \bar{D} = \bar{D}_{e_k} \bar{D}_{e_k} - \bar{D}_{D_{e_k} e_k}$  and  $\bar{D}''^* \bar{D}'' = \bar{D}''_{e_k} \bar{D}''_{e_k} - \bar{D}''_{D_{e_k} e_k}$ . Then we have

$$\begin{aligned} \bar{D}''^* \bar{D}'' \tau_f(\phi \circ \psi) &= \bar{D}''^* \bar{D}''(d\phi \circ \tau_f(\psi)) \\ &= \bar{D}''_{e_k} \bar{D}''_{e_k} (d\phi \circ \tau_f(\psi)) - \bar{D}''_{D_{e_k} e_k} (d\phi \circ \tau_f(\psi)). \end{aligned} \quad (15)$$

We derive from (13) that

$$\begin{aligned} \bar{D}''_{e_k} (d\phi \circ \tau_f(\psi)) &= (\hat{D}'_{\hat{D}_{e_j} d\psi(e_k)} d\phi)(\tau_f(\psi)) + d\phi \circ \bar{D}_{e_k}(\tau_f(\psi)) \\ &= d\phi \circ \bar{D}_{e_k} \tau_f(\psi) \end{aligned}$$

since  $\phi$  is totally geodesic. Therefore, we arrive at

$$\bar{D}_{e_k}'' \bar{D}_{e_k}'' (d\phi \circ \tau_f(\psi)) = \bar{D}_{e_k}'' (d\phi \circ \bar{D}_{e_k} \tau_f(\psi)) = d\phi \circ \bar{D}_{e_k} \bar{D}_{e_k} \tau_f(\psi) \quad (16)$$

and

$$\bar{D}_{D_{e_k} e_k}'' (d\phi \circ \tau(\psi)) = d\phi \circ \bar{D}_{D_{e_k} e_k} \tau_f(\psi). \quad (17)$$

Substituting (16), (17) into (16), we deduce

$$\bar{D}''^* \bar{D}'' \tau_f(\phi \circ \psi) = d\phi \circ \bar{D}^* \bar{D} \tau_f(\psi). \quad (18)$$

On the other hand, it follows from (14) that

$$\begin{aligned} R^{M_3}(d(\phi \circ \psi)(e_i), \tau_f(\phi \circ \psi))d(\phi \circ \psi)(e_i) \\ = R^{\phi^{-1}TM_3}(d\psi(e_i), \tau_f(\psi))d\phi(d\psi(e_i)) \\ = d\phi \circ R^{M_2}(d\psi(e_i), \tau_f(\psi))d\psi(e_i). \end{aligned} \quad (19)$$

By (18) and (19), we obtain

$$\begin{aligned} \bar{D}''^* \bar{D}'' \tau_f(\phi \circ \psi) + R^{M_3}(d(\phi \circ \psi)(e_i), \tau_f(\phi \circ \psi))d(\phi \circ \psi)(e_i) \\ = d\phi \circ [\bar{D}^* \bar{D} \tau_f(\psi) + R^{M_2}(d\psi(e_i), \tau_f(\psi))d\psi(e_i)]. \end{aligned} \quad (20)$$

Consequently, if  $\tau_f(\psi)$  is a Jacobi field, then  $\tau_f(\phi \circ \psi)$  is a Jacobi field. ■

**Corollary 5** ([8]). *If  $\psi : (M_1, g) \rightarrow (M_2, h)$  is a biharmonic map between two Riemannian manifolds and  $\phi : (M_2, h) \rightarrow (M_3, k)$  is totally geodesic, then  $\phi \circ \psi : (M_1, g) \rightarrow (M_3, k)$  is a biharmonic map.*

**Proof:** If  $f = 1$  and  $\psi : (M_1, g) \rightarrow (M_2, h)$  is a biharmonic map of two Riemannian manifolds, then  $\tau_f(\psi) = \tau(\psi)$  is a Jacobi field. We can apply the analogous arguments as Theorem 3.3, and (20) becomes

$$\begin{aligned} \bar{D}''^* \bar{D}'' \tau(\phi \circ \psi) + R^{M_3}(d(\phi \circ \psi)(e_i), \tau(\phi \circ \psi))d(\phi \circ \psi)(e_i) \\ = d\phi \circ [\bar{D}^* \bar{D} \tau(\psi) + R^{M_2}(d\psi(e_i), \tau(\psi))d\psi(e_i)] \end{aligned}$$

i.e.,  $\tau_2(\phi \circ \psi) = d\phi \circ (\tau_2(\psi))$ , where  $\tau_2(\psi)$  is the bi-tension field of  $\psi$ . Hence, we can conclude the result. ■

#### 4. Stress $f$ -bienergy Tensors

Let  $\psi : (M_1, g) \rightarrow (M_2, h)$  be a smooth map between two Riemannian manifolds. The stress energy tensor is defined by Baird and Eells [3] as

$$S(\psi) = e(\psi)g - \psi^*h$$

where  $e(\psi) = \frac{|d\psi|^2}{2}$ . Thus we have  $\text{div}S(\psi) = -\langle \tau(\psi), d\psi \rangle$ . Hence, if  $\psi$  is harmonic, then  $\psi$  satisfies the conservation law for  $S$  (i.e.,  $\text{div}S(\psi) = 0$ ). In

[27], the **stress  $f$ -energy tensor** of the smooth map  $\psi : M_1 \rightarrow M_2$  was similarly defined as

$$S^f(\psi) = fe(\psi)g - f\psi^*h$$

and they obtained

$$\operatorname{div} S^f(\psi) = -\langle \tau_f(\psi), d\psi \rangle + e(\psi)df.$$

In this case, an  $f$ -harmonic map usually does not satisfy the conservation law for  $S^f$ . In particular, by letting  $f = F'(\frac{d\psi|^2}{2})$ , then  $S^f(\psi) = F'(\frac{d\psi|^2}{2})e(\psi)g - F'(\frac{d\psi|^2}{2})\psi^*h$ . It is different than following Ara's idea [1] to define  $S^F(\psi) = F(\frac{d\psi|^2}{2})g - F'(\frac{d\psi|^2}{2})\psi^*h$ , and we have

$$\operatorname{div} S^F(\psi) = -\langle \tau_F(\psi), d\psi \rangle.$$

It implies that if  $\psi : M_1 \rightarrow M_2$  is an  $F$ -harmonic map between Riemannian manifolds, then it satisfies the conservation law for  $S^F$ .

The stress bienergy tensors and the conservation laws of biharmonic maps between Riemannian manifolds were first studied by Jiang [22] in 1987. Following his notions, we define the stress  $f$ -bienergy tensor of a smooth map as follows.

**Definition 6.** Let  $\psi : (M_1, g) \rightarrow (M_2, h)$  be a smooth map between two Riemannian manifolds. The **stress  $f$ -bienergy tensor** of  $\psi$  is defined by

$$\begin{aligned} S_2^f(X, Y) &= \frac{1}{2}|\tau_f(\psi)|^2\langle X, Y \rangle + \langle d\psi, \bar{D}(\tau_f(\psi)) \rangle \langle X, Y \rangle \\ &\quad - \langle d\psi(X), \bar{D}_Y \tau_f(\psi) \rangle - \langle d\psi(Y), \bar{D}_X \tau_f(\psi) \rangle \end{aligned} \quad (21)$$

for all  $X, Y \in \Gamma(TM_1)$ .

Remark that if  $\psi : (M_1, g) \rightarrow (M_2, h)$  is an  $f$ -biharmonic map between two Riemannian manifolds, then  $\psi$  does not necessarily satisfy the conservation law for the stress  $f$ -bienergy tensor  $S_2^f$ . Instead, we obtain the following theorem.

**Theorem 7.** If  $\psi : (M_1, g) \rightarrow (M_2, h)$  be a smooth map between two Riemannian manifolds, then we have

$$\operatorname{div} S_2^f(Y) = \pm \langle J_{\tau_f(\psi)}(Y), d\psi(Y) \rangle \quad \text{for all } Y \in \Gamma(TM_1) \quad (22)$$

where  $J_{\tau_f(\psi)}$  is the Jacobi field of  $\tau_f(\psi)$ .

**Proof:** For the map  $\psi : M_1 \rightarrow M_2$  between two Riemannian manifolds, set  $S_2^f = K_1 + K_2$ , where  $K_1$  and  $K_2$  are  $(0, 2)$ -tensors defined by

$$\begin{aligned} K_1(X, Y) &= \frac{1}{2}|\tau_f(\psi)|^2\langle X, Y \rangle + \langle d\psi, \bar{D}\tau_f(\psi) \rangle \langle X, Y \rangle \\ K_2(X, Y) &= -\langle d\psi(X), \bar{D}_Y \tau_f(\psi) \rangle - \langle d\psi(Y), \bar{D}_X \tau_f(\psi) \rangle. \end{aligned}$$



Let  $\{e_i\}$  be the geodesic frame at a point  $a \in M_1$ , and write  $Y = Y^i e_i$  at the point  $a$ . We first compute

$$\begin{aligned}
\operatorname{div} K_1(Y) &= \sum_i (\bar{D}_{e_i} K_1)(e_i, Y) = \sum_i (e_i(K_1(e_i, Y)) - K_1(e_i, \bar{D}_{e_i} Y)) \\
&= \sum_i (e_i(\frac{1}{2} |\tau_f(\psi)|^2 Y^i) + \sum_k \langle d\psi(e_k), \bar{D}_{e_k} \tau_f(\psi) \rangle Y^i) \\
&\quad - \frac{1}{2} |\tau_f(\psi)|^2 Y^i e_i - \sum_k \langle d\psi(e_k), \bar{D}_{e_k} \tau_f(\psi) \rangle Y^i e_i) \\
&= \langle \bar{D}_Y \tau_f(\psi), \tau_f(\psi) \rangle + \sum_i \langle d\psi(Y, e_i), \bar{D}_{e_i} \tau_f(\psi) \rangle \\
&\quad + \sum_i \langle d\psi(e_i), \bar{D}_Y \bar{D}_{e_i} \tau_f(\psi) \rangle \\
&= \langle \bar{D}_Y \tau_f(\psi), \tau_f(\psi) \rangle + \operatorname{Tr} \langle \bar{D} d\psi(Y, \cdot), \bar{D} \cdot \tau_f(\psi) \rangle \\
&\quad + \operatorname{Tr} \langle d\psi(\cdot), \bar{D}^2 \tau_f(\psi)(Y, \cdot) \rangle.
\end{aligned} \tag{23}$$

We then compute

$$\begin{aligned}
\operatorname{div} K_2(Y) &= \sum_i (\bar{D}_{e_i} K_2)(e_i, Y) = \sum_i (e_i(K_2(e_i, Y)) - K_2(e_i, \bar{D}_{e_i} Y)) \\
&= -\langle \bar{D}_Y \tau_f(\psi), \tau_f(\psi) \rangle - \sum_i \langle \bar{D} d\psi(Y, e_i), \bar{D}_{e_i} \tau_f(\psi) \rangle \\
&\quad - \sum_i \langle d\psi(e_i), \bar{D}_{e_i} \bar{D}_Y \tau_f(\psi) - \bar{D}_{D_{e_i} Y} \tau_f(\psi) \rangle \\
&\quad + \langle d\psi(Y), \Delta \tau_f(\psi) \rangle = -\langle \bar{D}_Y \tau_f(\psi), \tau_f(\psi) \rangle \\
&\quad - \operatorname{Tr} \langle \bar{D} d\psi(Y, \cdot), \bar{D} \cdot \tau_f(\psi) \rangle \\
&\quad - \operatorname{Tr} \langle d\psi(\cdot), \bar{D}^2 \tau_f(\psi)(\cdot, Y) \rangle + \langle d\psi(Y), \Delta \tau_f(\psi) \rangle.
\end{aligned} \tag{24}$$

Adding (24) and (25), we arrive at

$$\begin{aligned}
\operatorname{div} S_2^f(Y) &= \pm \langle d\psi(Y), \Delta \tau_f(\psi) \rangle + \sum_i \langle d\psi(e_i), R'(Y, e_i) \tau_f(\psi) \rangle \\
&= \pm \langle J_{\tau_f(\psi)}(Y), d\psi(Y) \rangle
\end{aligned} \tag{25}$$

where  $J_{\tau_f(\psi)}$  is the Jacobi field of  $\tau_f(\psi)$ . ■

**Corollary 8.** *If  $\tau_f(\psi)$  is a Jacobi field (i.e.,  $J_{\tau_f(\psi)} = 0$ ) for a map  $\psi : M_1 \rightarrow M_2$ , then it satisfies the conservation law (i.e.,  $\operatorname{div} S_2^f = 0$ ) for the stress  $f$ -bienergy tensor  $S_2^f$ .*

**Theorem 9** ([22]). *If  $\psi : (M_1, g) \rightarrow (M_2, h)$  is biharmonic between two Riemannian manifolds, then it satisfies the conservation law for stress bienergy tensor  $S_2$*

**Proof:** If  $f = 1$  and  $\psi : (M_1, g) \rightarrow (M_2, h)$  is biharmonic, then (26) yields to

$$\begin{aligned} \operatorname{div} S_2(Y) &= \pm \langle d\psi, \Delta\tau(\psi) + \sum_i (d\psi(e_i), R'(Y, X_i)\tau(\psi)) \rangle \\ &= \pm \langle J_{\tau(\psi)}(Y), d\psi(Y) \rangle = \pm \langle \tau_2(\psi), d\psi(Y) \rangle \end{aligned}$$

where  $\tau_2(\psi)$  is the bi-tension field of  $\psi$  (i.e.,  $\tau(\psi)$  is a Jacobi field). Hence, we can conclude the result. ■

**Proposition 10.** *Let  $\psi : (M_1, g) \rightarrow (M_2, h)$  be a submersion such that  $\tau_f(\psi)$  is basic, i.e.,  $\tau_f(\psi) = W \circ \psi$  for  $W \in \Gamma(TM_2)$ . Suppose that  $W$  is Killing and  $|W|^2 = c^2$  is non-zero constant. If  $M_1$  is non-compact, then  $\tau_f(\psi)$  is a non-trivial Jacobi field.*

**Proof:** Since  $\tau_f(\psi)$  is basic

$$\begin{aligned} S_2^f(X, Y) &= \left(\frac{c^2}{2} + \langle d\psi, \bar{D}\tau_f(\psi) \rangle\right)(X, Y) - \langle d\psi(X), \bar{D}_Y\tau_f(\psi) \rangle \\ &\quad - \langle d\psi(Y), \bar{D}_X\tau_f(\psi) \rangle \end{aligned} \tag{26}$$

where  $X, Y \in \Gamma(TM_1)$ . Let  $a$  be a point in  $M_1$  with the orthonormal frame  $\{e_i\}_{i=1}^m$  such that  $\{e_j\}_{j=1}^n$  are in  $T_a^H M_1 = (T_a^V M_1)^\perp$  and  $\{e_k\}_{k=n+1}^m$  are in  $T_a^V M_1 = \operatorname{Ker} d\psi(a)$ . Because  $W$  is Killing, we have

$$\begin{aligned} \langle d\psi, \bar{D}\tau_f(\psi) \rangle(a) &= \sum_j \langle d\psi_a(e_j), \bar{D}_{e_j}\tau_f(\psi) \rangle + \sum_k \langle d\psi_a(e_k), \bar{D}_{e_k}\tau_f(\psi) \rangle \\ &= \sum_j \langle d\psi_a(e_j), D_{d\psi_a(e_j)}^{M_2} W \rangle = 0. \end{aligned} \tag{27}$$

Therefore,

$$\begin{aligned} S_2^f(a)(X, Y) &= \frac{c^2}{2}(X, Y) + \langle d\psi_a(X), D_{d\psi_a(Y)}^{M_2} W \rangle \\ &\quad - \langle d\psi_a(Y), D_{d\psi_a(X)}^{M_2} W \rangle = \frac{c^2}{2}(X, Y). \end{aligned}$$

If  $M_1$  is not compact,  $S_2^f = \frac{c^2}{2}g$  is divergence free and  $\tau_f(\psi)$  is a non-trivial Jacobi field due to  $c \neq 0$ . ■

**Proposition 11.** *If  $\psi : (M_1^2, g) \rightarrow (M_2, h)$  is a map from a surface with  $S_2^f = 0$ , then  $\psi$  is  $f$ -harmonic.*

**Proof:** Since  $S_2^f = 0$ , it implies

$$0 = \text{Tr } S_2^f = |\tau_f(\psi)|^2 + 2\langle \bar{D}\tau_f(\psi), d\psi \rangle - 2\langle \bar{D}\tau_f(\psi), d\psi \rangle = |\tau_f(\psi)|^2. \quad \blacksquare$$

**Proposition 12.** *If  $\psi : (M_1^m, g) \rightarrow (M_2, h)$  ( $m \neq 2$ ) with  $S_2^f = 0$ , then*

$$\frac{1}{m-2}|\tau_f(\psi)|^2(X, Y) + \langle \bar{D}_X\tau_f(\psi), d\psi(Y) \rangle + \langle \bar{D}_Y\tau_f(\psi), d\psi(X) \rangle = 0 \quad (28)$$

for  $X, Y \in \Gamma(T(M_1))$ .

**Proof:** Suppose that  $S_2^f = 0$ , it implies  $\text{Tr } S_2^f = 0$ . Therefore

$$\langle \bar{D}\tau_f(\psi), d\psi \rangle = -\frac{m}{2(m-2)}|\tau_f(\psi)|^2, \quad m \neq 2. \quad (29)$$

Substituting it into the definition of  $S_2^f$ , we arrive at

$$\begin{aligned} 0 = S_2^f(X, Y) &= -\frac{1}{m-2}|\tau_f(\psi)|^2(X, Y) \\ &\quad -\langle \bar{D}_X\tau_f(\psi), d\psi(Y) \rangle - \langle \bar{D}_Y\tau_f(\psi), d\psi(X) \rangle. \end{aligned} \quad (30)$$

**Corollary 13.** *If  $\psi : (M_1, g) \rightarrow (M_2, h)$  ( $m > 2$ ) with  $S_1^f = 0$  and  $\text{rank } \psi \leq m-1$ , then  $\psi$  is  $f$ -harmonic.*

**Proof:** Since  $\text{rank } \psi(a) \leq m-1$ , for a point  $a \in M_1$  there exists a unit vector  $X_a \in \text{Ker } d\psi_a$ . Letting  $X = Y = X_a$ , (28) gives to  $\tau_f(\psi) = 0$ .  $\blacksquare$

**Corollary 14.** *If  $\psi : (M_1, g) \rightarrow (M_2, h)$  is a submersion ( $m > n$ ) with  $S_2^f = 0$ , then  $\psi$  is  $f$ -harmonic.*

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