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HARMONIC ANALYSIS ON LAGRANGIAN MANIFOLDS OF INTEGRABLE HAMILTONIAN SYSTEMS

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Abstract. For an integrable Hamiltonian system we construct a representation of the phase space symmetry algebra over the space of functions on a Lagrangian manifold. The representation is a result of the canonical quantization of the integrable system using separating variables. The variables are chosen in such way that half of them parameterizes the Lagrangian manifold, which coincides with the Liouville torus of the integrable system. The obtained representation is indecomposable and non-exponentiated.

1. Introduction

The problem of quantization on a Lagrangian manifold has arisen from the theory of geometric quantization [4]. But the question how to choose a proper Lagrangian manifold remains open. Dealing with a dynamical system we use its Liuoville torus as a Lagrangian manifold. This choice guarantees that the representation space consists of holomorphic functions - functions on the special Lagrangian manifold whose complexification serves as a phase space of the system.

According to the orbit method one can construct an integrable soliton hierarchy (hierarchy of equations of soliton type) on the orbits of a loop group [3]. Finite gap phase spaces for the integrable hierarchy appeared to consist of orbits of finite quotient algebras corresponding to the loop group. On such phase spaces one can introduce canonical separating variables (Darboux coordinates), which represent points of a spectral curve [2]. The curve is hyperelliptic for many interesting integrable systems. A half of the separating variables parametrizes the Lagrangian manifold which is the Liouville torus for the integrable system in question, and the complexified Lagrangian manifold serves as a generalized Jacobian of the spectral curve.

Canonical quantization in terms of the variables of separation gives rise to a representation for the symmetry group of the phase space. We construct such representation in the space of holomorphic functions on the complexified Lagrangian manifold, and perform a harmonic analysis of the representation for the system of isotropic Landau-Lifshits equation (for a finite gap phase space).

2. Preliminaries

We deal with systems on orbits of the loop algebra $\widetilde{\mathfrak{g}} = \mathfrak{sl}(2,\mathbb{C}) \times \mathcal{P}(z, z^{-1})$. In particular, on these orbits one can construct the integrable heirarchies of modified Korteweg-de Vries equation, sin(sh)-Gordon equation, nonlinear Schrödinger equation, and isotropic Landau-Lifshits equation, for more details see [2]. The systems obey the Lax equation

$$\frac{\mathrm{d}L(z)}{\mathrm{d}t} = [A(z), L(z)], \qquad \widetilde{\mathfrak{g}}^* \ni L(z) = \begin{pmatrix} \alpha(z) & \beta(z) \\ \gamma(z) & -\alpha(z) \end{pmatrix}$$

$$\alpha(z) = \sum_{j=0}^{N} \alpha_j z^j, \quad \beta(z) = \sum_{j=0}^{N} \beta_j z^j, \quad \gamma(z) = \sum_{j=0}^{N} \gamma_j z^j$$

where α_N, β_N, γ_N are constant. The matrix $A \in \widetilde{\mathfrak{g}}$ defines a heirarchy. For example, the hierarchy of the Landau-Lifshits equation is obtained by means of the matrix

$$A(z) = -\frac{1}{z}\begin{pmatrix} \alpha_1 & \beta_1 \\ \gamma_1 & -\alpha_1 \end{pmatrix} - \frac{1}{z^2}\begin{pmatrix} \alpha_0 & \beta_0 \\ \gamma_0 & -\alpha_0 \end{pmatrix}.$$

2.1. Phase Space of the Integrable System

According to the Kostant-Adler scheme, the coadjoint action of finite quotient algebra $\mathfrak{g} \times \mathcal{P}(z^{\nu}, \ldots, z^{\nu+N-1})$ over the finite subspace $\mathcal{M}^N \equiv \mathfrak{g}^* \times \mathcal{P}(1,\, z,\, \ldots,\, z^N)$ of $\widetilde{\mathfrak{g}}^*$ produces a set of orbits $\mathcal{O}^N \in \mathcal{M}^N$, which serves as an N-gap phase space of an integrable system. Choosing different ν, one can construct different Hamiltonian systems generated by a series of Poisson structures.

The Lax equation guarantees that evolution of a system preserves the spectrum of matrix L. Thus the quantities $\mathrm{Tr}\, L^k$ are automatically constants of motion, and one gets as many as the order of L. A half of these constants defines an orbit $\mathcal{O}^N$, the rest forms a complete set of integrals of motion, which we call Hamiltonians.

All such systems are algebraic integrable, that is integrable in Kowalewska sense: every solution of the system admits a holomorphic continuation in time. So every solution is associated with a Riemannian surface $\mathcal{R}$. The constant spectrum of L provides existence of a spectral curve, which is usually defined by the equation

$$\det(L(z) - w) = 0.$$

The spectral curve serves as the Riemannian surface $\mathcal{R}$ from the definition of integrability in Kowalewska sense.

As mentioned above, the orbits $\mathcal{O}^N$ form the phase space of an integrable system. On the other hand, the phase space is the Abelian torus arising as a complexification of the Liouville torus of the system. The complexified Liouville torus coincides with a generalized Jacobian of the mentioned Reimannian surface $\mathcal{R}$

$$\widetilde{\mathrm{Jac}}(\mathcal{R}) = \mathrm{Symm}_N \underbrace{\mathcal{R} \times \mathcal{R} \times \cdots \times \mathcal{R}}_{N}, \qquad N > g$$

where g is the genus of $\mathcal{R}$. The necessity of generalization arises in hierarchies of soliton type equations because the number N of gaps is usually greater than g, see [7] for finite gap systems of the nonlinear Schrödinger hierarchy.

2.2. Separation of Variables and Quantization

Original variables in the phase space are coefficients of the polynomials γ, β, α which are the entries of the matrix L. The set of coefficients $\{\gamma_j\,; j=0,\dots,N\}$ are eliminated by means of orbit equations. So $\{\beta_j,\ \alpha_j\,; j=0,\dots,N-1\}$ serve as independent variables, and normally they are not canonically conjugate.

In order to construct a Lagrangian manifold, it is suitable to find conjugate variables. We use the scheme from [2] and the idea is the following. Let $\{z_k,\ w_k\,; k=1,\dots,N\}$ be a set of separation variables. If one requires every conjugate pair $(z_k,\ w_k)$ to be a point of the spectral curve, then $\{z_k\}$ should be the roots of the polynomial β.

The proposed scheme enables to construct variables of separation. Then we define a **Lagrangian manifold** as the submanifold parameterized by $\{z_k\,; k=1,\dots,N\}$ (all w_k are fixed), it coincides with the Liouville torus of the system in question.

Quantization in the Schrödinger picture

$$z_k \mapsto \hat{z}_k, \qquad w_k \mapsto \hat{w}_k = -\mathrm{i}\frac{\partial}{\partial z_k}, \qquad \{z_k, w_l\} = \delta_{kl} \mapsto [\hat{z}_k, \hat{w}_l] = \mathrm{i}\delta_{kl}\,\mathbb{I}$$

in a very natural way gives a representation of the algebra corresponding to the phase space symmetry group, which we call the phase space symmetry algebra. The obtained algebra representation is realized by differential operators of high order (higher than one), and so can not be exponentiated to a group. This happens because we restrict the domain of functions from the phase space to a Lagrangian manifold. This is the difference from the standard geometric quantization.

3. The Integrable System of Isotropic Landau-Lifshits Equation

Here we consider the two-gap system from the hierarchy of isotropic **Landau-Lifshits equation**, also called the continuous Heisenberg magnetic chain

$$\frac{\partial \boldsymbol{\mu}}{\partial t} = \frac{1}{2c_0}\left[\boldsymbol{\mu}, \frac{\partial^2 \boldsymbol{\mu}}{\partial x^2}\right] + \frac{c_1}{2c_0}\frac{\partial \boldsymbol{\mu}}{\partial x} \tag{1}$$

where the vector $\boldsymbol{\mu}$ describes the magnetization and c_0, c_1 are constants and $[.,.]$ denotes the cross product.

3.1. The Phase Space and its $\mathfrak{e}(3)$ Structure

The Lax matrix L looks as follows

$$\begin{aligned} L(z) &= \begin{pmatrix} \mathrm{i}\mu_3(z) & \mu_1(z) - \mathrm{i}\mu_2(z) \\ -\mu_1(z) - \mathrm{i}\mu_2(z) & -\mathrm{i}\mu_3(z) \end{pmatrix} \\ \mu_{1,2}(z) &= \sum_{j=0}^{N-1} \mu_{1,2}^{(j)} z^j, \qquad \mu_3(z) = \frac{1}{2}\, z^N + \sum_{j=0}^{N-1} \mu_3^{(j)} z^j. \end{aligned}$$

The vector $\left(\mu_1^{(0)}, \mu_2^{(0)}, \mu_3^{(0)}\right) = \boldsymbol{\mu}$ obeys the Landau-Lifshits equation (1). In the case of two-gap system ($N = 2$) one has

$$\begin{aligned} \mu_1(z) &= \mu_1^{(0)} + \mu_1^{(1)} z \\ \mu_2(z) &= \mu_2^{(0)} + \mu_2^{(1)} z \\ \mu_3(z) &= \mu_3^{(0)} + \mu_3^{(1)} z + z^2/2. \end{aligned}$$

The coefficients $\{\mu_{1,2,3}^{(0)}, \mu_{1,2,3}^{(1)}\}$ serve as *dynamical variables*, they form a phase space, which we equip with the Poisson structure

$$\{\mu_k^{(0)}, \mu_l^{(0)}\} = 0, \qquad \{\mu_k^{(0)}, \mu_l^{(1)}\} = \varepsilon_{klj}\mu_j^{(0)}, \qquad \{\mu_k^{(1)}, \mu_l^{(1)}\} = \varepsilon_{klj}\mu_j^{(1)}. \tag{2}$$

This is $\mathfrak{e}(3)$ algebra structure, therefore the Euclidian group $\mathrm{E}(3)$ serves as a **phase space symmetry group** of the system. We also call $\mathfrak{e}(3)$ the **phase space symmetry algebra**.

The invariance of the spectrum of the matrix L spectrum provides constants of motion: h_0, h_1, h_2, h_3 obtained from the equation

$$z^4/4 + h_3 z^3 + h_2 z^2 + h_1 z + h_0 = -\operatorname{Tr} L^2(z) = \mathrm{const}$$

$$\begin{aligned} h_0 &= (\boldsymbol{\mu}^{(0)}, \boldsymbol{\mu}^{(0)}) & \boldsymbol{\mu}^{(0)} &\equiv (\mu_1^{(0)}, \mu_2^{(0)}, \mu_3^{(0)}) \\ h_1 &= 2(\boldsymbol{\mu}^{(1)}, \boldsymbol{\mu}^{(0)}) & \boldsymbol{\mu}^{(1)} &\equiv (\mu_1^{(1)}, \mu_2^{(1)}, \mu_3^{(1)}) \\ h_2 &= (\boldsymbol{\mu}^{(1)}, \boldsymbol{\mu}^{(1)}) + \boldsymbol{\mu}_3^{(0)} & & \\ h_3 &= \boldsymbol{\mu}_3^{(1)}. & & \end{aligned}$$

The functions h_0, h_1 annihilate the Lie-Poisson bracket, they define an orbit $\mathcal{O}$

$$h_0 = c_0, \qquad h_1 = c_1$$

where c_0 and c_1 are arbitrary constants. The functions h_2, h_3 serve as integrals of motion called Hamiltonians.

The spectral curve, which is the Riemannian surface $\mathcal{R}$, is of genus two

$$z^4 w^2 = z^4/4 + h_3 z^3 + h_2 z^2 + c_1 z + c_0.$$

In what follows we change notations from $\boldsymbol{\mu}^{(0)}$ and $\boldsymbol{\mu}^{(1)}$ to $\boldsymbol{p}$ and $\boldsymbol{L}$ vectors

$$\boldsymbol{\mu}^{(0)} \equiv \boldsymbol{p}, \qquad \boldsymbol{\mu}^{(1)} \equiv \boldsymbol{L}.$$

Then the orbit equations get the form

$$\boldsymbol{p}^2 = c_0, \qquad (\boldsymbol{p}, \boldsymbol{L}) = c_1/2. \tag{3}$$

Evidently, the orbit is a bundle of the planes specified by the second equation in (3), over the sphere given by the first equation. Using different values of c_0 and c_1 one obtains a set of orbits. All such orbits form the phase space of the system. There exists a degenerate orbit collapsed into the point $\boldsymbol{p} = \boldsymbol{0}$, that corresponds to the case $c_0 = 0$, $c_1 = 0$.

In the new notations the Hamiltonians look as follows

$$h_2 = \boldsymbol{L}^2 + p_3, \qquad h_3 = L_3.$$

3.2. Canonical Quantization

In order to obtain a representation of the phase space symmetry algebra we use the canonical quantization (see Preliminaries). By separation of variables we prepare the system for quantization, which gives a representation over the space of functions on the Lagrangian manifold formed by a half of conjugate variables.

Separating variables are obtained in the following way, for more details see [2]. According to the scheme, the variables z_1, z_2 are roots of the polynomial β. But this is a polynomial of degree one in our case. The situation is improved by means of the similarity transformation

$$P^{-1}L(z)P = \begin{pmatrix} \mathrm{i}\mu_2(z) & \mu_1(z)+\mathrm{i}\mu_3(z) \\ -\mu_1(z)+\mathrm{i}\mu_3(z) & -\mathrm{i}\mu_2(z) \end{pmatrix}, \qquad P = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}.$$

Now the polynomial $\mu_1(z)+\mathrm{i}\mu_3(z)$ has two roots - z_1 and z_2. The conjugate variables are calculated by the formula $w_k = \mathrm{i}\mu_2(z_k)/z_k^2$. Explicit expressions for all dynamic variables are given by the formulas

$$\begin{aligned}
p_1 &= \mathrm{i}\left(\frac{z_1z_2}{4} - \frac{c_0}{z_1z_2} - \frac{z_1z_2(z_1w_1 - z_2w_2)^2}{(z_1-z_2)^2}\right) \\
p_2 &= \mathrm{i}z_1z_2\frac{z_1w_1 - z_2w_2}{z_1-z_2} \\
p_3 &= \frac{z_1z_2}{4} + \frac{c_0}{z_1z_2} + \frac{z_1z_2(z_1w_1 - z_2w_2)^2}{(z_1-z_2)^2} \\
L_1 &= \mathrm{i}\left(-\frac{z_1+z_2}{4} - \frac{c_1}{z_1z_2} - \frac{c_0(z_1+z_2)}{z_1^2z_2^2} + \frac{z_1^2w_1^2 - z_2^2w_2^2}{z_1-z_2}\right) \\
L_2 &= -\mathrm{i}\frac{z_1^2w_1 - z_2^2w_2}{z_1-z_2} \\
L_3 &= -\frac{z_1+z_2}{4} + \frac{c_1}{z_1z_2} + \frac{c_0(z_1+z_2)}{z_1^2z_2^2} - \frac{z_1^2w_1^2 - z_2^2w_2^2}{z_1-z_2}.
\end{aligned}$$

After the canonical quantization: $z_k \mapsto \hat{z}_k$, $w_k \mapsto \hat{w}_k = -\mathrm{i}\partial/\partial z_k$ and checking commutation relations we come to a representation of the Lie algebra $\mathfrak{e}(3)$. We write the algebra in the form

$$\begin{gathered}
\mathfrak{e}(3) = \{\hat{L}_3,\ \hat{L}_\pm = \hat{L}_1 \pm \mathrm{i}\hat{L}_2,\ \hat{p}_3,\ \hat{p}_\pm = \hat{p}_1 \pm \mathrm{i}\hat{p}_2\} \\
[\hat{L}_3, \hat{L}_\pm] = \pm\hat{L}_\pm, \quad [\hat{L}_+, \hat{L}_-] = 2\hat{L}_3, \quad [\hat{p}_3, \hat{p}_\pm] = 0, \quad [\hat{p}_+, \hat{p}_-] = 0 \\
[\hat{L}_3, \hat{p}_\pm] = [\hat{p}_3, \hat{L}_\pm] = \pm\hat{p}_\pm, \quad [\hat{L}_+, \hat{p}_-] = [\hat{p}_+, \hat{L}_-] = 2\hat{p}_3.
\end{gathered}$$

The representation of $\mathfrak{e}(3)$ is the following

$$\hat{L}_3 = \frac{z_1^2}{z_1 - z_2}\left(\frac{\partial^2}{\partial z_1^2} - \frac{1}{4} - \frac{c_1}{z_1^3} - \frac{c_0}{z_1^4}\right) - \frac{z_2^2}{z_1 - z_2}\left(\frac{\partial^2}{\partial z_2^2} - \frac{1}{4} - \frac{c_1}{z_2^3} - \frac{c_0}{z_2^4}\right)$$

$$\hat{L}_\pm = \frac{\mathrm{i} z_1^2}{z_1 - z_2}\left(-\frac{\partial^2}{\partial z_1^2} - \frac{1}{4} + \frac{c_1}{z_1^3} + \frac{c_0}{z_1^4} \mp \frac{\partial}{\partial z_1}\right)$$
$$\frac{\mathrm{i} z_2^2}{z_1 - z_2}\left(-\frac{\partial^2}{\partial z_2^2} - \frac{1}{4} + \frac{c_1}{z_2^3} + \frac{c_0}{z_2^4} \mp \frac{\partial}{\partial z_2}\right)$$

$$\hat{p}_3 = -\frac{z_1 z_2}{(z_1 - z_2)^2}\left(z_1^2\frac{\partial^2}{\partial z_1^2} + z_2^2\frac{\partial^2}{\partial z_2^2} - 2z_1 z_2\frac{\partial^2}{\partial z_1 \partial z_2}\right) + \frac{z_1 z_2}{4} + \frac{c_0}{z_1 z_2}$$
$$+\frac{2z_1^2 z_2^2}{(z_1 - z_2)^3}\left(\frac{\partial}{\partial z_1} - \frac{\partial}{\partial z_2}\right)$$

$$\hat{p}_\pm = \mathrm{i}\left[\frac{z_1 z_2}{(z_1 - z_2)^2}\left(z_1^2\frac{\partial^2}{\partial z_1^2} + z_2^2\frac{\partial^2}{\partial z_2^2} - 2z_1 z_2\frac{\partial^2}{\partial z_1 \partial z_2}\right) + \frac{z_1 z_2}{4} - \frac{c_0}{z_1 z_2}\right.$$
$$\left. -\frac{2z_1^2 z_2^2}{(z_1 - z_2)^3}\left(\frac{\partial}{\partial z_1} - \frac{\partial}{\partial z_2}\right) \pm \frac{z_1 z_2}{z_1 - z_2}\left(z_1\frac{\partial}{\partial z_1} - z_2\frac{\partial}{\partial z_2}\right)\right].$$

One can easily see that $L_\pm$ and L_3 admit separation of variables, but $p_\pm$ and p_3 have not a good structure for separation.

Then we calculate the Hamiltonians, which also allow separation of variables

$$\hat{h}_2 = -\frac{z_1^2 z_2}{z_1 - z_2}\left(\frac{\partial^2}{\partial z_1^2} - \frac{1}{4} - \frac{c_0}{z_1^4} - \frac{c_1}{z_1^3}\right) + \frac{z_1 z_2^2}{z_1 - z_2}\left(\frac{\partial^2}{\partial z_2^2} - \frac{1}{4} - \frac{c_0}{z_2^4} - \frac{c_1}{z_2^3}\right)$$

$$\hat{h}_3 = \frac{z_1^2}{z_1 - z_2}\left(\frac{\partial^2}{\partial z_1^2} - \frac{1}{4} - \frac{c_0}{z_1^4} - \frac{c_1}{z_1^3}\right) - \frac{z_2^2}{z_1 - z_2}\left(\frac{\partial^2}{\partial z_2^2} - \frac{1}{4} - \frac{c_0}{z_2^4} - \frac{c_1}{z_2^3}\right).$$

The obtained representation of $\mathfrak{e}(3)$ is realized by differential operators of the second order, therefore it can not be exponentiated to a group. This is a representation over the space of smooth symmetric functions on the Lagrangian manifold.

4. Degenerate Orbit

Now we come to the harmonic analysis, which we develop with respect to the subalgebra $\mathfrak{sl}(2) \subset \mathfrak{e}(3)$. Firstly we consider the simplest case of degenerate orbit, collapsed into a point

$$\boldsymbol{p}^2 = 0, \qquad (\boldsymbol{p}, \boldsymbol{L}) = 0.$$

Its spectral curve $\mathcal{R}$ is reduced to genus one: $z^2w^2 = z^2/4 + h_3 z + h_2$. As a result the operators $\hat{L}_3$, $\hat{L}_\pm$ can be decomposed to one-particle operators. Then we investigate the case of a generic orbit

$$\boldsymbol{p}^2 = c_0, \qquad (\boldsymbol{p}, \boldsymbol{L}) = c_1/2.$$

We construct a representation space and obtain quantization conditions.

4.1. Representation Space

We start from an action of $\mathfrak{sl}(2) = \{\hat{L}_+, \hat{L}_-, \hat{L}_3\}$. Thus, we solve the equation

$$\hat{L}_3 f(z_1, z_2) = m f(z_1, z_2) \tag{4}$$

by the method of separation of variables: $f(z_1, z_2) = W_1(z_1)W_2(z_2)$. Both functions W_1, W_2 obey the same equation

$$W'' + \left(-\frac{1}{4} - \frac{m}{z} - \frac{C}{z^2}\right) W = 0$$

which is the **Whittaker equation** with $C = \mu^2 - 1/4$ and solutions $W_{-m,\mu}$. If $C = m(m+1)$, the function f also serves as an eigenfunction of $\hat{L}^2$. We fix a value of m and denote it by J, then $\mu = \pm(J + 1/2)$. At $\mu = -(J + 1/2)$ the Whittaker function has a very simple form: $W_{-J,-J-1/2}(z) = z^{-J}\mathrm{e}^{-z/2}$. This brings to the function

$$\begin{gathered} f_{JJ}(z_1, z_2) = (z_1 z_2)^{-J}\mathrm{e}^{-(z_1+z_2)/2} \\ \hat{L}_3 f_{JJ} = J f_{JJ}, \qquad \hat{L}^2 f_{JJ} = J(J+1) f_{JJ} \end{gathered} \tag{5}$$

which is annihilated by $\hat{L}_+$. We obtain the highest weight vector of the $\mathfrak{sl}(2)$ **Verma module** $\mathcal{M}^J$ produced by the action of $\hat{L}_-$

$$f_{Jm}(z_1, z_2) = \mathrm{i}^{J-m}(J-m)!(z_1 z_2)^{-J}\mathrm{e}^{-(z_1+z_2)/2}\mathcal{L}_{J-m}^{-2J-1}(z_1 + z_2)$$

where $\mathcal{L}_n^\alpha$ denotes the associated Laguerre polynomial [10] and $m = J,\ J-1,\ \ldots$. Using the well known formula

$$\mathcal{L}_n^\alpha(z_1)\mathcal{L}_n^\alpha(z_2) = \sum_{k=0}^{n}(\alpha + k + 1)\cdots(\alpha + n)\frac{(z_1 z_2)^k}{k!}\mathcal{L}_{n-k}^{\alpha+2k}(z_1 + z_2) \tag{6}$$

one can expand every function f_{Jm} into a sum of products $W_{-m,\mu}(z_1)W_{-m,\mu}(z_2)$ over μ from $-(J + 1/2)$ to $-(m + 1/2)$, in accordance with the variable separation method. The algebra $\{\hat{L}_+, \hat{L}_-, \hat{L}_3\}$ acts in the following way

$$\hat{L}_3 f_{Jm} = m f_{Jm}, \ \ \hat{L}_- f_{Jm} = f_{J,m-1}, \ \ \hat{L}_+ f_{Jm} = (J-m)(J+m+1) f_{J,m+1}.$$

The obtained Verma module has the invariant subspace $\mathcal{M}^{-J-1}$ with the highest weight vector $f_{J,-J-1}$. Thus, a representation over the quotient $\mathcal{V}=\mathcal{M}^J/\mathcal{M}^{-J-1}$ is irreducible.

4.2. 'Unitarization' of $\mathfrak{sl}(2)$ Representation

The obtained representation is *not canonical*. Reduction to a canonical representation we call 'unitarization', because normally this procedure brings to a unitary group. On account of inability to exponentiate the proposed representation we use quotation marks.

A canonical representation can be constructed by means of the intertwining operator $\hat{A}$ defined as follows

$$\begin{aligned}\tilde{f}_{Jm}\equiv\hat{A}f_{Jm} &= \sqrt{\frac{\Gamma(J+m+1)}{\Gamma(J-m+1)}}f_{Jm}\\ &= \mathrm{i}^{J-m}\sqrt{\Gamma(J+m+1)\Gamma(J-m+1)}(z_1z_2)^{-J}\mathrm{e}^{-(z_1+z_2)/2}\mathcal{L}_{J-m}^{-2J-1}(z_1+z_2).\end{aligned}$$

Indeed, one easily checks that $\mathfrak{sl}(2)$ algebra acts in canonical way

$$\hat{L}_\pm\tilde{f}_{Jm} = \sqrt{(J\mp m)(J\pm m+1)}\,\tilde{f}_{J,m\pm1}, \qquad \hat{L}_3\tilde{f}_{Jm} = m\tilde{f}_{Jm}.$$

Also we make the basis $\{\tilde{f}_{Jm}\,; -J\leqslant m\leqslant J,\ J=0,\ 1,\ \dots\}$ orthonormal by introducing the inner product

$$\begin{aligned}\langle\tilde{f}_{Jm},\tilde{f}_{Jn}\rangle = \int_0^\infty\int_0^\infty &\frac{\tilde{f}^*_{Jm}(z_1,z_2)\tilde{f}_{Jn}(z_1,z_2)}{\Gamma(J-m+1)\Gamma(J+m+1)}\\ &\times\frac{\mathrm{d}z_1\mathrm{d}z_2}{z_1^{-J+1}z_2^{-J+1}\sum_{i=0}^{J-n}\frac{\Gamma(-J+i)}{i!}\frac{\Gamma(-n-i)}{(J-n-i)!}} = \delta_{nm}.\end{aligned}$$

Here we use the summation theorem and the orthogonal relation from [1]. One can observe that 'unitarization' by means of the intertwining operator is equivalent to the Shapovalov formula [8].

4.3. Action of $\hat{p}_3$, $\hat{p}_\pm$

With respect to the canonical representation one gets the following action of the operators $\hat{p}_3$, $\hat{p}_\pm$

$$\begin{aligned}\hat{p}_+\tilde{f}_{Jm} &= -\mathrm{i}\sqrt{(J-m)(J-m-1)}\,\tilde{f}_{J-1,m+1}\\ \hat{p}_3\tilde{f}_{Jm} &= -\mathrm{i}\sqrt{(J-m)(J+m)}\,\tilde{f}_{J-1,m}\\ \hat{p}_-\tilde{f}_{Jm} &= \mathrm{i}\sqrt{(J+m)(J+m-1)}\,\tilde{f}_{J-1,m-1}\end{aligned}$$

in agreement with the abstract action formulas for $\mathfrak{e}(3)$.

5. Generic Orbit

5.1. Representation Space

In the similar way we deal with a generic orbit. Again we start with the equation (4), and come to a more complicate equation for the functions W_1, W_2

$$W'' + \left(-\frac{1}{4} - \frac{m}{z} - \frac{C}{z^2} - \frac{c_1}{z^3} - \frac{c_0}{z^4}\right) W = 0. \tag{7}$$

Requiring $\hat{L}_+ W(z_1)W(z_2) = 0$, we find the following solution of (7)

$$W(z) = z^{-m}\mathrm{e}^{-z/2+a/z}$$

with an arbitrary a. In order to make this function an eigenfunction of $\hat{L}^2$ we should assign $C = J(J+1) + a$, $c_0 = a^2$, $c_1 = 2a(J+1)$, we again use J for the highest value of m. Then the highest weight vector has the form

$$f_{JJ}(z_1, z_2) = (z_1 z_2)^{-J}\mathrm{e}^{-(z_1+z_2)/2+a/z_1+a/z_2}.$$

By the action of $\hat{L}_-$ we produce the $\mathfrak{sl}(2)$ Verma module $\mathcal{M}^J$

$$f_{Jm}(z_1, z_2) = \mathrm{i}^{J-m}(J-m)!(z_1 z_2)^{-J}\mathrm{e}^{-(z_1+z_2)/2+a/z_1+a/z_2}\mathcal{L}_{J-m}^{-2J-1}(z_1+z_2)$$

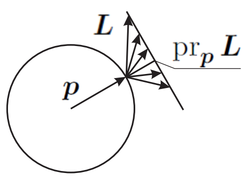
Being applied to the function f_{Jm} the formula (6) does not lead to a separation variable expansion, because the function $z^{-J}\mathrm{e}^{-z/2+a/z}\mathcal{L}_{J-m}^{-2J-1}$ with $m < J$ does not obey (7).

Nevertheless, we obtain a proper representation of the algebra $\mathfrak{sl}(2)$. Indeed, one can easily check

$$\hat{L}_3 f_{Jm} = m f_{Jm}, \quad \hat{L}_- f_{Jm} = f_{J,m-1}, \quad \hat{L}_+ f_{Jm} = (J-m)(J+m+1) f_{J,m+1}$$

that coincides with the action formulas in the case of degenerate orbit ($a = 0$).

5.2. Quantization



As shown above, one can quantize only certain orbits: with an arbitrary value $c_0 = a^2$ one should take the fixed value $c_1 = 2a(J+1)$. The latter means that a projection of $\boldsymbol{L}$ along $\boldsymbol{p}$ quantizes

$$\mathrm{pr}_{\boldsymbol{p}}\, \boldsymbol{L} = J + 1.$$

This result agrees with [5], where it is proven that a phase space admits quantization if its symplectic form is integer

$$\frac{1}{4\pi}\int_{\mathcal{S}^2}\omega \in \mathbb{Z}\,.$$

Indeed, after restriction to the orbit (3) the Poisson bracket (2) becomes nonsingular, and the restricting two-form ω is symplectic. Moreover, it is shown in [6] that

$$\frac{1}{4\pi}\int_{\mathcal{S}^2}\omega = \frac{c_1}{2\sqrt{c_0}} = J+1$$

for the same Poisson structure on the same orbit as we consider.

5.3. 'Unitarization' of $\mathfrak{sl}(2)$ Representation

Again we need to reduce the obtained representation to the canonical form, for this purpose we use the same intertwining operator $\hat{A}$

$$\tilde{f}_{Jm} \equiv \hat{A} f_{Jm} = \sqrt{\frac{\Gamma(J+m+1)}{\Gamma(J-m+1)}}\, f_{Jm} = \mathrm{i}^{J-m}\sqrt{\Gamma(J+m+1)}\times$$
$$\times\sqrt{\Gamma(J-m+1)}(z_1 z_2)^{-J}\mathrm{e}^{-(z_1+z_2)/2+a/z_1+a/z_2}\mathcal{L}_{J-m}^{-2J-1}(z_1+z_2).$$

The representation space becomes a Hilbert one after introducing the inner product

$$\langle \tilde{f}_{Jm}, \tilde{f}_{Jn}\rangle = \int_0^\infty\int_0^\infty \frac{\tilde{f}^*_{Jm}(z_1,z_2)\tilde{f}_{Jn}(z_1,z_2)}{\Gamma(J-m+1)\Gamma(J+m+1)}$$
$$\times\frac{\mathrm{e}^{-2a/z_1-2a/z_2}\,\mathrm{d}z_1\mathrm{d}z_2}{z_1^{-J+1}z_2^{-J+1}\sum_{i=0}^{J-n}\frac{\Gamma(-J+i)}{i!}\frac{\Gamma(-n-i)}{(J-n-i)!}} = \delta_{nm}.$$

5.4. Action of $\hat{p}_3$, $\hat{p}_\pm$

With respect to the canonical representation one obtains the action of $\hat{p}_3$, $\hat{p}_\pm$

$$\hat{p}_+\tilde{f}_{Jm} = -\mathrm{i}\left(1+\frac{a(z_1+z_2)}{Jz_1z_2}\right)\sqrt{(J-m)(J-m-1)}\,\tilde{f}_{J-1,m+1}+ \\ +\frac{a}{J}\sqrt{(J-m)(J+m+1)}\,\tilde{f}_{J,m+1}$$

$$\hat{p}_3\tilde{f}_{Jm} = -\mathrm{i}\left(1+\frac{a(z_1+z_2)}{Jz_1z_2}\right)\sqrt{(J-m)(J+m)}\,\tilde{f}_{J-1,m}+\frac{a}{J}\,m\tilde{f}_{J,m}$$

$$\hat{p}_-\tilde{f}_{Jm} = \mathrm{i}\left(1+\frac{a(z_1+z_2)}{Jz_1z_2}\right)\sqrt{(J+m)(J+m-1)}\,\tilde{f}_{J-1,m-1}+ \\ +\frac{a}{J}\sqrt{(J+m)(J-m+1)}\,\tilde{f}_{J,m-1}$$

which does not match with the abstract action formulas for $\mathfrak{e}(3)$. This situation is probably caused by the mentioned absence of a separation variable expansion.

6. Conclusion

A combination of algebraic geometry and Lie algebras representation theory methods applied to integrable Hamiltonian systems gives a new approach to harmonic analysis on Lagrangian manifolds. Dealing with an integrable system we have a definite rule how to chose a Lagrangian manifold - it should coincides with the Liouville torus of the system. This provides holomorphic functions as a representation space. Restriction of the function domain to the Lagrangian manifold entails that the phase space symmetry algebra is represented by differential operators of order higher than one and so it can not be exponentiated to a group. Nevertheless, there are a lot of integrable systems, among them Gaudin's model [9], where the proposed scheme gives a good basis in the phase space.

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References

[1] Abramowitz M. and Stegun I., *Handbook of Mathematical Functions,* Dover, New York 1972.

[2] Bernatska J. and Holod P., *On Separation of Variables for Integrable Equations of Soliton Type*, J. Nonlinear Math. Phys. **14** (2007) 353–374.

[3] Holod P., *Hamiltonian Systems on the Orbits of Affine Lie Groups and Finite-Band Integration of Nonlinear Equations*, Nonlinear and Turbulebt Processes in Physics vol. 3, R. Sagdeev (Ed), Harwood Academic Publishers, New York 1984, pp 1361–1367.

[4] Karasev M., *Quantization by Membranes and Integral Representations of Wave Functions,* In: Quantization and Infinite-Dimensional Systems, S. Ali, J.-P. Antoine, W. Lisiecki, I. Mladenov and A. Odzijewicz (Eds), Plenum, New York 1994, pp. 9-19.

[5] Kostant B., *Quantization and Unitary Representation,* Lecture Notes in Math. **170** (1970) 87–208.

[6] Novikov S. and Shmel'tser I., *Periodic Solutions of Kirchhoff's Equations for the Free Motion of a Rigid Body in a Fluid and the Extended Theory of Lyusternik-Shnirel'man - Morse* (*LSM*) *I*, Funct. Anal. Appl. **15** (1981) 197–207.

[7] Previato E., *Hyperelliptic Quasi-Periodic and Soliton Solution of the Nonlinear Schrodinger Equation*, Duke Math. J. **52** (1985) 323–332.

[8] Shapovalov N., *On a Bilinear Form on the Universal Enveloping Algebra of a Complex Semisimple Lie Algebra,* Funct. Anal. Appl. **6** (1972) 307–312.

[9] Sklyanin E., *Separation of Variables in the Gaudin Model,* J. Sov. Math. **47** (1989) 2473–2488.

[10] Szego G., *Orthogonal Polynomials*, Amer. Math. Soc., Rhode Island 1939.