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EXPANSIONS OVER ADJOINT SOLUTIONS FOR THE CAUDREY-BEALS-COIFMAN SYSTEM WITH $\mathbb{Z}_p$ REDUCTIONS OF MIKHAILOV TYPE

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Abstract. We consider the Caudrey-Beals-Coifman linear problem and the theory of the Recursion Operators (Generating Operators) related to it in the presence of $\mathbb{Z}_p$ reduction of Mikhailov type

1. Introduction

1.1. The Generalized Zakharov-Shabat and Caudrey-Beals-Coifman Systems

As it is well known **nonlinear evolution equations** (NLEEs) of soliton type are equations (systems) that can be written into the form $[L, A] = 0$ (Lax represensta-tion) where L, A are linear operators on ∂_x, ∂_t depending also on some functions $q_\alpha(x,t)$, $1 \leq \alpha \leq s$ (called **potentials**) and the **spectral parameter** λ. The corresponding system is of a course system of partial differential equations on $q_\alpha(x,t)$. Usually the equation is a part of a hierarchy of NLEEs related to $L\psi = 0$ (auxiliary linear problem) which consists of the equations that can be obtained by changing A and fixing L, [7, 15]. The soliton equations possess many interesting properties but for our purposes we shall mention only that they can be solved explicitly through various schemes, most of which share the property that the Lax representation permits to pass from the original evolution to the evolution of some spectral data related to the problem $L\psi = 0$. The **Caudrey-Beals-Coifman** (CBC) system, called the **Generalized Zakharov-Shabat** (GZS) system in the case when the element J is real, is one of the best known auxiliary linear problems

$$L\psi = (\mathrm{i}\partial_x + q(x) - \lambda J)\,\psi = 0. \tag{1}$$

The system has a long history of study and generalizations see [2–6,29,30], finally it has been realized that one can assume that $q(x)$ and J belong to a fixed simple Lie algebra $\mathfrak{g}$ in some finite dimensional irreducible representation, [17]. Then the element J should be regular, that is $\ker(\mathrm{ad}_J)$ ($\mathrm{ad}_J(X) \equiv [J, X]$, $X \in \mathfrak{g}$) is a **Cartan subalgebra** $\mathfrak{h} \subset \mathfrak{g}$ and $q(x)$ should belong to the orthogonal complement $\mathfrak{h}^\perp \equiv \bar{\mathfrak{g}}$ of $\mathfrak{h}$ with respect to the Killing form: $\langle X, Y\rangle = \mathrm{tr}(\mathrm{ad}_X\, \mathrm{ad}_Y)$, $X, Y \in \mathfrak{g}$. Thus $q(x) = \sum_{\alpha\in\Delta} q_\alpha(x)E_\alpha$ where E_α are the root vectors, Δ is the root system of $\mathfrak{g}$ with respect to $\mathfrak{h}$. (We use notation and normalizations as in [20].) The scalar functions $q_\alpha(x)$ are defined on $\mathbb{R}$, are complex valued, smooth and tend to zero as $x \to \pm\infty$. We shall assume that they are Schwartz-type functions. **Classical Zakharov-Shabat** system is obtained for $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$, $J = \mathrm{diag}(1, -1)$.

1.2. The AKNS Approach to the Soliton Equations

Let us construct the so-called adjoint solutions of the system L, that is functions of the type $w = mXm^{-1}$ where $X = \mathrm{const}$, $X \in \mathfrak{g}$ and m is fundamental solution of $Lm = 0$. They satisfy the equation

$$[L, w] = (\mathrm{i}\partial_x w + [q(x) - \lambda J, w]) = 0.$$

Let $w^{\mathrm{a}} = \pi_0$, $w^{\mathrm{d}} = (\mathrm{id} - \pi_0)w$ where π_0 is the orthogonal projector (with respect to the Killing form) of w over $\mathfrak{h}^\perp$ and $\mathfrak{h}$ respectively. We cannot go in detail into the AKNS approach, its history and generalizations, we just mention the seminal work [1] according to which the approach has been named and refer to [15] for all the details. Very roughly speaking the main facts are the following

- If a suitable set of adjoint solutions $(w_i(x, \lambda))_i$ is taken, for λ on the spectrum of L the functions $w_i^{\mathrm{a}}(x, \lambda)$ form a complete set in the space of potentials $q(x)$.
- If one expands the potential over $(w_i(x, \lambda))_i$ as coefficients one gets the minimal scattering data for L.

2. Recursion Operators

Relation to the expansions over adjoint solutions. From the above follows that passing from the potentials to the scattering data can be considered as **Generalized Fourier Transform**(GFT). For it the functions $w_i^{\mathrm{a}}(x, \lambda)$ play the role the exponents play in the Fourier transform. The **Recursion Operators** (**Generating Operators**, **Λ-operators**) are the operators for which the adjoint solutions $w_i^{\mathrm{a}}(x, \lambda)$ introduced above are eigenfunctions and therefore for the GFT they play the same role as the differentiation operator in the Fourier transform method.

For the above reason recursion operators are theoretical tools containing most of the information about the NLEEs associated with L. Through them can be obtained

i) The hierarchies of the nonlinear evolution equations solvable through L.

ii) The conservation laws for these NLEEs.

iii) The hierarchies of Hamiltonian structures for these NLEEs.

It is not hard to get that the recursion operators related to L have the form

$$\Lambda_{\pm}(X(x)) = \operatorname{ad}_J^{-1}\left(\mathrm{i}\partial_x X + \pi_0[q, X] + \mathrm{i}\operatorname{ad}_q \int\limits_{\pm\infty}^{x} (\mathrm{id} - \pi_0)[q(y), X(y)]\mathrm{d}y \right) \tag{2}$$

where of course $\operatorname{ad}_q(X) = [q, X]$ and X is a smooth, fast decreasing function with values in $\mathfrak{h}^{\perp}$.

Relation to recursion identities. The name recursion operators has the following origin. If for the NLEEs such that $[L, A] = 0$ the operator A is of the form

$$A = \mathrm{i}\partial_t + \sum_{k=0}^{n} \lambda^k A_k, \qquad A_n \in \mathfrak{h}, \qquad A_n = \mathrm{const}, \qquad A_{n-1} \in \mathfrak{h}^{\perp}$$

then first $A_{n-1} = \operatorname{ad}_J^{-1}[q, A]$ and for $0 < k < n-1$ one gets the recursion relations

$$\pi_0 A_{k-1} = \Lambda_{\pm}(\pi_0 A_k), \qquad (\mathrm{id} - \pi_0)A_k = \mathrm{i}(\mathrm{id} - \pi_0)\int\limits_{\pm\infty}^{x} [q, \pi_0 A_k](y)\mathrm{d}y. \tag{3}$$

Moreover, the NLEEs related to L can be written into one of the two forms

$$\mathrm{i}\operatorname{ad}_J^{-1} q_t + \Lambda_{\pm}^n \left(\operatorname{ad}_J^{-1}[A_n, q]\right) = 0. \tag{4}$$

Thus the recursion operators could be introduced also algebraically as the operators solving the above recursion relations.

Geometric Interpretation. The recursion operators have interesting geometric interpretation as dual objects to a Nijenhuis tensors N on the manifold of potentials on which it is defined a special geometric structure, Poisson-Nijenhuis structure [15, 22]. The corresponding NLEEs are fundamental fields of that structure.

Summarizing, the recursion operators have three important aspects

- They appear naturally by considering the recursion relations arising from the Lax representations of the NLEEs related with L.
- In the generalized Fourier expansions they play the role similar of the role of differentiation in the Fourier expansions.
- Their adjoint operators are Nijenhuis tensors for some special geometric structure on the manifold of potentials - Poisson-Nijenhuis structures.

In this work we shall discuss the implications of the Mikhailov-type reductions on the theory of recursion operators. The topic has been considered recently in several papers, for example [12–14, 25–27]. The case treated in these papers is of the CBC system in pole gauge. The CBC system in canonical gauge (the one we discuss now) subject to reductions has been considered earlier. For example, in [18, 19] were investigated the implications to the scattering data. In [16] the recursion operators in the presence of reductions has been considered from spectral theory viewpoint. General result about the geometry of the recursion operators for L in canonical gauge is presented in [28]. From the other side, though there are number of papers treating what happens with the spectral expansions related with the recursion operators in concrete situations with $\mathbb{Z}_p$ reductions, there has been no general treatment and we shall try to fill this gap.

3. Fundamental Solutions to the CBC System

If $q(x) = \sum_{\alpha\in\Delta} q_\alpha(x)E_\alpha$ we define: $\|q\|_1 = \sum\limits_{\alpha\in\Delta}\int\limits_{-\infty}^{+\infty} |q_\alpha(x)|\mathrm{d}x$. Potentials for which $\|q\|_1 < \infty$ form a Banach space $L^1(\bar{\mathfrak{g}}, \mathbb{R})$. Some important facts about the solutions of (1) with $q \in L^1(\bar{\mathfrak{g}})$ in some irreducible matrix representation defined on a space V are obtained in [17]. We remind them in this and the next section. Let $m(x,\lambda) = \psi(x,\lambda)\exp \mathrm{i}\lambda Jx$ where ψ satisfies CBC system. Then

$$\mathrm{i}\partial_x m + q(x)m - \lambda Jm + \lambda mJ = 0, \qquad \lim_{x\to-\infty} m = \mathbf{1}_V. \tag{5}$$

Theorem 1. *Suppose that for a fixed λ the bounded fundamental solution $m(x,\lambda)$, satisfying the equation* (5) *exists. Suppose that λ does not belong to the bunch of straight lines $\Sigma = \cup_{\alpha\in\Delta} l_\alpha$ where*

$$l_\alpha = \{\lambda;\ \mathrm{Im}(\lambda\alpha(J)) = 0\}. \tag{6}$$

Then the solution $m(x,\lambda)$ is unique. (*In the above* Im *denotes the imaginary part.*)

The connected components of $\mathbb{C}\setminus\Sigma$ are open sectors in the λ-plain. In every such sector either $\mathrm{Im}[\lambda\alpha(J)]$, $\alpha\in\Delta$ is identically zero or it has the same sign. We denote these sectors by Ω_ν and order them anti-clockwise. Clearly ν takes values from one to some even number $2M$. The boundary of the sector Ω_ν consists of two rays - L_ν and $L_{\nu+1}$ (L_ν comes before $L_{\nu+1}$ when we turn anti-clockwise) so that $\bar{\Omega}_\nu \cap \bar{\Omega}_{\nu-1} = L_\nu$. Of course, we understand the number ν modulo $2M$.

For small potentials ($\|q\|_1 < 1$) in any representation of $\mathfrak{g}$ there is no discrete spectrum and in each sector Ω_ν there exists unique fundamental solution $m_\nu(x,\lambda)$ of (5), analytic in λ. The solution admits extension by continuity to the boundary of Ω_ν, that is to the rays L_ν and $L_{\nu+1}$. For potentials that are not small the typical approach is to consider potentials on compact support and then to pass to Lebesgue integrable potentials. The situation is complicated, there is discrete spectrum etc.,

[17]. For our purposes however we shall limit ourselves to the situation when there is no discrete spectrum.

4. Expansions Over Adjoint Solutions

We first define in each Ω_ν analytic solutions $\chi_\nu(x,\lambda)$ of equation (1)

$$m_\nu(x,\lambda) = \chi_\nu(x,\lambda)\mathrm{e}^{\mathrm{i}\lambda Jx} \tag{7}$$

and then we set

$$e^\nu_\alpha(x,\lambda) = \pi_0(\chi_\nu(x,\lambda)E_\alpha\chi_\nu^{-1}(x,\lambda)), \qquad \lambda\in\bar{\Omega}_\nu. \tag{8}$$

This notation is better to be changed because for $\lambda\in L_\nu$ it will be good to retain the index ν to refer to the ray L_ν. Then it becomes necessary to distinguish from which sector the solution is extended. So for $\lambda\in L_\nu$ we shall write $e^{(+;\nu)}_\alpha(x,\lambda)$ if the solution is extended from the sector $\Omega_{\nu-1}$ and $e^{(-;\nu)}_\alpha(x,\lambda)$ if the solution is extended from the sector Ω_ν. In other words, for $\lambda\in L_\nu$

$$\begin{aligned} e^{\nu;+}_\alpha(x,\lambda) &= \pi_0(\chi_{\nu-1}(x,\lambda)E_\alpha\chi^{-1}_{\nu-1}(x,\lambda)) \\ e^{\nu;-}_\alpha(x,\lambda) &= \pi_0(\chi_\nu(x,\lambda)E_\alpha\chi^{-1}_\nu(x,\lambda)). \end{aligned} \tag{9}$$

In order to write the completeness relations, let is denote

$$\Pi_0 = \sum_{\gamma\in\Delta}\frac{|\,\gamma\rangle\langle\gamma\,|}{\gamma(J)}, \qquad \delta^\pm_\nu = \Delta^\pm_\nu\cap\delta_\nu \tag{10}$$

$$\delta_\nu = \{\alpha\in\Delta\ ;\ \mathrm{Im}(\lambda\alpha(J)) = 0 \text{ for } \lambda\in L_\nu\}. \tag{11}$$

Let us also assume that the rays L_ν are oriented from 0 to ∞. Then the completeness relations (no discrete spectrum) amount to the formula

$$\begin{aligned} &\Pi_0\delta(x-y) \\ &\quad = \frac{1}{2\pi}\sum_{\nu=1}^{2M}\int\limits_{L_\nu}\mathrm{d}\lambda\{\sum_{\alpha\in\delta^+_\nu}[e^{(-;\nu)}_\alpha(x)\otimes e^{(-;\nu)}_{-\alpha}(y) - e^{(+;\nu)}_{-\alpha}(x)\otimes e^{(+;\nu)}_\alpha(y)]\} \end{aligned} \tag{12}$$

where we have omitted the dependence on λ in order to be able to write the relation (12) more nicely. The above formula should be understood in the following way: first, it is assumed that $\mathfrak{g}^*$ is identified with $\mathfrak{g}$, assuming that the pairing is given by the Killing form. So for example, for $X,Y,Z\in\mathfrak{g}$ making a contraction of $X\otimes Y$ with Z on the right we obtain $X\langle Y,Z\rangle$ and making contraction on the left we get $\langle Z,X\rangle Y$. Next, the formula for Π_0 implies that making a contraction with Π_0 on the right we get $\Pi_0 X = \mathrm{ad}_J^{-1}\pi_0 X$ and similarly on the left $X\Pi_0 = -\mathrm{ad}_J^{-1}\pi_0 X$. (On the space $\bar{\mathfrak{g}}$ the operator ad_J is invertible.) Finally, if we have a L^1-integrable function $h : \mathbb{R} \mapsto \bar{\mathfrak{g}}$ then making a contraction of $\operatorname{ad}_J h = [J, h]$ with (12) from the right (left) and integrating over y from $-\infty$ to $+\infty$ we get

$$h(x) = \frac{\epsilon}{2\pi} \sum_{\nu=1}^{2M} \int_{L_\nu} \{ \sum_{\alpha\in\delta_\nu^+} [e_{\epsilon\alpha}^{(-;\nu)}(x) \langle\langle e_{-\epsilon\alpha}^{(-;\nu)}, [J, h]\rangle\rangle \\ - e_{\epsilon\alpha}^{(+;\nu)}(x) \langle\langle e_{-\epsilon\alpha}^{(+;\nu)}, [J, h]\rangle\rangle]\} \mathrm{d}\lambda. \quad (13)$$

We have two expansions here for $\epsilon = +1$ and $\epsilon = -1$ and we adopted the notation

$$\langle\langle e_\alpha^{(\pm;\nu)}, [J, h]\rangle\rangle = \int_{-\infty}^{+\infty} \langle e_\alpha^{(\pm;\nu)}(x), [J, h(x)]\rangle\rangle \mathrm{d}x. \quad (14)$$

We must make some comments here

1. It can be shown that the expansion (13) converges in the same sense as the Fourier expansions for $h(x)$. These are the so-called Generalized Fourier Expansions and the functions $e_\alpha^{\pm;\nu}(x, \lambda)$ are the Generalized Exponents. When one expands over the Generalized Exponents the potential $q(x)$ one gets as coefficients the minimal scattering data.
2. One can prove that

$$(\Lambda_- - \lambda) e_\alpha^{(-;\nu)} = 0, \qquad (\Lambda_- - \lambda) e_{-\alpha}^{(+;\nu)} = 0, \qquad \alpha \in \delta_\nu^+ \quad (15)$$

$$(\Lambda_+ - \lambda) e_{-\alpha}^{(-;\nu)} = 0, \qquad (\Lambda_+ - \lambda) e_\alpha^{(+;\nu)} = 0, \qquad \alpha \in \delta_\nu^+ \quad (16)$$

and therefore the expansions (13) are in fact the spectral decompositions for the operators Λ_- and Λ_+, that is they play for these expansions the role that $\mathrm{i}\partial_x$ plays for the Fourier expansion.

5. $\mathbb{Z}_p$ Reductions in the CBC System Defined by an Automorphism

We shall consider now special type of linear problems of the type (1) in which the potential function $q(x)$ and the element J obey some special requirements resulting from Mikhailov-type reductions, [21,23,24]. We shall consider the case when the Mikhailov reduction group G_0 is generated by one element, which we denote by H. It acts on the fundamental solutions in the following way

$$H(\psi(x, \lambda)) = \mathcal{K}(\psi(x, \omega^{-1}\lambda)) \quad (17)$$

where $\omega = \exp \dfrac{2\pi\mathrm{i}}{p}$ and $\mathcal{K}$ is automorphism of order p of the Lie group corresponding to the algebra $\mathfrak{g}$. $\mathcal{K}$ generates an automorphism of $\mathfrak{g}$ which we shall denote by the same letter $\mathcal{K}$. We shall require in the above situation that the automorphism leaves invariant the Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ to which the element J in the CBC system belongs. We proceed with some general remarks and technical results.

1. Suppose $\mathcal{K}$ is an automorphism of $\mathfrak{g}$ and $\mathcal{K}^p = \mathrm{id}$, $\mathcal{K}\mathfrak{h} \subset \mathfrak{h}$. (In case of Coxeter automorphisms p is the **Coxeter number**.) The Coxeter automorphisms are internal that is each $\mathcal{K}$ is internal and can be represented as $\mathcal{K} = \mathrm{Ad}(K)$, K belonging to the corresponding group G with algebra $\mathfrak{g}$.
2. The automorphisms leave the Killing form invariant, a fact that we shall use constantly.
3. The algebra $\mathfrak{g}$ splits into a direct sum of eigenspaces of $\mathcal{K}$, that is
$$\mathfrak{g} = \oplus_{s=0}^{p-1}\mathfrak{g}^{[s]} \tag{18}$$
where for each $X \in \mathfrak{g}^{[s]}$ we have $\mathcal{K}X = \omega^s X$ and the spaces $\mathfrak{g}^{[s]}$, $\mathfrak{g}^{[k]}$ for $k \neq s$ are orthogonal with respect to the Killing form.
4. Because $\mathcal{K}$ leaves $\mathfrak{h}$ invariant, it leaves invariant also the orthogonal complement $\bar{\mathfrak{g}}$ of $\mathfrak{h}$. Thus each $\mathfrak{g}^{[s]}$ splits into $\bar{\mathfrak{g}}^{[s]} \oplus \mathfrak{h}^{[s]}$ and
$$\bar{\mathfrak{g}} = \oplus_{s=0}^{p-1}\bar{\mathfrak{g}}^{[s]}, \qquad \mathfrak{h} = \oplus_{s=0}^{p-1}\mathfrak{h}^{[s]}. \tag{19}$$
The spaces $\bar{\mathfrak{g}}^{[k]}$ and $\mathfrak{h}^{[s]}$ are orthogonal for arbitrary k and s. We shall denote the projectors over the space $\bar{\mathfrak{g}}^{[k]}$ by $\pi_0^{[s]}$.

After the above preliminaries, let us assume that the set of fundamental solutions for the spectral problem (1) is invariant under G_0. Then as it is easy to see that we must have
$$\mathcal{K}(J) = \omega J, \qquad \mathcal{K}q = q \tag{20}$$
that is, $J \in \mathfrak{g}^{[1]}$, $q(x) \in \mathfrak{g}^{[0]}$. In fact, suppose we have a Lax representation $[L, A] = 0$ where A has the form
$$A = \mathrm{i}\partial_t + \sum_{k=0}^{n} \lambda^k A_k, \qquad A_n \in \mathfrak{h}, \qquad A_n = \mathrm{const}, \qquad A_{n-1} \in \bar{\mathfrak{g}}.$$
If the common fundamental solutions for $L\psi = 0, A\psi = 0$ are invariant under G_0 then we also have
$$\mathcal{K}(A_s) = \omega^s A_s, \qquad s = 0, 1, 2, \dots n. \tag{21}$$
The above reductions are compatible with the evolution in the sense that if at the moment $t = 0$ we have (20), (21) we have the same relations at arbitrary moment t. The invariance of the set of the fundamental solutions can be additionally specified if we take the solutions $m_\nu(x, \lambda)$ defined in the sectors Ω_ν, $\nu = 1, 2, \dots 2M$ defined by the straight lines $l_\alpha = \{\lambda\ ;\ \mathrm{Im}(\lambda\alpha(\mathrm{J})) = 0, \alpha \in \Delta\}$. (Of course, one obtains the same line for α and $-\alpha$ but it can happen that $\alpha \neq \beta$ and $l_\alpha = l_\beta$.)

Taking into account the uniqueness of the solutions $m(x, \lambda)$ we get that $\mathcal{K}(m(x, \lambda))$ is equal to $m(x, \omega\lambda)$. Consequently, we obtain that
$$\mathcal{K}(\chi(x, \lambda)) = \mathcal{K}(m(x, \lambda)\mathrm{e}^{-\mathrm{i}Jx\lambda}) = m(x, \omega\lambda)\mathrm{e}^{-iJx\omega\lambda} = \chi(x, \omega\lambda) \tag{22}$$

is analytic in $\omega\Omega_\nu$. If l_α, l_β form the boundary of Ω_ν then $\omega l_\alpha, \omega l_\beta$ are the straight lines defining the boundary of $\omega\Omega_\nu$.

Let us define $\hat{\mathcal{K}} : \mathfrak{h} \mapsto \mathfrak{h}$ by $\hat{\mathcal{K}} = (\mathcal{K}^*)^{-1}$. The map $\hat{\mathcal{K}}$ defines the coadjoint action of $\mathcal{K}$ on $\mathfrak{h}^*$. Naturally $\hat{\mathcal{K}}^p = \mathrm{id}$ and

$$\langle \hat{\mathcal{K}}\xi, \mathcal{K}H\rangle = \langle \xi, H\rangle, \qquad \xi \in \mathfrak{h}^*, H \in \mathfrak{h}. \tag{23}$$

It is a general fact from the theory of the automorphisms is that for all roots we have $\mathcal{K}E_\alpha = q(\alpha)E_{\hat{\mathcal{K}}\alpha}$, where $q(\alpha) = \pm 1$, $q(\alpha)q(-\alpha) = 1$, $q(\alpha)q(\beta) = q(\alpha+\beta)$ if $\alpha + \beta \in \Delta$. *One easily gets that* $\omega l_\alpha = l_{\hat{\mathcal{K}}^{-1}\alpha}$. *Thus we have an action of the automorphism* $\mathcal{K}$ *(the group* $\mathbb{Z}_p$*) on the bunch of lines* $\{l_\alpha\}_{\alpha\in\Delta}$ *defined by* $\hat{\mathcal{K}}^{-1}$ *and similarly the action on the set of sectors* Ω_ν*,* $\nu = 1, 2, \ldots, 2M$. We have

Proposition 2. *The representatives from the different orbits of* $\mathbb{Z}_p$ *on the set of sectors* Ω_ν*,* $\nu = 1, 2\ldots, a$ *can be taken to be adjacent, which we shall always assume.*

6. Expansions in Presence of Reductions Defined by Automorphisms

6.1. $\mathbb{Z}_p$ Reductions of General Type

Consider the general case of automorphism $\mathcal{K}$ of order p, let Ω_1, Ω_2,...,Ω_a be the fundamental sectors (moving anticlockwise when we go from Ω_1 to Ω_a) and let us label the rays that form the boundaries of the sectors in such a way that Ω_ν is locked between the rays L_ν and $L_{\nu+1}$ that are oriented from zero to infinity. Since multiplication by ω^p is identity (turning by angle 2π) the number of sectors is $2M = pa$. Multiplying by ω we get from the sector Ω_ν the sector $\Omega_{a+\nu}$ and multiplying by ω^{2M} we get again Ω_ν so we shall understand the labels modulo $2M$. Naturally, $L_{a+\nu} = \omega L_\nu$. For each $\alpha \in \Delta$ we have $\mathcal{K}(E_\alpha) = q(\alpha)E_{\hat{\mathcal{K}}\alpha}$, where $q(\alpha)$ are numbers, such that $q(\alpha) = \pm 1$, $q(\alpha)q(-\alpha) = 1$ and $q(\alpha)q(\beta) = q(\alpha+\beta)$ if $\alpha + \beta \in \Delta$. It is not hard to obtain that

$$\begin{aligned}[\mathcal{K}\circ\pi_0](\chi_\nu(x,\lambda)E_\alpha\chi_\nu^{-1}(x,\lambda)) &= \pi_0(\chi_{\nu+a}(x,\omega\lambda)\mathcal{K}(E_\alpha)\chi_{\nu+a}^{-1}(x,\omega\lambda)) \\ &= q(\alpha)\pi_0(\chi_{\nu+a}(x,\omega\lambda)E_{\hat{\mathcal{K}}\alpha}\chi_{\nu+a}^{-1}(x,\omega\lambda))\end{aligned}$$

and as a consequence

$$\mathcal{K}(e_\alpha^\nu(x,\lambda)) = q(\alpha)e_{\hat{\mathcal{K}}\alpha}^{\nu+a}(x,\omega\lambda). \tag{24}$$

Changing the variables for the integrals over the rays that do not belong to the set $\{L_1, L_2, \dots L_a\}$ we transform expansion (12) into

$$\begin{aligned}\Pi_0\,\delta(x-y) = \frac{1}{2\pi}\sum_{\nu=1}^{a}\sum_{k=1}^{p}\int_{L_\nu}\{\sum_{\alpha\in\delta_\nu^+}[\omega^k\mathcal{K}^k\otimes\mathcal{K}^k(e_{\alpha}^{(-;\nu)}(x)\otimes e_{-\alpha}^{(-;\nu)}(y))\\ -\sum\omega^k\mathcal{K}^k\otimes\mathcal{K}^k(e_{-\alpha}^{(+;\nu)}(x)\otimes e_{\alpha}^{(+;\nu)}(y))]\}\mathrm{d}\lambda\end{aligned} \tag{25}$$

where $(\mathcal{K}\otimes\mathcal{K})(X\otimes Y) = \mathcal{K}(X)\otimes\mathcal{K}(Y)$. Note that the numbers $q(\alpha)$ do not appear any more, this occurs because we apply $\mathcal{K}$ always on products of the type $E_\alpha\otimes E_{-\alpha}$. The rays L_ν are orientated from 0 to ∞ and the index ν is understood modulo a.

The expansions of a function $h(x)$ over the adjoint solutions can be simplified further, if for arbitrary x the value $h(x)\in\mathfrak{g}^{[s]}$, where $\mathfrak{g}^{[s]}$ is the eigenspace corresponding to the eigenvalue ω^s. As the Killing form is invariant with respect to the action of the automorphism, we get

$$\begin{aligned}\langle\mathcal{K}^k(e_\alpha^\nu(x,\lambda)),[J,h(x)]\rangle &= \langle e_\alpha^\nu(x,\lambda),\mathcal{K}^{-k}([J,h(x)])\rangle\\ &= \omega^{-k(s+1)}\langle e_\alpha^\nu(x,\lambda),[J,h(x)]\rangle.\end{aligned}$$

The expansions over the adjoint solutions run as follows

$$\begin{aligned}h(x) = \frac{\epsilon}{2\pi}\sum_{\nu=1}^{a}\int_{L_\nu}\{\sum_{\alpha\in\delta_\nu^+}[\sum_{k=1}^{p}\omega^{-ks}\mathcal{K}^k(e_{\epsilon\alpha}^{(-;\nu)}(x,\lambda))\langle\langle e_{-\epsilon\alpha}^{(-;\nu)},[J,h]\rangle\rangle\\ -\sum_{k=1}^{p}\omega^{-ks}\mathcal{K}^k(e_{-\epsilon\alpha}^{(+;\nu)}(x,\lambda))\langle\langle e_{\epsilon\alpha}^{(+;\nu)},[J,h]\rangle\rangle]\}\mathrm{d}\lambda.\end{aligned} \tag{26}$$

Actually here we have two expansions, one for $\epsilon=+1$ and the other for $\epsilon=-1$ and the index ν is understood modulo a.Thus we see that $h(x)$ is expanded over the functions

$$e_\alpha^{(\pm;\nu;s)}(x,\lambda) = \sum_{k=1}^{p}\omega^{-ks}\mathcal{K}^k(e^{(\pm;\nu)}(x,\lambda))\in\mathfrak{g}^{[s]},\qquad \nu=1,2,\dots,a \tag{27}$$

since for arbitrary $X\in\mathfrak{g}$ we have $\sum_{k=1}^{p}\omega^{-ks}\mathcal{K}^k(X)\in\mathfrak{g}^{[s]}$. We shall denote by $e_\alpha^{(\nu;s)}(x,\lambda)$ the expressions

$$e_\alpha^{(\nu;s)}(x,\lambda) = \sum_{k=1}^{p}\omega^{-ks}\mathcal{K}^k(e_\alpha^\nu(x,\lambda)),\qquad \lambda\in\Omega_\nu. \tag{28}$$

Clearly, $e_\alpha^{(\pm;\nu;s)}(x,\lambda)$ are just the limits of $e_\alpha^{(\nu-1;s)}(x,\lambda)$ and $e_\alpha^{(\nu;s)}(x,\lambda)$ when λ approaches one of the rays L_ν from one or the other side. If as before $h(x)\in\mathfrak{g}^{[s]}$, we get $\langle e_{\alpha}^{(\nu;s+1)}(x,\lambda),[J,h(x)]\rangle = p\langle e_{\alpha}^{\nu},[J,h(x)]\rangle$ and the expansions (26) can be cast into the form

$$\begin{aligned} h(x) = \frac{\epsilon}{2\pi p}\sum_{\nu=1}^{a}\int_{L_\nu}\{\sum_{\alpha\in\delta_\nu^+}[e_{\epsilon\alpha}^{(-;\nu;s)}(x,\lambda)\langle\langle e_{-\epsilon\alpha}^{(-;\nu;s+1)},[J,h]\rangle\rangle \\ -e_{-\epsilon\alpha}^{(+;\nu;s)}(x,\lambda))\langle\langle e_{+\epsilon\alpha}^{(+;\nu;s+1)},[J,h]\rangle\rangle]\}\mathrm{d}\lambda. \end{aligned} \tag{29}$$

(As before we have two expansions, for $\epsilon = +1$ and for $\epsilon = -1$.)

6.2. Coxeter Automorphisms Reductions

Coxeter automorphisms are the automorphisms for which

$$\hat{\mathcal{K}} = S_{\alpha_1}S_{\alpha_2}\dots S_{\alpha_r}, \qquad \mathcal{K}^p = \mathrm{id}, \qquad p-\text{the Coexter number}$$

and S_{α_i} are the **Weyl reflections** corresponding to the simple roots $\alpha_1,\alpha_2,\dots\alpha_r$ of $\mathfrak{g}$. We are able to prove the following

Theorem 3. *Assume we have the* CBC *problem for the classical series of simple Lie algebras and the $\mathbb{Z}_p$ reduction is defined as in the above using the Coxeter automorphism $\mathcal{K}$. Then we have two adjacent fundamental sectors of analyticity for the fundamental analytic solutions $m_\nu(x,\lambda)$ and they can be chosen to be*

$$\begin{aligned} \Omega_0 &= \{\lambda\ ;\ \frac{\pi}{2} < \arg(\lambda) < \frac{\pi}{2}+\frac{\pi}{p}\} \\ \Omega_1 &= \{\lambda\ ;\ \frac{\pi}{2}+\frac{\pi}{p} < \arg(\lambda) < \frac{\pi}{2}+\frac{2\pi}{p}\}. \end{aligned} \tag{30}$$

For a reduction defined by Coxeter automorphism of order p on some fixed algebra from the classical series of simple Lie algebras the expansion we considered specify even further. First, for the sake of symmetry we label the fundamental sectors by 0 and 1, that is they are Ω_0 and Ω_1 (as in the above). Their boundaries are formed by the rays L_0, L_1, L_2. Next, if $\alpha\in\delta_\nu^+$ then

- $\nu = 2k$ leads to $\hat{\mathcal{K}}^{-k}\alpha\in\delta_0^+ = \delta_{2p}^+$
- $\nu = 2k+1$ leads to $\hat{\mathcal{K}}^{-k}\alpha\in\delta_1^+$.

The completeness relations we have considered, namely the general formula (25) and the expansions (26),(29) can be written easily for the case of Coxeter automorphism reduction. The only thing one needs to do is not to sum over ν instead from 1 to a but from 0 to 1. Of course, p is then the Coxeter number.

7. Recursion Operators and $\mathbb{Z}_p$ Reductions Related to Automorphisms

7.1. Algebraic Aspects

When we have $\mathbb{Z}_p$ reductions of the type we consider the algebra splits in a direct sum, see (18) and $q \in \mathfrak{g}^{[0]}$ while $J \in \mathfrak{h}^{[1]}$. In particular, this means that

$$\mathrm{ad}_J(\bar{\mathfrak{g}}^{[s]}) \subset \bar{\mathfrak{g}}^{[s+1]}, \qquad \mathrm{ad}_J^{-1}(\bar{\mathfrak{g}}^{[s]}) \subset \bar{\mathfrak{g}}^{[s-1]} \tag{31}$$

(the superscripts are understood modulo p.) Also, if $X \in \bar{\mathfrak{g}}^{[s]}$ then $\partial_x X \in \bar{\mathfrak{g}}^{[s]}$, $\partial_x^{-1} X \in \bar{\mathfrak{g}}^{[s]}$, $[q, X] \in \bar{\mathfrak{g}}^{[s]}$ and

$$\Lambda_{\pm} X = \mathrm{ad}_J^{-1}\{\mathrm{i}\partial_x X + \pi_0[q, X] + \mathrm{ad}_q\, \partial_x^{-1}(\mathbf{1} - \pi_0)[q, X]\} \in \bar{\mathfrak{g}}^{[s-1]}. \tag{32}$$

If we use the projectors $\pi_0^{[s]}$ introduced earlier the above expression can also be written as

$$\Lambda_{\pm} X = \mathrm{ad}_J^{-1}\{\mathrm{i}\partial_x + \pi_0\, \mathrm{ad}_q + \mathrm{ad}_q\, \partial_x^{-1}(\mathbf{1} - \pi_0)\, \mathrm{ad}_q\}\pi_0^{[s]} X. \tag{33}$$

Further on we will denote

- by $\mathfrak{F}(\bar{\mathfrak{g}})$ the space of smooth, rapidly decreasing $\bar{\mathfrak{g}}$-valued functions.
- by $\mathfrak{F}(\bar{\mathfrak{g}}^{[s]})$ the space of smooth, rapidly decreasing $\bar{\mathfrak{g}}^{[s]}$-valued functions.
- by $\Lambda_{\pm;s}$ the operator $\Lambda_{\pm}\pi_0^{[s]}$, that is $\Lambda_{\pm;s} X = \Lambda_{\pm} X$ if $X \in \mathfrak{F}(\bar{\mathfrak{g}}^{[s]})$.

The spaces $\mathfrak{F}(\bar{\mathfrak{g}}^{[s]})$ are mapped by $\Lambda_{\pm}$ and are invariant under the action of $\Lambda_{\pm}^p$

$$\Lambda_{\pm}|_{\mathfrak{F}(\bar{\mathfrak{g}}^{[s]})} = \Lambda_{\pm;s}|_{\mathfrak{F}(\bar{\mathfrak{g}}^{[s]})}, \qquad \Lambda_{\pm;s}\mathfrak{F}(\bar{\mathfrak{g}}^{[s]}) \subset \mathfrak{F}(\bar{\mathfrak{g}}^{[s-1]}). \tag{34}$$

Also

$$\Lambda_{\pm}^p|_{\mathfrak{F}(\bar{\mathfrak{g}}^{[s]}} = \Lambda_{\pm;s-p+1} \dots \Lambda_{\pm;s-1}\Lambda_{\pm;s} \tag{35}$$

and the indexes are understood modulo p. In particular

$$\Lambda_{\pm}^p|_{\mathfrak{F}(\bar{\mathfrak{g}}^{[0]})} = \Lambda_{\pm;1} \dots \Lambda_{\pm;p-2}\Lambda_{\pm;p-1}\Lambda_{\pm;p}. \tag{36}$$

Recall that the recursion operators arise naturally when looking for the NLEEs that have Lax representation $[L, A] = 0$ with L being the CBC system operator and A is the form

$$A = \mathrm{i}\partial_t + \sum_{k=0}^{n} \lambda^k A_k, \qquad \mathfrak{h} \ni A_n = \mathrm{const}, \qquad A_{n-1} \in \bar{\mathfrak{g}}. \tag{37}$$

Then from $[L, A] = 0$ we first obtain $A_{n-1} = \mathrm{ad}_J^{-1}[q, A]$ and next for $0 < k < n - 1$ the recursion relations $\pi_0 A_{k-1} = \Lambda_{\pm}(\pi_0 A_k)$ and the NLEEs (4).

Assume that we have $\mathbb{Z}_p$ reduction. Then $q \in \bar{\mathfrak{g}}^{[0]}$, $J \in \mathfrak{h}^{[0]}$ and we must have $\mathcal{K}(A_s) = \omega^s A_s$. Assume that $A_n \in \mathfrak{h}^{[n]}$. Then $A_{n-1} \in \bar{\mathfrak{g}}^{[n-1]}$ and we see that $A_s \in \mathfrak{g}^{[s]}$. Therefore the reduction requirements will be satisfied automatically when we choose $A_n \in \mathfrak{h}^{[n]}$. Since n is a natural number let us write it into the form $n = kp + m$ where k, p, m are natural numbers and $0 \leq m < p$. Then

$$\begin{aligned}\Lambda_\pm^n \operatorname{ad}_J^{-1}[A_n, q] &= \Lambda_\pm^{kp} \Lambda_\pm^m \operatorname{ad}_J^{-1}[A_n, q] \\ &= (\Lambda_{\pm;0} \ldots \Lambda_{\pm;p-2} \Lambda_{\pm;p-1})^k \Lambda_{\pm;0} \ldots \Lambda_{\pm;m-2} \Lambda_{\pm;m-1} \operatorname{ad}_J^{-1}[A_n, q].\end{aligned}$$

Starting from the works [8,9] it is frequently said that when reductions are present the recursion operator becomes of higher order in the derivative ∂_x and factorizes into a product of first order operators with respect to ∂_x. The above has been used by some authors to justify the claim that the recursion operators $R_\pm$ in the presence of $\mathbb{Z}_p$ reduction factorize to become

$$R_\pm = \Lambda_{\pm;0} \ldots \Lambda_{\pm;p-2} \Lambda_{\pm;p-1}. \tag{38}$$

To our opinion more accurate would be simply to say that they are restrictions of the recursion operator in general position on some subspaces

$$\mathfrak{F}(\bar{\mathfrak{g}}^{[p]}) = \mathfrak{F}(\bar{\mathfrak{g}}^{[0]}) \xrightarrow{\Lambda_{\pm;0}} \mathfrak{F}(\bar{\mathfrak{g}}^{[p-1]}) \xrightarrow{\Lambda_{\pm;p-1}} \ldots \xrightarrow{\Lambda_{\pm;1}} \mathfrak{F}(\bar{\mathfrak{g}}^{[0]}) = \mathfrak{F}(\bar{\mathfrak{g}}^{[p]}). \tag{39}$$

The above shows that the role of the recursion operators in case of $\mathbb{Z}_p$ reductions is taken now by $\Lambda_\pm^p$. This view is supported also by the geometric picture, [28], since the operators $(\Lambda_\pm^p)^*$ are also Nijenhuis tensors.

7.2. Expansions Over Adjoint Solutions

Let us see how the operators we introduced act on the set of functions (27), (28) over which the expansions (26) are written. Using the properties of the automorphism $\mathcal{K}$ (the fact that it commutes with the projection π_0 on $\mathfrak{h}$) and the facts that $\mathcal{K}q = q$ and $\mathcal{K}J = \omega J$ we easily get

Lemma 4. *If $\mathcal{K}$ is an automorphism of order p defining the $\mathbb{Z}_p$ reduction then*

$$\Lambda_\pm \circ \mathcal{K} = \omega \mathcal{K} \circ \Lambda_\pm. \tag{40}$$

As a consequence

$$\Lambda_\pm^p \circ \mathcal{K} = \mathcal{K} \circ \Lambda_\pm^p. \tag{41}$$

Then for $\lambda \in \Omega_\nu$ we immediately obtain

$$\Lambda_\pm e_\alpha^{(\nu;s)}(x, \lambda) = \lambda \sum_{k=1}^{p} \omega^{-k(s-1)} \mathcal{K}^k \Lambda_\pm (e_\alpha^\nu(x, \lambda)) = \lambda e_\alpha^{(\nu;s-1)}(x, \lambda). \tag{42}$$

After some calculations we get that

$$\Lambda_- e_\alpha^{(-;\nu;s)} = \lambda e_\alpha^{(-;\nu,s-1)}, \qquad \Lambda_- e_\alpha^{(+;\nu,s)} = \lambda e_\alpha^{(+;\nu.s-1)}, \quad \alpha \in \delta_\nu^+ \tag{43}$$

$$\Lambda_+ e_{-\alpha}^{(-;\nu,s)} = \lambda e_{-\alpha}^{(-;\nu,s-1)}, \qquad \Lambda_+ e_\alpha^{(+;\nu,s)} = \lambda e_\alpha^{(+;\nu,s-1)}, \quad \alpha \in \delta_\nu^+. \tag{44}$$

As a corollary

$$\Lambda_-^p e_\alpha^{(-;\nu;s)} = \lambda^p e_\alpha^{(-;\nu,s)}, \qquad \Lambda_-^p e_{-\alpha}^{(+;\nu,s)} = \lambda^p e_{-\alpha}^{(+;\nu.s)}, \quad \alpha \in \delta_\nu^+ \tag{45}$$

$$\Lambda_+^p e_{-\alpha}^{(-;\nu,s)} = \lambda^p e_{-\alpha}^{(-;\nu,s)}, \qquad \Lambda_+^p e_\alpha^{(+;\nu,s)} = \lambda^p e_\alpha^{(+;\nu,s)}, \quad \alpha \in \delta_\nu^+ \tag{46}$$

and we have

Theorem 5. *For the expansions* (26) *the role of the recursion operators are played by the* p*-th powers of the operators* $\Lambda_\pm$.

8. Conclusions

- The above considerations show that both from recursion relations viewpoint and expansion over adjoint solutions viewpoint the role of the recursion operators in case of $\mathbb{Z}_p$ reductions is played by the operators $\Lambda_\pm^p$.
- The same conclusion is drawn from the geometric considerations [28] so the theory now is complete in all aspects - algebraic, spectral and geometric.

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