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QUASICLASSICAL AND QUANTUM DYNAMICS OF SYSTEMS OF ANGULAR MOMENTA

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Abstract. Here we use the mathematical structure of group algebras and H^{+}-algebras for describing certain problems concerning the quantum dynamics of systems of angular momenta, including also the spin systems. The underlying groups are $\mathrm{SU}(2)$ and its quotient $\mathrm{SO}(3,\mathbb{R})$. The proposed scheme is considered in two different contexts.

Firstly, the purely group-algebraic framework is applied to the system of angular momenta of arbitrary origin, e.g., orbital and spin angular momenta of electrons and nucleons, systems of quantized angular momenta of rotating extended objects like molecules and etc.

Secondly, the other promising area of applications is Schrödinger quantum mechanics of rigid body with its often rather unexpected and very interesting features. Even within this Schrödinger framework the algebras of operators related to group algebras are a very useful tool.

Finally, we investigate also some problems of composed systems and the quasiclassical limit obtained as the asymptotics of "large" quantum numbers, i.e., "quickly oscillating" wave functions on groups. They are related in an interesting way to the geometry of the coadjoint orbits of the Lie group $\mathrm{SU}(2)$.

The presentation is based on the general ideas of applying group-algebraic methods and extesive use of the Lie group structure. The papers ends with consideration of the special case of the group SU(2) and its quotient SO(3,R), which is the main subject in this paper, i.e., angular momentum problems. Formally, the scheme could be applied to the isospin systems.

However, it is rather hard to imagine realistic quasiclassical isospin problems.

CONTENTS

1. Introduction

Many physical systems have geometric background based on some groups or their byproducts like homogeneous spaces, Lie algebras and co-algebras, co-adjoint orbits, etc. Those group structures are relevant both for classical and quantum theories. They are basic tools for fundamental theoretical studies. They provide us also with the very effective tool for practical calculations. According to some views [19], such a purely group-theoretical background is characteristic for almost all physical models, or at least for realistic and viable ones.

Let us mention a funny fact known to everybody from the process of learning or teaching quantum mechanics. After the primary struggle with elementary introduction to quantum theory, first of all to atomic and molecular physics, students are often convinced that the properties of quantum angular momentum, e.g., its composition rules, so important in atomic spectroscopy and nuclear phenomena, are some mysterious and obscure dynamical laws. And only later on, they are very surprised that there is nothing but group theory there, namely, the theory of unitary irreducible representations of the three-dimensional rotation group $\mathrm{SO}(3,\mathbb{R})$ or its covering group $\mathrm{SU}(2)$ [17, 31, 33]. And the really dynamical model assumptions are placed elsewhere.

Below we concentrate on certain quantum and quasiclassical problems based on some group-theoretic apriori, first of all on the theory of quantum angular momenta and their systems, including systems of spins. There were various views and various answers to the question: "What is quantum mechanics?" What is to be used as its proper and most adequate mathematical language? Hilbert space, rigged Hilbert space, operator algebra, wave mechanics, matrix mechanics, quantum logic, orthomodular lattices, etc. [2, 10, 19]? We suppose there is no answer to this question, in any case, there is none as yet. This work unifies the results presented in [27–29].

2. Group Algebras as Framework for Quantum-Mechanical Models with Symmetries

In this paper we follow some working hypothesis, idea by Schroeck [19], that every really fundamental and viable model in quantum, but also in classical, mechanics is always based on some apriori chosen group and its representations, cf. also [4–7, 20–24, 26, 31, 32]. In flat-space theories, i.e., ones without gravitation, they are Euclidean, pseudo-Euclidean and affine groups (and other Lie groups, e.g., in gauge theories). When working in a manifold, i.e., when gravitation is taken into account, everything is based on the infinite-dimensional group of all diffeomorphisms. Incidentally, this group is also fundamental in certain geometric models of nonlinear quantum mechanics [25].

In our treatment the main mathematical tool is the theory of group algebras. And we follow the idea of Tulczyjew [30] and Weyl [31] about group algebra as the interesting, in a sense aprioric, model of quantum mechanics. We begin with a short review of necessary mathematical preliminaries and prerequisites. This review is less than being far from completeness and it cannot be anything more here. It is just quoted to remind some elementary concepts and to fix notations. More details and systematic exposition can be found in [1, 9, 11–13].

2.1. H^+-Algebras

Let us begin with the concept of H^+-algebra as introduced by Ambrose [1]. This is a special case of the Banach algebra with involution, but not necessarily with the identity. Let us mention, incidentally, that any Banach algebra B without the identity may be reinterpreted as a maximal ideal in the unital Banach algebra $B\times\mathbb{C}$ with the product rule

$$(x, \lambda)\,(y, \mu) := (xy + \lambda y + \mu x, \lambda\mu)$$

and the norm

$$\|\,(x, \lambda)\,\| := \|x\|_B + |\lambda|.$$

Then the element $(0, 1)$ becomes the identity and $x \mapsto (x, 0)$ is just the mentioned injection of B into $B \times \mathbb{R}$, its image $(B, 0)$ being a maximal ideal. Let us remind that the involution, denoted by $x \mapsto x^+$, is assumed to satisfy

$$x^{++} = x, \qquad (\lambda x + \mu y)^+ = \overline{\lambda}x^+ + \overline{\mu}y^+, \qquad (xy)^+ = y^+x^+.$$

The bar-symbol above denotes the complex conjugation. We avoid to use the star-symbol for it, because this would badly interfere with stars used in our paper in the different context.

Often, but not necessarily, one assumes also

$$\|xx^+\| = \|x\|^2.$$

The algebra of bounded operators in a Hilbert space, with the usual definition of the operator norm and with the Hermitian conjugation as an involution, is a typical and very important example.

An H^+-algebra is a consistent hybrid of two structures: a Banach algebra with involution and a Hilbert space. The underlying linear space will be denoted by B and the scalar product of elements $x, y \in B$ will be denoted by (x, y) and it is assumed to obey the usual Hilbert space axioms. Let us remind what is meant by the compatibility of those structures.

- The Banach and Hilbert norms are identical

$$\|x\|^2 = (x, x)\,.$$

- The involution, referred to as Hermitian conjugation, is compatible with the Hermitian conjugation of linear operators acting in B. This means that for any $w \in B$ the Hermitian conjugation of the left regular translation $L_w : B \to B$ is identical with the left regular translation $L_{w^+} : B \to B$ by the involution of w, $(L_w)^+ = L_{w^+}$, i.e.,

$$(wx, y) = \big(x, w^+y\big) \tag{1}$$

  for any $x, y \in B$.

- The involution is a norm-preserving operation, i.e.,

$$\|x^+\| = \|x\|$$

  for any $x \in B$

$$x \neq 0 \quad \Rightarrow \quad x^+x \neq 0.$$

All these axioms imply in particular that the involution is an antiunitary operator

$$\left(x^{+}, y^{+}\right) = (y, x) = \overline{(x, y)}.$$

Therefore, it is an isometry of B as a metric space

$$d\left(x^{+}, y^{+}\right) = \|y^{+} - x^{+}\| = \|y - x\| = d\left(x, y\right).$$

Of course, being antilinear, the involution cannot be unitary. Another important consequence is that the analogue of (1) holds also for the right translations R_w

$$(xw, y) = \left(x, yw^{+}\right) \tag{2}$$

for any $x, y \in B$.

It is just (1) and (2) that enable one to use the same symbol for the involution in B and Hermitian conjugation in $L(B)$, the algebra of linear operators on B. There is no danger of confusion.

One deals very often with some special situations, when the Hilbert structure of a Banach algebra with involution is a byproduct of something more elementary, i.e., a linear functional $T : B \to \mathbb{C}$ such that

$$T(xy) = T(yx), \qquad T\left(x^{+}\right) = \overline{T(x)}, \qquad x \neq 0 \Rightarrow T\left(x^{+}x\right) > 0.$$

The scalar product is then defined as

$$(x, y) = T\left(x^{+}y\right). \tag{3}$$

The most elementary example, which at the same time provides some, so-to-speak, comparison pattern for all more general situations, is the associative algebra $L(H)$ of all linear operators acting on a finite-dimensional unitary space H. Scalar product of vectors $\varphi, \psi \in H$ will be denoted by $\langle\varphi|\psi\rangle$ while the involution in $B = L(H)$ is defined by the usual formula

$$\langle x\varphi|\psi\rangle = \langle\varphi|x^{+}\psi\rangle$$

for the Hermitian conjugation. Then T is just the trace operation, $T(x) = \mathrm{Tr}\ x$, and the scalar product (3) is given by the standard formula

$$(x, y) = \mathrm{Tr}\left(x^{+}y\right). \tag{4}$$

Let e_i be some basic elements of H and e^i be the corresponding dual elements of the conjugate space H^*, thus

$$\langle e^i, e_j\rangle = e^i\left(e_j\right) = \delta^i{}_j$$

where the symbol $\langle f, u\rangle$, used as a popular abbreviation for $f(u)$, denotes the evaluation of the linear function $f \in H^*$ on the vector $u \in H$. The **Gramm matrix** assigned to the basis $(\ldots, e_i, \ldots)$ has elements

$$\Gamma_{ij} = \langle e_i|e_j\rangle$$

then for any vectors $u = u^i e_i$, $v = v^j e_j$ we have

$$\langle u|v\rangle = \Gamma(u, v) = \Gamma_{ij}\overline{u}^i v^j.$$

Warning: There are some subtle problems concerning the complex conjugation of vectors or, more generally, tensors. The point is that there is no well-defined complex conjugation as an operation acting within an abstract linear space over $\mathbb{C}$. There is a well-defined concept of the complex space $\overline{V}$ complex-conjugated to V. Then the bar-operation acts from V to $\overline{V}$, not from V to V. Such problem does not appear in $\mathbb{C}^n$ or more generally when some distinguished basis is fixed. The corresponding detailed description would take too much space and introduce superfluous discussion, for details cf. [25]. Hence, any time when in this article we write the complex conjugation symbol over vectors or tensors, we mean the complex conjugation of their *components* as *numbers*.

The inverse matrix element will be denoted by Γ^{ij}

$$\Gamma^{ik}\Gamma_{kj} = \delta^i{}_j.$$

The scalar product of linear functions $f = f_i e^i$, $g = g_j e^j \in H^*$ is given by

$$\langle f, g\rangle = \Gamma^{ij} f_i \overline{g}_j.$$

Apparently, this expression is correctly defined, i.e., independent on the choice of basis in H. Usually one prefers the choice of orthonormal bases, when

$$\Gamma_{ij} = \langle e_i|e_j\rangle = \Gamma\left(e_i, e_j\right) = \delta_{ij}, \qquad \Gamma^{ij} = \delta^{ij}.$$

Any basis $(\ldots, e_i, \ldots)$ in a linear space H gives rise to the corresponding adapted basis $\left(\ldots, e_j{}^i, \ldots\right)$ in $L(H)$, where

$$e_j{}^i := e_j \otimes e^i, \qquad \text{i.e.,} \qquad e_j{}^i e_k = \delta^i{}_k e_j.$$

Therefore, the matrix elements of $e_j{}^i$ with respect to the basis $(\ldots, e_i, \ldots)$ are given by

$$\left(e_j{}^i\right)^a{}_b = \delta^i{}_b \delta^a{}_j.$$

It is easy to see that

$$e_j{}^i e_r{}^s = \delta^i{}_r e_j{}^s, \qquad \operatorname{Tr} e_j{}^i = \delta_j{}^i. \tag{5}$$

Introducing the modified basic elements

$$e_{ji} := \Gamma_{ik} e_j{}^k = e_j{}^k \overline{\Gamma}_{ki}$$

we have

$$\begin{aligned} e_{ik}e_{lj} &= \Gamma_{kl}e_{ij} && (6)\\ \mathrm{Tr}\,(e_{ij}) &= \Gamma_{ji} = \overline{\Gamma}_{ij} && (7)\\ e_{ij}^{+} &= e_{ji} && (8)\\ e_i{}^{j+} &= \Gamma_{ia}\Gamma^{bj}e_b{}^a = \overline{\Gamma}^{jb}e_b{}^a\overline{\Gamma}_{ai} && (9) \end{aligned}$$

where the contravariant upper-case Γ is reciprocal to the covariant lower-case one,

$$\Gamma^{ac}\Gamma_{cb} = \delta^a{}_b, \qquad \Gamma_{ac}\Gamma^{cb} = \delta_a{}^b.$$

It is clear that also the following holds

$$\overline{\Gamma}^{ac}\,\overline{\Gamma}_{cb} = \delta^a{}_b, \qquad \overline{\Gamma}_{ac}\,\overline{\Gamma}^{cb} = \delta_a{}^b.$$

The basic scalar products of operators have the following form

$$\begin{aligned} \left(e_i{}^j, e_a{}^b\right) &= \Gamma_{ia}\Gamma^{bj} = \Gamma_{ia}\overline{\Gamma}^{jb} && (10)\\ (e_{ij}, e_{ab}) &= \Gamma_{ia}\Gamma_{bj} = \Gamma_{ia}\overline{\Gamma}_{jb}. && (11) \end{aligned}$$

These are "orthogonality" relations for the operators $e_a{}^b$, e_{ab}.

Some of the above formulas become remarkably simpler, if the basis $(\ldots, e_i, \ldots)$ is orthonormal

$$\Gamma_{ij} = \delta_{ij}, \qquad \Gamma^{ij} = \delta^{ij}.$$

However, it is sometimes convenient to separate the "metrical" concepts from the weaker "affine" ones as far as possible.

It is instructive and convenient for the analysis of quantum problems to mention and make use of the Dirac notation in H, $B = L(H)$. The basic vectors e_i are then denoted by $|i\rangle$ and the basic operators e_{ij} are then given by

$$e_{ij} = |i\rangle\langle j|.$$

Certainly, the above notation is adapted just to the situation when we choose the basis $(\ldots, e_i, \ldots)$ to be orthonormal. Perhaps the notation

$$P_{ij} := |i\rangle\langle j|$$

is then more adequate than e_{ij}. The diagonal elements

$$P_i := P_{ii} = |i\rangle\langle i|$$

are then orthogonal projections onto $\mathbb{C}$-one-dimensional subspaces with bases $e_i = |i\rangle$ and

$$\sum_i |i\rangle\langle i| = \mathrm{Id}_H.$$

For a general basis we have the following **completeness relation**

$$\sum_{a,b} \Gamma^{ab} e_{ab} = \sum_{a,b} \Gamma^{ab} P_{ab} = \mathrm{Id}_H$$

or using the "summation convention"

$$\Gamma^{ab} e_{ab} = \overline{\Gamma}^{ba} e_{ab} = \mathrm{Id}_H.$$

The corresponding non-metrical, **"affine" completeness relation** is given by

$$\sum_a e^a{}_a = e^a{}_a = \mathrm{Id}_H.$$

Let us notice that the operators $e_{\underline{i}}{}^{\underline{i}}$ (the underlining of indices means that no summation convention is used here!) are idempotents, cf. equation (5). When the basis $(\dots, e_i, \dots)$ is not orthonormal, they are not Hermitian, however, they are such if the basis is orthonormal, $\Gamma_{ij} = \delta_{ij}$, see equation (9). Unlike this, the "diagonal" elements $e_{\underline{ii}}$ (no summation convention!) are always Hermitian and in the case of orthonormal basis in H they are also idempotents, cf. equations (6) and (8), so we have then that

$$e^+_{\underline{ii}} = e_{\underline{ii}}, \qquad e_{\underline{ii}} e_{\underline{ii}} = e_{\underline{ii}}$$

(no summation convention!). So in this case we obtain the orthonormal decomposition of the identity operator

$$\mathrm{Id}_H = \sum_a e_{\underline{aa}} = e^a{}_a$$

(the summation convention meant on the extreme right-hand side).

It is easy to see that for any fixed j, the linear span of elements e_{ij}, i.e., equivalently, the linear span of elements $e_i{}^j = e_{ik}\Gamma^{jk}$, forms a minimal left ideal $L(H)e_{ij} = L(H)e_i{}^j$ in $L(H)$, so $L(H)$ is a direct sum of $n = \dim H$ such ideals. One can easily show that such left ideals are generated by the operators $e_{\underline{j}}{}^{\underline{j}}$, or equivalently, by $e_{\underline{jj}}$ (no summation convention!). Let us denote those left ideals by

$$M_j = L(H) e_{\underline{jj}} = L(H) e_{\underline{j}}{}^{\underline{j}}.$$

Let us stress that: M_j is the set of linear combinations of the form $\alpha^i e_i{}^j = \beta^i e_{ij}$, where α, β are arbitrary.

Similarly, we have the minimal right ideals

$${}_jM := e_{\underline{jj}} L(H) = e_{\underline{j}}{}^{\underline{j}} L(H).$$

They are obtained as the sets of linear combinations of the form $\alpha_i e_j{}^i = \beta^i e_{ji}$, where α, β are again arbitrary.

As mentioned, $L(H)$ splits into the direct sum of ideals M_j or ${}_jM$

$$L(H) = M_1 \oplus \cdots \oplus M_n = {}_1M \oplus \cdots \oplus {}_nM.$$

This is the orthogonal splitting in the sense of (4), i.e., M_i is orthogonal to M_j if $i \neq j$, and the same is true for ${}_iM$, ${}_jM$.

Any finite-dimensional H^+-algebra B is an H^+-subalgebra of the above $L(H)$ with the induced structures. It may be uniquely decomposed into the direct sum of minimal two-sided ideals $M(\alpha)$, $\alpha = 1, \ldots, k$, every one of them being isomorphic to some $L(V)$ with the structure of H^+-algebra as described above. Therefore, $\dim M(\alpha) = n_\alpha^2$ and

$$\sum_{\alpha=1}^{k} n_\alpha^2 = \dim B.$$

Every $M(\alpha)$ is generated by some Hermitian idempotent $e(\alpha)$

$$M(\alpha) = Be(\alpha)B$$

and the following holds

$$e(\alpha)e(\beta) = 0 \quad \text{if} \quad \alpha \neq \beta, \qquad (e(\alpha), e(\beta)) = 0 \quad \text{if} \quad \alpha \neq \beta$$
$$e(\alpha)e(\alpha) = e(\alpha), \qquad e(\alpha)^+ = e(\alpha).$$

The minimal two-sided ideals $M(\alpha)$, $M(\beta)$ are orthogonal when $\alpha \neq \beta$.

In $L(H)$ there are only two ideals $M(\alpha)$, the improper ones, namely, $L(H)$ itself and $\{0\}$. And, evidently, in $L(H)$ the corresponding Hermitian idempotent is just the identity element

$$e = \mathrm{Id}_H = e^a{}_a = \Gamma^{ab} e_{ab}.$$

In a general finite-dimensional H^+-algebra B, we have that the minimal two-sided ideals $M(\alpha)$, being isomorphic with $\mathrm{L}\,(n_\alpha, \mathbb{C}) \simeq \mathbb{C}^{n_\alpha^2}$, are direct sums of $n(\alpha)$ left minimal ideals $M(\alpha)_j$, each one of dimension $n(\alpha)$. Of course, they are also representable as direct sums of $n(\alpha)$ right minimal ideals ${}_jM(\alpha)$, every of dimension $n(\alpha)$. The label j runs the range of naturals from 1 to $n(\alpha)$. And, on analogy to $L(H)$, we choose some special bases $e(\alpha)_i{}^j$, $e(\alpha)_{ij}$ in B, where, for a fixed α, i, j run over the natural range from 1 to $n(\alpha)$ (to avoid the crowd of symbols we simply write i, j instead of $i(\alpha)$, $j(\alpha)$ as we in principle should have done). Those bases are assumed to have, for a fixed α, the properties analogous to that described in equations (5)–(11). More precisely, it is so when B is not an abstract H^+-algebra but some H^+-subalgebra of $L(H)$. Otherwise some comments would be necessary concerning the coefficients $\Gamma(\alpha)_{ij}$. In any case, to avoid discussion, one can put them to be the Kronecker symbols.

Nevertheless, in a general H^+-algebra there are situations when (10), (11) are modified, e.g., that for any α there exists its own $\Gamma(\alpha)$. For instance, analytically the coefficients $\Gamma(\alpha)_{rs}$ are there proportional to δ_{rs} with coefficients depending on α. This is not the case in (6), (7). One can show (cf. [1]) that in general the canonical ε-basis may be chosen in such a way that

$$\begin{aligned}
\varepsilon(\alpha)_{ik}\varepsilon(\alpha)_{jl} &= \delta_{kj}\varepsilon(\alpha)_{il} \\
(\varepsilon(\alpha)_{ik}, \varepsilon(\alpha)_{jl}) &= 0 \quad \text{unless} \quad i = j \quad \text{and} \quad k = l \\
(\varepsilon(\alpha)_{ik}, \varepsilon(\alpha)_{ik}) &= (\varepsilon(\alpha)_{11}, \varepsilon(\alpha)_{11}) \\
\varepsilon(\alpha)^{+}_{ij} &= \varepsilon(\alpha)_{ji} \\
\varepsilon(\alpha) &= \sum_i \varepsilon(\alpha)_{ii}.
\end{aligned}$$

The diagonal elements $\varepsilon(\alpha)_{ii}$ are irreducible Hermitian idempotents (any of them is not a sum of two idempotents), and their sum equals the idempotent $\varepsilon(\alpha)$ generating the two-sided ideal $M(\alpha)$. For different α, β the corresponding ε-elements are mutually orthogonal and annihilate each other under multiplication

$$\begin{aligned}
(\varepsilon(\alpha)_{ij}, \varepsilon(\beta)_{rs}) &= 0 \quad \text{if} \quad \alpha \neq \beta \\
\varepsilon(\alpha)_{ik}\varepsilon(\beta)_{rs} &= 0 \quad \text{if} \quad \alpha \neq \beta \\
(\varepsilon(\alpha), \varepsilon(\alpha)) &= (\varepsilon(\alpha)_{11}, \varepsilon(\alpha)_{11}) \dim M(\alpha).
\end{aligned}$$

Unlike the minimal two-sided ideals $M(\alpha)$, the left and right ideals $M(\alpha)_i$ and ${}_iM(\alpha)$ are not unique.

Similarly, the basic elements $e(\alpha)_i{}^j$ or $e(\alpha)_{ij}$ are not unique. However, their "index-traces"

$$e(\alpha) = e(\alpha)^i{}_i = \Gamma(\alpha)^{ij} e(\alpha)_{ij} \tag{12}$$

are unique and just coincide, as denoted, with the generating idempotents $e(\alpha)$. The "diagonal" idempotents $e(\alpha)_{\underline{i}}{}^{\underline{i}}$ or $e(\alpha)_{\underline{ii}}$ (no summation convention!) are not unique, however, the "trace" (12) is so, and their sum is just the identity element of B

$$\sum_\alpha e(\alpha) = e.$$

The idempotent $e(\alpha)$ is referred to as the induced unit of $M(\alpha)$.

2.2. Infinite-Dimensional Situation

Those roughly referred examples provide some "reference frame" for understanding the general theory. Nevertheless, the general case, when the infinite dimension of B is admitted, is much more complicated, and many finite-dimensional analogies are misleading. Many important H^+-algebras are non-unital. Instead of direct sums, some direct integrals of Hilbert spaces must be used. In the infinite dimension many structures taken from finite-dimensional operator algebras diffuse, one must oscillate between various subsets like those of trace-class operator, Hilbert-Schmidt operators, etc.

The simplest infinite-dimensional situations are group algebras on locally compact topological groups. In particular, group algebras on compact subgroups are relatively similar to the finite-dimensional case, e.g., all minimal ideals are finite-dimensional.

Let G be a locally compact topological group. Although we are interested mainly in finite-dimensional Lie groups, nevertheless, there is a hierarchy of structures based on more general ideas and only later on, on the level of applications, assuming more and more specialized concepts. Let μ_l, μ_r denote respectively the left- and right-invariant Haar measures on G. They are unique up to constant normalization factors, but in general they do not coincide. Nevertheless, the right-shifted left-invariant measure is still left-invariant, i.e., roughly speaking

$$\mathrm{d}\mu_l(gh) = \Delta(h)\mathrm{d}\mu_l(g)$$

where $\Delta(h)$ is a positive factor, and iterating those right transforms one can easily show that

$$\Delta(hk) = \Delta(h)\Delta(k) = \Delta(kh).$$

So, Δ is a homomorphism of G into $\mathbb{R}^+$ as a multiplicative group and μ_l, μ_r may be different only when G does possess a nontrivial homomorphism into $\mathbb{R}^+$. If they are identical, we say that G is unimodular. Compact and Abelian Lie groups, and so their direct and semidirect products, are unimodular. If G is unimodular, the measure elements $\mathrm{d}\mu_l(g) = \mathrm{d}\mu_r(g)$ are denoted simply by $\mathrm{d}g$. If G is unimodular, then not only

$$\mu(Ah) = \mu(hA) = \mu(A)$$

but also $\mu\left(A^{-1}\right) = \mu(A)$ for any measurable subset $A \subset G$. From now on it will be always assumed that G is unimodular, so we write symbolically

$$\mathrm{d}(gh) = \mathrm{d}(hg), \qquad \mathrm{d}\left(g^{-1}\right) = \mathrm{d}g.$$

In the space $L^1(G)$ of integrable functions one defines the **convolution** operation

$$\left(A * B\right)(g) = \int A(h)B\left(h^{-1}g\right)\mathrm{d}h = \int A\left(gk^{-1}\right)B(k)\mathrm{d}k. \tag{13}$$

It is defined on the total $L^1(G) \times L^1(G)$ and produces from elements of $L^1(G)$ the elements of $L^1(G)$, so $L^1(G)$ is an algebra under the convolution,

$$L^1(G) * L^1(G) \subset L^1(G).$$

We see that the convolution $*$ turns $L^1(G)$ into $L^1(G)$, so that the axioms of associative Banach C^*-algebra hold in $L^1(G)$, e.g.

$$||f * g|| \leq ||f||\,||g||$$

where the $L^1(G)$-norm is meant, and the involution is defined as

$$\left(f^+\right)(x) = \overline{f\left(x^{-1}\right)}. \tag{14}$$

The linear functional Tr is defined as the value at the group identity

$$\mathrm{Tr}(f) := f(1) \tag{15}$$

and the scalar product of functions on G is defined as

$$(\varphi, \psi) = \mathrm{Tr}\left(\varphi^{+} * \psi\right) = \int \overline{\varphi(x)}\psi(x)\mathrm{d}x.$$

It is positive, i.e.,

$$\mathrm{Tr}\left(\varphi^{+}\varphi\right) = \int \overline{\varphi(x)}\varphi(x)\mathrm{d}x > 0 \quad \text{if} \quad 0 \neq \varphi \in L^2(G).$$

These expressions lead us to the space $L^2(G)$ and algebraic structures there. All these structures fit together so as to result in the structure of H^{+}-algebra in $L^2(G)$. Such structures in $L^1(G)$, $L^2(G)$ are referred to as group algebras. Everything is particularly simple when G is a discrete group. Then we have that

$$\langle \varphi, \psi \rangle = \sum_{x \in G} \overline{\varphi(x)}\psi(x).$$

If G is compact, then we usually normalize the measure such that

$$\mu(G) = 1.$$

In particular, if it is a finite, N-element group, we put

$$\int f(x)\mathrm{d}x = \frac{1}{N} \sum_{x \in G} f(x).$$

However, this normalization is not always used and not always is convenient.
If G is compact, then $L^2(G)$ splits uniquely into the direct sum of minimal two-sided ideals $M(\alpha)$, where α runs over some discrete set of labels Ω

$$L^2(G) = \underset{\alpha \in \Omega}{\oplus} M(\alpha).$$

These ideals are mutually orthogonal

$$M(\alpha) \perp M(\beta) \quad \text{if} \quad \alpha \neq \beta, \quad (F, G) = 0 \quad \text{if} \quad F \in M(\alpha),\ G \in M(\beta)$$

and then

$$M(\alpha) * M(\beta) = \{0\} \quad \text{if} \quad \alpha \neq \beta, \quad F * G = 0 \quad \text{if} \quad F \in M(\alpha),\ G \in M(\beta).$$

They are generated by Hermitian idempotents $\varepsilon(\alpha)$, therefore

$$M(\alpha) = \varepsilon(\alpha) * L^2(G) = L^2(G) * \varepsilon(\alpha)$$

and the following holds

$$(\varepsilon(\alpha), \varepsilon(\beta)) = 0 \quad \text{if} \quad \alpha \neq \beta, \quad \varepsilon(\alpha) * \varepsilon(\beta) = \delta_{\alpha\beta}\varepsilon(\beta)n^2(\beta)$$

(no summation convention in the last expression!).

The convolution with $\varepsilon(\alpha)$ acts as the orthogonal projection of $L^2(G)$ onto $M(\alpha)$. In particular, it is a generated unit of $M(\alpha)$

$$\varepsilon(\alpha) * F = F * \varepsilon(\alpha) = F$$

for any $F \in M(\alpha)$. If G is a finite group with N elements, then $L^1(G) = L^2(G)$ is a unital algebra under convolution. Its identity ε is proportional to the "delta" function

$$\mathrm{Id}(g) = N\delta(g), \quad \delta(x) := \delta_{xe} = 1 \quad \text{if} \quad x = e, \quad \delta(x) = 0 \quad \text{if} \quad x \neq e$$

where $e \in G$ denotes the group identity. The δ-type convolution identity does exist for any discrete group. It is just δ itself for finite groups.

Apparently, δ is identical with the sum of idempotents $\varepsilon(\alpha)$

$$\delta = \sum_{\alpha \in \Omega} \varepsilon(\alpha). \tag{16}$$

If G is a continuous group, this expression is a divergent series and there is no identity in group algebra. It does exist in any (in general, non-minimal) two-sided ideal obtained from (16) when the summation is extended over a finite subset of Ω. The procedure of the formally introduced identity in a one-dimensionally extended group algebra would be rather artificial, nevertheless. If G is a Lie group, it is much more natural to introduce the "identity" represented by the Dirac-delta distribution. The more so, some derivatives of "delta" distribution represent some very important physical quantities.

Let us stress that the minimal two-sided Hermitian idempotents $\varepsilon(\alpha)$ span the centre of group algebra. More precisely, they form a complete system in the subspace of convolution-central functions. All such central functions are constant on the classes of conjugate elements, i.e., on the orbits of inner automorphisms of G. If we admit the unit element of group algebra as represented by the Dirac delta distribution, then formally this δ is given by the series (16). As a function series it is divergent, however the limit does exist in the distribution sense. And, as a functional on the appropriate function space, δ assigns to any function $F : G \to \mathbb{C}$ its value at the unit element e of G

$$\langle \delta, f \rangle = f(e).$$

It is shown that the set of minimal two-sided ideals is identical with the set Ω of unitary irreducible representations of G, pairwise non-equivalent ones. More precisely, Ω is the set of equivalence classes of unitary irreducible representations. Due to the compactness of G, all those representations are finite dimensional. Let

$$D(\alpha) : G \to \mathrm{U}(n(\alpha)) \subset \mathrm{GL}(n(\alpha), \mathbb{C})$$

denote the α-th unitary irreducible representation, more precisely, some representant of the corresponding class. Clearly, $n(\alpha)$ is the dimension of the corresponding representation space $\mathbb{C}^{n(\alpha)}$. All these representation spaces are assumed to be unitary in the sense of the standard scalar products

$$\langle u, v\rangle = \sum_{a=1}^{n(\alpha)} \overline{u}^a v^a = \delta_{ab}\overline{u}^a v^b.$$

The minimal two-sided ideals $M(\alpha)$ are spanned by the matrix elements of the representations α, $D(\alpha)_{ij}$. Moreover, one can show that after appropriate normalization the functions $D(\alpha)_{ij}$ form the canonical basis, more precisely, the canonical complete system of the H^+-algebra $L^2(G)$. This follows from the following properties of matrix elements, well-known from the representation theory

$$D(\alpha)_{ij} * D(\alpha)_{kl} = \frac{1}{n(\alpha)}\delta_{jk}D(\alpha)_{il} \tag{17}$$

$$D(\alpha)_{ij} * D(\beta)_{rs} = 0 \qquad \text{if} \quad \beta \neq \alpha \tag{18}$$

$$(D(\alpha)_{ij}, D(\alpha)_{kl}) = \frac{1}{n(\alpha)}\delta_{ik}\delta_{jl} \tag{19}$$

$$(D(\alpha)_{ij}, D(\beta)_{rs}) = 0 \qquad \text{if} \quad \beta \neq \alpha. \tag{20}$$

These equations are valid when the Haar measure on G is normed to unity

$$\mu(G) = \int_G \mathrm{d}\mu = 1.$$

If other normalization is fixed, then on the right-hand sides of (17)–(20) the volume of G, $\mu(G)$, appears as a factor.

Let $\chi(\alpha)$ and $\varepsilon(\alpha)$ denote the character of $D(\alpha)$ and the corresponding trace of $\varepsilon(\alpha) = n(\alpha)D(\alpha)$ respectively

$$\chi(\alpha) = \sum_i D(\alpha)_{ii}, \qquad \varepsilon(\alpha) = \sum_i \varepsilon(\alpha)_{ii} = n(\alpha)\chi(\alpha).$$

Then

$$\chi(\alpha) * \chi(\alpha) = \frac{1}{n(\alpha)}\chi(\alpha), \qquad \chi(\alpha) * \chi(\beta) = 0 \quad \text{if} \quad \alpha \neq \beta$$

$$(\chi(\alpha), \chi(\alpha)) = 1, \qquad (\chi(\alpha), \chi(\beta)) = 0 \quad \text{if} \quad \alpha \neq \beta$$

and similarly

$$\varepsilon(\alpha) * \varepsilon(\alpha) = \varepsilon(\alpha), \qquad \varepsilon(\alpha) * \varepsilon(\beta) = 0 \quad \text{if} \quad \alpha \neq \beta \tag{21}$$

$$(\varepsilon(\alpha), \varepsilon(\alpha)) = n^2(\alpha), \qquad (\varepsilon(\alpha), \varepsilon(\beta)) = 0 \quad \text{if} \quad \alpha \neq \beta. \tag{22}$$

The above properties tell us that the canonical basis is given by functions

$$\varepsilon(\alpha)_{ij} = n(\alpha)D(\alpha)_{ij}. \tag{23}$$

They really satisfy all above-quoted structural properties of canonical bases in H^+-algebras

$$\begin{aligned}
\varepsilon(\alpha)_{ij} * \varepsilon(\alpha)_{kl} &= \delta_{jk}\varepsilon(\alpha)_{il} &(24)\\
\varepsilon(\alpha)_{ij} * \varepsilon(\beta)_{rs} &= 0 \qquad \text{if} \quad \beta \neq \alpha &(25)\\
(\varepsilon(\alpha)_{ij}, \varepsilon(\alpha)_{kl}) &= \delta_{ik}\delta_{jl}n(\alpha) &(26)\\
(\varepsilon(\alpha)_{ij}, \varepsilon(\beta)_{rs}) &= 0 \qquad \text{if} \quad \beta \neq \alpha &(27)\\
{\varepsilon(\alpha)_{ij}}^{+} &= \varepsilon(\alpha)_{ji} &(28)\\
\mathrm{Tr}\, \varepsilon(\alpha)_{ij} &= \delta_{ij}. &(29)
\end{aligned}$$

Let us stress that the above symbols "$+$" and "Tr" are used in the sense of equations (14) and (15) and do not concern directly the operations performed on indices i, j. However, they concern them indirectly. Namely, every function $F \in L^2(G)$ may be expanded as a function series with respect to the above complete system

$$F = \sum_{\alpha\in\Omega,\ n,m=1,\ldots,n(\alpha)} F(\alpha)_{nm}\varepsilon(\alpha)_{nm}. \tag{30}$$

Let us describe relationships in equations (24)–(29) in terms of the coefficients used here. For binary operations we analogously expand the other function

$$G = \sum_{\alpha\in\Omega,\ n,m=1,\ldots,n(\alpha)} G(\alpha)_{nm}\varepsilon(\alpha)_{nm}. \tag{31}$$

It follows from the above rules (24)–(29) that the convolution of F, G is represented by the system of matrices

$$(F(\alpha)G(\alpha))_{nm} = \sum_k F(\alpha)_{nk}G(\alpha)_{km}$$

i.e.,

$$F * G = \sum_{\alpha\in\Omega,\ n,m=1,\ldots,n(\alpha)} (F(\alpha)G(\alpha))_{nm}\varepsilon(\alpha)_{nm}. \tag{32}$$

Similarly, the Hermitian conjugate and the Tr-functional are represented by the usual matrix Hermitian conjugation and trace

$$\begin{aligned}
F^{+} &= \sum_{\alpha\in\Omega,\ n,m=1,\ldots,n(\alpha)} (F(\alpha)^{+})_{nm}\varepsilon(\alpha)_{nm}\\
\mathrm{Tr}\, F &= \sum_{\alpha\in\Omega} \mathrm{Tr}\, F(\alpha)
\end{aligned}$$

and in particular, for the scalar product we have that

$$(F, G) = \sum_{\alpha \in \Omega} \operatorname{Tr}\big(F(\alpha)^{+} G(\alpha)\big)\, n(\alpha). \tag{33}$$

This is just the explanation of apparently strange definitions of those operations.

2.3. Algebraic Formulation of Quantum Mechanics

In a sense, the group algebra over G may be considered as an arena for some type of the algebraic, operator-type formulation of quantum mechanics [30]. We are given the associative convolution product and all necessary equipment of H^{+}-algebra. So, we have everything that is necessary for the algebraic formulation of quantum mechanics. Physical quantities are $+$-self-adjoint elements of the group algebra, $A^{+} = A$. Density operators are self-adjoint elements, $\rho = \rho^{+}$, satisfying in addition the normalization condition

$$\operatorname{Tr} \rho = 1 \tag{34}$$

and the positive-definiteness condition

$$\big(\rho, A^{+} * A\big) > 0$$

for any element A of the group algebra. Pure states are described by Hermitian idempotents. Thus, in addition to the above conditions the following must hold

$$\rho\rho = \rho, \qquad \rho^{+} = \rho.$$

Expectation value of the physical quantity $A = A^{+}$ on the state ρ is given by

$$\langle A \rangle_{\rho} = \operatorname{Tr}(A\rho) = \big(A^{+}, \rho\big) = (A, \rho).$$

If ρ_0 is some pure state, then the probability that the measurement performed on the general state ρ will detect the state ρ_0 is given by

$$\operatorname{Tr}(\rho\rho_0) = (\rho, \rho_0).$$

Statistical interpretation may be also assigned to the non-normalized states ρ, i.e., those which do not fulfill (34). Then one can speak only about relative probabilities. However, there are yet no wave functions and no superposition principle. It is a good thing to have also some space of wave functions. The most natural candidates are L^2-spaces on the group G and its homogeneous spaces. Before going any further in this direction one should quote some comments concerning invariance problems.

Everything above was based on the convolution product (13). It is evidently associative

$$(F * G) * H = F * (G * H).$$

It is so for any $L^1(G)$-functions on any locally compact topological group G. The group structure of G brings about the question concerning the G-invariance of the convolution, in any case the question concerning the sense of such invariance. On the group manifold of G the group G itself acts through three natural transformation groups: left translations, right translations and inner automorphisms. These actions are given respectively as follows

$$x \mapsto L_g(x) := gx, \qquad x \mapsto R_g(x) := xg$$
$$x \mapsto A_g(x) := gxg^{-1}, \qquad A_g = L_g \circ R_{g^{-1}} = R_{g^{-1}} \circ L_g.$$

With this convention $g \mapsto L_g$, $g \mapsto A_g$, $g \mapsto R_g$ are respectively two realizations and one anti-realization of the group G

$$L_{g_1 g_2} = L_{g_1} \circ L_{g_2}, \qquad A_{g_1 g_2} = A_{g_1} \circ A_{g_2}, \qquad R_{g_1 g_2} = R_{g_2} \circ R_{g_1}. \tag{35}$$

Substituting g^{-1} instead of g we replace realizations by anti-realizations and conversely.

The above operations induce the pointwise actions on functions on G, namely

$$L[g]f := f \circ L_{g^{-1}}, \qquad R[g]f := f \circ R_{g^{-1}} \tag{36}$$
$$A[g]f := f \circ A_{g^{-1}}, \qquad A[g] = L[g]R\left[g^{-1}\right] = R\left[g^{-1}\right]L[g]. \tag{37}$$

Via replacement of g by g^{-1}, we obtain in this way two linear representations and one anti-representation of G in function spaces on G itself

$$L[g_1 g_2] = L[g_1]L[g_2], \quad A[g_1 g_2] = A[g_1]A[g_2], \quad R[g_1 g_2] = R[g_2]R[g_1].$$

All these transformations preserve the spaces $L^1(G)$, $L^2(G)$. They preserve also the scalar products and the corresponding statistical statements concerning measurements. However, the left and right regular translations $L[g]$, $R[g]$ do not preserve the convolution. Unlike this, internal automorphisms do preserve this algebraic structure. Indeed

$$L[g](F * G) = (L[g]F) * G \neq (L[g]F) * (L[g]G) \tag{38}$$
$$R[g](F * G) = F * (R[g]G) \neq (R[g]F) * (R[g]G) \tag{39}$$
$$A[g](F * G) = (A[g]F) * (A[g]G). \tag{40}$$

Physically the associative product has to do with spectra, eigenvalues and eigenstates. This is just that part of physical statements for which the left and right regular translations in G are not physical automorphisms in the H^+-algebraic formulation of quantum mechanics. Concerning the connection with spectra and eigenproblems: in an algebraic formulation, including the H^+-algebraic one, the number λ does belong to the spectrum of the function F if the convolution inverse of $(F - \lambda\delta)$ does not exist, i.e., if there is no function H satisfying

$$H * (F - \lambda\delta) = \delta.$$

This is easily expressible in terms of the expansion (30), namely, $\lambda \in \operatorname{Sp} F$ if and only if there exists $\alpha \in \Omega$ such that

$$\det\left(F(\alpha) - \lambda \mathrm{Id}_{n(\alpha)}\right) = 0.$$

All this is equivalent to the statement that there exists such a **density matrix** $\rho \in L^2(G)$ for which the following **eigenequation** holds

$$A * \rho = \lambda \rho.$$

When $A^+ = A$, this is equivalent to the right-hand-side "eigenequation"

$$\rho * A = \lambda \rho.$$

These physically interpretable statements are based on the associative convolution product, therefore, the left and right regular translations, which do not preserve it, are not physical symmetries of the quantum-mechanical formulation based on the group algebra of G. Some light is shed on such problems when convolutions are interpreted as linear shells of regular translations. And at the same time some natural link is established then with the concepts of wave functions, superpositions, etc.

Let us follow one of finite-dimensional patterns outlined above. Namely, we begin with the linear space H of wave functions on G, in principle the Hilbert space $L^2\left(G, \mathrm{d}g\right)$, although in practical problems the Hilbert space language is often too narrow, e.g., one must admit distributions or non-normalizable wave functions (in the non-compact case). The following sets are relevant for quantum theory: the Banach algebra $B(H)$ of bounded linear operators on H and H^+-algebraic structures in appropriate subspaces of $B(H)$. Of course, in practical problems some non-bounded operators, elements of $L(H)$ are admissible and, when properly and carefully treated, just desirable. The point is that some very important physical quantities, e.g., momenta, angular momenta and so on, are represented by differential operators, of course, non-bounded ones.

However, let us begin with bounded operators describing G-symmetries, namely, described by equations (36) and (37)

$$L[g], \qquad R[g], \qquad A[g] = L[g]R\left[g^{-1}\right] = R\left[g^{-1}\right]L[g].$$

These operators are unitary in $L^2\left(G, \mathrm{d}g\right)$ and, being unitary, they are bounded. The linear shell of the family of operators $\{L[g]\,;\, g \in G\}$ is just the group algebra of G. Namely, if we take two functions $F, H \in L^1(G)$ and the corresponding linear operators

$$L\{F\} = \int F(g)L[g]\mathrm{d}g, \qquad L\{H\} = \int H(g)L[g]\mathrm{d}g \tag{41}$$

then it may be easily shown that

$$L\{H\}L\{F\} = L\{H * F\}. \tag{42}$$

Of course, (41) is a rather symbolic way of writing, i.e., this formula is meant in the following sense

$$L\{F\}f = \int F(g)L[g]f\mathrm{d}g = F * f. \tag{43}$$

Therefore, all operations performed on operators of the type $L\{F\}$ are represented by the corresponding, described above, operations on functions F as elements of the Lie algebra over G. Let us stress that the operators of the form (41) are very special, albeit important elements of $L(H)$. Let us formally substitute for F in (41) the "delta distribution" δ_h concentrated at $h \in G$, i.e., symbolically

$$\delta_h(g) = \delta\left(gh^{-1}\right) = \delta\left(hg^{-1}\right).$$

Then

$$L\{\delta_h\} = L[h], \quad \delta_h * f = L[h]f$$

i.e., the formal convolution with δ_h is the h-translation of f. In particular, $\delta_e = \delta$ is the convolution identity. Something similar may be done with the right translations. One obtains then another family of linear operators acting on wave functions. Namely, let us take again the linear shell of right regular translations, in particular the operators

$$R\{F\} = \int F(g)R[g]\mathrm{d}g, \qquad R\{H\} = \int H(g)R[g]\mathrm{d}g \tag{44}$$

give, with the definition analogues to (43),

$$R\{F\}f = f * F.$$

And again after simple calculations we obtain the following superposition rule

$$R\{F\}R\{H\} = R\{H * F\}. \tag{45}$$

Unlike the representation rule (42), this is anti-representation of the convolution group algebra on G into the algebra of all operators acting on "wave functions" on B, in particular, on $L^1(G)$, $L^2(G)$. To obtain the representation property also for the R-objects, one should define them in the "transposed" way

$$\begin{aligned} R^T\{F\} &:= \int F^T(g)R[g]\mathrm{d}g = \int F\left(g^{-1}\right)R[g]\mathrm{d}g \\ &= \int F(g)R\left[g^{-1}\right]\mathrm{d}g \\ F^T(g) &:= F\left(g^{-1}\right). \end{aligned}$$

Then

$$R^T\{F\}R^T\{H\} = R^T\{F * H\}.$$

The transformation rules (38), (39) and the representation rules (42), (45) tell us that the convolution is not invariant under regular translations, i.e., convolution of translates differs from the translate of convolution. Nevertheless, this is a very peculiar non-invariance and something is invariant, in a sense. Namely, the left-translated convolution is identical with the convolution in which the left factor is left-translated and the right one is kept unchanged. And conversely, the right-translated convolution is the convolution in which the right factor is right-translated (but only this one).

Concerning translational non-invariance of the convolution, let us notice that $F*H$ may be symbolically expressed with the use of the Dirac distribution

$$\begin{aligned}(F*H)(g) &= \int \delta\left(x^{-1}gy^{-1}\right) F(x)H(y)\mathrm{d}x\mathrm{d}y \\ &= \int \delta\left(yg^{-1}x\right) F(x)H(y)\mathrm{d}x\mathrm{d}y.\end{aligned} \tag{46}$$

Let us write down a binary multilinear operation on functions on G in the integral form, maybe symbolic one

$$(F \perp H)(g) = \int K(g;x,y)F(x)H(y)\mathrm{d}x\mathrm{d}y.$$

This operation is invariant under right or left translations, i.e., respectively the following holds

$$(F \perp H)(gh) = \int K(g;x,y)F(xh)H(yh)\mathrm{d}x\mathrm{d}y \tag{47}$$

$$(F \perp H)(hg) = \int K(g;x,y)F(hx)H(hy)\mathrm{d}x\mathrm{d}y \tag{48}$$

when

$$K(g;x,y) = K_r\left(gx^{-1}, gy^{-1}\right), \qquad K(g;x,y) = K_l\left(x^{-1}g, y^{-1}g\right)$$

respectively for (47) and (48). Certainly,

$$K(g;x,y) = \delta\left(x^{-1}gy^{-1}\right)$$

does not satisfy any of conditions (47), (48). Moreover, if G is non-Abelian, those two conditions are rather incompatible.

Let us consider also the total linear shell of translation operators. Let us take a function $F : G \times G \to \mathbb{C}$ and construct the operator

$$T_t\{F\} := \int F\left(g_1, g_2\right) L[g_1]R\left[g_2^{-1}\right]\mathrm{d}g_1\mathrm{d}g_2. \tag{49}$$

One can show that multiplication of such operators results in convolution of functions on the direct product $G \times G$

$$T_t\{F\}T_t\{H\} = T_t\{F*H\} \tag{50}$$

where

$$(F * H)(g_1, g_2) = \int F(h_1, h_2) H\left({h_1}^{-1} g_1, {h_2}^{-1} g_2\right) \mathrm{d}h_1 \mathrm{d}h_2. \tag{51}$$

One could also proceed like in (44), namely, define

$$T\{F\} = \int F(g_1, g_2) L[g_1] R[g_2] \mathrm{d}g_1 \mathrm{d}g_2. \tag{52}$$

Then the convolution in the second argument is "transposed". By that we mean the operation

$$f *^t g := g * f$$

and thus

$$T\{F\} T\{H\} = T\{F \left(*^t\right) H\} \tag{53}$$

where

$$\left(F \left(*^t\right) H\right)(g_1, g_2) = \int F(h_1, h_2) H\left(h_1^{-1} g_1, g_2 h_2^{-1}\right) \mathrm{d}h_1 \mathrm{d}h_2. \tag{54}$$

Evidently, the difference between the expressions (52), (53), (54) and respectively (49), (50), (51) is of a rather cosmetical nature.

Operators of the convolution form (49), (52) are very special linear operations acting on the wave functions $\Psi : G \to \mathbb{C}$. They are "smeared out" in G, essentially non-local, if F is a "usual", "good" function. To obtain very important operators of geometrically distinguished physical quantities or unitary operators of left and right regular translations one must use distributions. Let us mention, e.g., the obvious, trivial examples

$$L[h] = L\{\delta_h\}, \qquad R[h] = R\{\delta_h\}, \qquad L[h]R[k] = F\{\delta_{(h,k)}\} \tag{55}$$

where, let us repeat

$$\delta_h(g) = \delta\left(g h^{-1}\right), \quad \delta_{h,k}(g_1, g_2) = \delta\left(g_1 h^{-1}, g_2 k^{-1}\right) = \delta_h(g_1) \delta_k(g_2). \tag{56}$$

A more detailed analysis, including the description of important physical quantities, may be performed only when one deals with Lie groups and makes use of their differential and analytical structure. Here we stress only the fact that when the algebraic scheme of group algebras is used, then the regular translations fail to be automorphisms of the theory. Physical automorphisms are given by the operators

$$A[g] = L[g] R\left[g^{-1}\right] = R\left[g^{-1}\right] L[g].$$

Therefore, the minimal two-sided ideals $M(\alpha)$ must be further decomposed into direct sums of minimal subspaces invariant under inner automorphisms. This leads to operator algebras invariant under unitary similarity transformations acting in the space $L^2(G)$ of wave functions.

Let us stress that the Hilbert space operations in $L^2(G)$ as the space of wave functions are compatible with the H^+-algebra operations in the sense that

$$L\{F\}^+ = L\left\{F^+\right\}, \qquad R\{F\}^+ = R\left\{F^+\right\}, \qquad T\{F\}^+ = T\left\{F^+\right\}$$

where the involutions on the right-hand sides of these equations are meant in the sense of (14), i.e.,

$$F^+(g) = \overline{F\left(g^{-1}\right)}, \qquad F^+\left(g_1, g_2\right) = \overline{F\left(g_1^{-1}, g_2^{-1}\right)}.$$

In particular, the mentioned operators are Hermitian if and only if the corresponding functions are involution-invariant.

Similarly, the operators of convolution are unitary

$$\begin{aligned} L\left\{F\right\}^+ L\left\{F\right\} &= \mathrm{Id}_{L^2(G)} \\ R\left\{F\right\}^+ R\left\{F\right\} &= \mathrm{Id}_{L^2(G)} \\ T\left\{H\right\}^+ T\left\{H\right\} &= \mathrm{Id}_{L^2(G\times G)} \end{aligned}$$

if and only if

$$F^+ * F = \delta_G, \qquad H^+ * H = \delta_{G\times G}.$$

Definitely, all translation operators in (36), (37), (55) and (56) are unitary in $L^2(G)$. This follows from the invariance of the Haar measure. They preserve the scalar product of wave functions on G, and automatically they preserve the scalar product in the H^+-algebra $L^2(G)$. It is interesting to stress again how they are represented in the algebra $L(L^2(G))$ of all operators in $L^2(G)$, and first of all, in the algebra of bounded operators $B(L^2(G))$. Of course, any invertible operator F in $L^2(G)$ acts in $L(L^2(G))$, $B(L^2(G))$ through the similarity transformations

$$A \to FAF^{-1}.$$

This concerns in particular unitary operators, like regular translations. It is interesting to see how they act in the linear shell of translations, i.e., in the algebras of convolution-type operators. One can easily show that

$$\begin{aligned} L[g]L\{F\}L[g]^{-1} &= L\{A[g]F\} \\ R[g]L\{F\}R[g]^{-1} &= L\{F\} \\ L[g]R\{F\}L[g]^{-1} &= R\{F\} \\ R[g]R\{F\}R[g]^{-1} &= R\{A[g^{-1}]F\} \end{aligned}$$

or, better to express

$$R[g^{-1}]R\{F\}R[g^{-1}]^{-1} = R\{A[g]F\}.$$

Therefore,

$$L[g_1]R[g_2]T\{F\}\left(L[g_1]R[g_2]\right)^{-1} = T\left\{F \circ \left(A[g_1] \times A[{g_2}^{-1}]\right)\right\}$$

or equivalently

$$L[g_1]R[g_2]T_t\{F\}\left(L[g_1]R[g_2]\right)^{-1} = T_t\left\{F \circ \left(A[g_1] \times A[g_2]\right)\right\}.$$

The Cartesian product of mappings $A : X \to U$, $B : Y \to V$, denoted by $A \times B : X \times Y \to U \times V$, is meant in the usual sense, i.e.,

$$\left(A \times B\right)(x, y) = \left(A(x), B(y)\right).$$

The message of these formulas is that regular translations acting on wave functions on G are represented in the group algebras over G or $G \times G$ by inner automorphisms. Because of this, classification of states and physical quantities in group algebras over G and $G \times G$ is based on the analysis of minimal subspaces invariant under inner automorphisms. One point must be stressed: the operators of the form $T\{F\}$ are not the most general operators acting in $L^2(G)$. The position operators, more precisely, the operators of pointwise multiplication of wave functions by functions on G do not belong to this class. Nevertheless, if G is non-Abelian, then some (not all!) position-like quantities are implicitly present in $T\{f\}$-type operators.

The main peculiarity of convolution-type operators and their singular special cases like the regular translations and automorphisms specified in (36), (37), (41), (44), (49) and (51) is that they preserve separately all subspaces/ideals $M(\alpha)$.

It is not the case with the position-like operators because they mix various subspaces (ideals) with each others. The points is interesting in itself, because it has to do with the relationship between two algebraic structures in function spaces over G. One of them is just the structure of the associative algebra under convolution in $L^1(G)$. It is non-commutative unless G is Abelian and has no literally meant unity unless G is discrete. The other one is the structure of commutative associative algebra under the pointwise multiplication of functions. Clearly, this is the unital algebra with the unity given by the constant function taking the value one on the whole G. In general there are some subtle points concerning the underlying sets of algebras and their mutual relationships. The convolution algebra is defined in $L^1(G)$, whereas the pointwise multiplicative algebra is defined in the set of all globally defined functions on G. No comments are necessary, and the both underlying sets coincide, only when G is finite.

The pointwise products of matrix elements of representations $D(\alpha)$, $D(\varrho)$ are expanded with respect to the orthogonal systems of functions $D(\varkappa)_{kl}$ according to the rule

$$D(\alpha)_{ab}D(\varrho)_{rs} = \sum_{\varkappa,k,l} \left(\alpha\varrho ar|\varkappa k\right)\left(\alpha\varrho bs|\varkappa l\right) D(\varkappa)_{kl}$$

where $(\alpha\varrho ar|\varkappa k)$ and so on are **Clebsch-Gordan coefficients** for the group G [17, 31, 33]. Summation over $\varkappa$ is extended over an appropriate range depending on (α, ϱ), and for the fixed κ, the range of k, l is given by the set of naturals

$1, \ldots, n(\varkappa)$. To be more precise, k, l run over some $n(\varkappa)$-element set. In the theory of angular momentum, when $G = \mathrm{SU}(2)$, it is convenient to use the convention: $\varkappa = 2j + 1$, where j runs over the set of non-negative integers and half-integers, and k, l run over the range $-j, -j+1, ..., j-1, j$, jumping by one. The expression (23) implies that

$$\varepsilon(\alpha)_{ab}\varepsilon(\varrho)_{rs} = \sum_{\varkappa,k,l} \frac{n(\alpha)n(\varrho)}{n(\varkappa)} \left(\alpha\varrho ar|\varkappa k\right) \left(\alpha\varrho bs|\varkappa l\right) \varepsilon(\varkappa)_{kl}.$$

The coefficients at $\varepsilon(\varkappa)_{kl}$ are structure constants of the commutative algebra of pointwise multiplication with respect to the canonical basis/complete system. They are bilinear in Clebsch-Gordon coefficients. The latter ones are meant in the usual sense of the procedure

i) Take two irreducible representations of G, $D(\alpha)$, $D(\varrho)$ acting respectively in $\mathbb{C}^{n(\alpha)}$, $\mathbb{C}^{n(\varrho)}$

$$\begin{aligned} \left(D(\alpha)(g)u(\alpha)\right)_a &= \sum_b D(\alpha)(g)_{ab}u(\alpha)_b \\ \left(D(\alpha)(g)u(\varrho)\right)_r &= \sum_s D(\alpha)(g)_{rs}u(\varrho)_s. \end{aligned}$$

ii) Take the tensor product of those representations

$$D(\alpha) \otimes D(\varrho): \quad G \times G \to L(\mathbb{C}^{n(\alpha)} \otimes \mathbb{C}^{n(\varrho)}) \simeq L(\mathbb{C}^{n(\alpha)n(\varrho)})$$

given by

$$\left[\left[D(\alpha)(g_1) \otimes D(\varrho)(g_2)\right] t(\alpha,\varrho)\right]_{ar} = \sum_{bs} D(\alpha)(g_1)_{ab}D(\varrho)(g_2)_{rs}t(\alpha,\varrho)_{bs}.$$

This representation is irreducible, if, as assumed, $D(\alpha)$, $D(\varrho)$ are irreducible.

iii) Take the direct product of $D(\alpha)$, $D(\varrho)$, i.e., restrict $D(\alpha) \otimes D(\varrho)$ to the diagonal $\{(g,g)\,;\, g \in G\} \subset G \times G$. One obtains some representation $D(\alpha) \times D(\varrho)$ of G in $\mathbb{C}^{n(\alpha)} \otimes \mathbb{C}^{n(\beta)} \simeq \mathbb{C}^{n(\alpha)n(\beta)}$.

In general, this representation is reducible and equivalent to the direct sum of some irreducible representations,

$$\bigoplus_{\varkappa \in \Omega(\alpha,\varrho)} D(\varkappa)$$

the direct sum performed over some subset of labels, $\Omega\left(\alpha,\varrho\right) \subset \Omega$. Evidently, this representation acts in the Cartesian product

$$\mathop{\times}_{\varkappa \in \Omega(\alpha,\varrho)} \mathbb{C}^{n(\varkappa)}. \tag{57}$$

Let U denote an equivalence isomorphism of $\mathbb{C}^{n(\alpha)} \otimes \mathbb{C}^{n(\varrho)}$ onto the representation space (57). Then, by definition, the Clebsch-Gordon coefficients are given by

$$U\left(u\left(\alpha\right)_a \otimes v\left(\varrho\right)_r\right) = \sum_{\varkappa,k} \left(\alpha\varrho ar|\varkappa k\right) w\left(\varkappa\right)_k \tag{58}$$

where $u\left(\alpha\right)_a,\ v\left(\varrho\right)_r,\ w\left(\varkappa\right)_k$ denote basis vectors of the representation spaces for $D\left(\alpha\right)$, $D\left(\varrho\right)$, $D\left(\varkappa\right)$. When the natural bases in $\mathbb{C}^{n(\alpha)}$, $\mathbb{C}^{n(\varrho)}$, $\mathbb{C}^{n(\varkappa)}$ are used, then $u\left(\alpha\right)_a$, $v\left(\varrho\right)_r$, $w\left(\varkappa\right)_k$ may be reinterpreted as components of the representation vectors $u\left(\alpha\right)$, $v\left(\varrho\right)$, $w\left(\varkappa\right)$. And then we simply write instead of (58) the following formulas

$$u(\alpha)_a v(\varrho)_r = \sum_{\varkappa,k} \left(\alpha\varrho ar|\varkappa k\right) w(\varkappa)_k.$$

These are just the implicit definitions of the Clebsch-Gordon coefficients.

There are two special cases when all minimal ideals $M(\alpha)$ are finite-dimensional, i.e., all irreducible unitary representations $D(\alpha)$ are finite-dimensional. These are when the topological group G is compact or Abelian. Of course, those are non-disjoint situations. In the Abelian case all $M(\alpha)$ are one-dimensional and one is dealing with Pontryagin duality [16, 18]. The set Ω of irreducible unitary representations has the natural structure of a locally compact Abelian group too, the so-called character group, denoted traditionally by $\widehat{G}$. The group operation in $\widehat{G}$ is meant as the pointwise multiplication of functions on G. In other words, the elements of $\widehat{G}$ are continuous homomorphisms of G into the group

$$U(1) = \{z \in \mathbb{C}\ ;\ |z| = 1\}$$

the multiplicative group of complex numbers of modulus one. If G is compact and Abelian, then $\widehat{G}$ is discrete, and the Peter-Weyl series expansion (30), (31) becomes a generalized Fourier series. If G is non-compact, one obtains generalized Fourier transforms and direct integrals of family of one-dimensional spaces.

According to the well-known Pontryagin theorem, the dual of $\widehat{G}$, e.g., the second dual $\widehat{\widehat{G}}$ of G, is canonically isomorphic with G itself [16, 18]. This resembles the relationship between duals of finite-dimensional linear spaces, $\left(V^*\right)^* \simeq V$.

The Fourier transform $\widehat{\Psi} : \widehat{G} \to \mathbb{C}$ of $\Psi : G \to \mathbb{C}$ is defined as follows

$$\widehat{\Psi}(\chi) = \int \overline{\langle\chi|g\rangle}\Psi(g)\mathrm{d}g = \int \langle\chi|g\rangle^{-1}\Psi(g)\mathrm{d}g \tag{59}$$

where $\mathrm{d}g$ again denotes the integration element of the Haar measure on G and $\langle\chi|g\rangle$ is the evaluation of $\chi \in \widehat{G}$ on $g \in G$. Equivalently, in virtue of Poincare duality, this is the evaluation of $g \in G \simeq \widehat{\widehat{G}}$ on $\chi \in \widehat{G}$. The inverse formula of (59) reads

$$\Psi(g) = \int \langle \chi | g \rangle \, \widehat{\Psi}(\chi) \mathrm{d}\chi \tag{60}$$

where $\mathrm{d}\chi$ denotes the element of Haar integration on $\widehat{G}$. The formulas (59), (60) fix the synchronization between normalizations of measures $\mathrm{d}g$, $\mathrm{d}\chi$. In principle, these formulas are meant in the sense of L^1-spaces over G, $\widehat{G}$, nevertheless, some more or less symbolic expressions are also admitted for other functions, as shorthands for longer systems of formulas. First of all, this concerns δ-distributions, just like in general situation of locally compact G. Of course, the correct definition of distributions and operations on them must be based on differential concepts, nevertheless, the Dirac distribution itself (but not its derivatives) may be introduced in principle on the basis of purely topological concepts, just like in the general case. Let us notice that

$$\Psi(g) = \int \mathrm{d}\chi \int \mathrm{d}h \Psi(h) \left\langle \chi | h g^{-1} \right\rangle .$$

The order of integration here is essential! But, of course, one cannot resist the temptation to change "illegally" this order and write symbolically

$$\Psi(g) = \int \mathrm{d}h \, \delta\left(h g^{-1}\right) \Psi(h), \qquad \delta(x) = \int \mathrm{d}\chi \, \langle \chi | x \rangle . \tag{61}$$

If G is discrete, then $\widehat{G}$ is compact (and conversely) and the second integral is well defined, namely

$$\delta(x) = \delta_{xe} \begin{cases} 1, & \text{if } x = e \\ 0, & \text{if } x \neq e \end{cases}$$

where e is the natural element (identity) of G. Then the first integral is literally true as a summation with the use of Kronecker delta. But when obeying some rules, we may safely use the formulas (61) also in the general case, when they are formally meaningless. So, we shall always write

$$\begin{aligned}
\delta(g) &= \int \langle \chi | g \rangle \, \mathrm{d}\chi = \delta\left(g^{-1}\right) \\
\delta(\chi) &= \int \overline{\langle \chi | g \rangle} \mathrm{d}g = \int \langle \chi | g \rangle \, \mathrm{d}g = \delta\left(\chi^{-1}\right) \\
\int \delta(g) f(g) \mathrm{d}g &= f\left(e(G)\right) \\
\int \delta(\chi) k(\chi) \mathrm{d}\chi &= k\left(e(\widehat{G})\right)
\end{aligned}$$

and $e(G)$, $e(\widehat{G})$ denote the units in G, $\widehat{G}$, respectively.

Convolution is defined by the usual formula (13), but the peculiarity of Abelian groups G is that convolution is a commutative operation

$$F * G = G * F.$$

Of course, Fourier transforms of convolution are pointwise products of Fourier transforms, and conversely

$$(F * G)\widehat{\ } = \widehat{F}\widehat{G}.$$

This is the obvious special case of the equation (32) - $\widehat{F}(\chi)$, $\widehat{G}(\chi)$ are 1×1 matrices $F(\alpha)$, $G(\alpha)$.

It is clear that just like in the general case, δ-distribution is the convolution identity

$$F * \delta = \delta * F = F.$$

And now, we may be a bit more precise. Namely, let $U \subset \widehat{G}$ be some compact measurable subset of $\widehat{G}$, and let $L\{U\}$ denote the linear subspace of functions (60) such that the Fourier transform $\widehat{\Psi}$ vanishes outside U and is L^1-class. Take the function $\delta\{U\}$ given by

$$\delta\{U\}(g) := \int\limits_U \langle \chi | g \rangle \, \mathrm{d}\chi.$$

It is clear that $\delta\{U\}$ is a convolution identity of the subspace $L\{U\}$. And now take an increasing sequence of subsets $V_i \subset \widehat{G}$ such that

$$V_i \supset V_j \quad \text{for} \quad i > j \qquad \bigcup_i V_i = \widehat{G}.$$

It is clear that for any function $F \subset L^1(G)$ we have

$$\lim_{i\to\infty} \delta\{V_i\} * F = F \tag{62}$$

although the limit of the sequence $\delta\{V_i\}$ does not exist in the usual sense of function sequences. However, it does exist in an appropriately defined functional sense. So, by abuse of language, we simply write

$$\delta = \lim_{i\to\infty} \delta\{V_i\}, \qquad \delta * F = F$$

as a shorthand for the rigorous relation (62).

Calculating formally the convolution of $\chi_1, \chi_2 \in \widehat{G}$, we obtain

$$\langle \chi_1 * \chi_2 | g \rangle = \delta\left(\chi_1 {\chi_2}^{-1}\right) \langle \chi_2 | g \rangle = \delta\left(\chi_1 {\chi_2}^{-1}\right) \langle \chi_1 | g \rangle$$

i.e., briefly

$$\chi_1 * \chi_2 = \delta\left(\chi_1 {\chi_2}^{-1}\right) \chi_2 = \delta\left(\chi_1 {\chi_2}^{-1}\right) \chi_1. \tag{63}$$

If G is compact, i.e., $\widehat{G}$ is discrete, this is the usual condition for irreducible idempotents (21), (22). Similarity, we have the orthogonality/normalization condition

$$(\chi_1, \chi_2) = \delta\left(\chi_1 {\chi_2}^{-1}\right) = \begin{cases} 1, & \text{if } \chi_1 = \chi_2 \\ 0, & \text{if } \chi_1 \neq \chi_2. \end{cases} \tag{64}$$

If G is not compact, i.e., $\widehat{G}$ is not discrete, then both normalization and idempotence rules (63), (64) are meant symbolically, just like the corresponding rules for Dirac distributions in $\mathbb{R}^n$

$$\delta_a * \delta_b = \delta(a-b)\delta_a = \delta(a-b)\delta_b \tag{65}$$

$$(\delta_a, \delta_b) = \delta(a-b). \tag{66}$$

Surely, $\delta_a(x) := \delta\left(x-a\right)$. Incidentally, (65), (66) is just the special case of (63), (64) when $G = \mathbb{R}^n$ and the addition of vectors is meant as a group operation.

The peculiarity of locally compact Abelian groups is that they offer some analogies to geometry of the classical phase spaces and some natural generalization of the Weyl-Wigner-Moyal formalism. Certain counterparts do exist also in non-Abelian groups, especially compact ones. However, they are radically different from the structures based on Abelian groups. And in the non-compact case the analogy rather diffuses.

Finally, let us remind that just like in the classical Fourier analysis, the Pontryagin Fourier transform is an isometry of $L^2(G)$ onto $L^2\left(\widehat{G}\right)$

$$\int \overline{A(g)} B(g) \mathrm{d}g = \int \overline{\widehat{A}(\chi)} B(\chi) \mathrm{d}\chi$$

in particular, the Plancherel theorem holds

$$\int |A(g)|^2 \,\mathrm{d}g = \int \left|\widehat{A}(\chi)\right|^2 \mathrm{d}\chi.$$

Compare with the formula (33) for compact topological groups and the corresponding expression for the norm $\|F\|$

$$\|F\|^2 = \sum_{\alpha\in\Omega} \mathrm{Tr}\left(F\left(\alpha\right)^+ F\left(\alpha\right)\right) n\left(\alpha\right).$$

2.4. Some Remarks Concerning Physical Interpretation

Let us finish this section with some comments concerning physical interpretation. It is impossible to answer definitely the question

What is the most fundamental mathematical structure underlying quantum mechanics?

There are approaches based on quantum logic, the usual Hilbert space formulations, operator algebras, etc. According to certain views [19], all non-artificial and viable models, both in quantum and classical mechanics, assume some groups as fundamental underlying structures. The framework of group algebra may appear in two different, nevertheless, somehow interrelated, ways.

1. The first scheme is one in the spirit of algebraic approaches. Namely, when some topological group G is assumed, one can simply state: "quantum mechanics based on G is the group algebra of G". In any case it works on compact groups and locally compact Abelian ones. Group algebras are particular H^+-algebras. The important operations of quantum mechanics are based on structures intrinsically built into them. Hermitian elements represent physical quantities, Hermitian and positive ones are quantum states in the sense of density operators, idempotent ones among them represent pure states. This is common to all H^+-algebras. The peculiarity of group algebras is that besides the convolution operation there exist also another composition rule, namely, the pointwise product of functions, associative one as well, and commutative (convolution is non-commutative if G is non-Abelian). The relationship between them is given by the Clebsch-Gordan coefficients. This structure is also physically interpretable, namely, it describes the properties of composed system, e.g., composition of angular momenta. Pointwise multiplication of functions representing quantum states describes the direct product of density operators of subsystems [30, 31].

   However, one important structure of quantum mechanics is missing here, namely, the superposition principle. The point is that wave functions do not fit this framework directly. Nevertheless, in a sense they are implicitly present. Namely, group algebra, as any associative algebra, acts on itself through the left or right regular translations

$$x \to a * x, \qquad x \to x * a. \tag{67}$$

   To be more precise, it is so in any H^+-algebra. In this way, the elements of group algebra become operators. By convention, we can choose the left regular translations. Group algebra becomes represented by algebra of linear operators. However, this representation is badly reducible. Namely, it is not only so that any ideal, in particular, any minimal ideal $M(\alpha)$, is invariant under left (and right too) regular translations (67). But within any minimal two-side ideal $M(\alpha)$ spanned by all $\varepsilon(\alpha)_{mn}$ functions, separately any minimal left ideal $M(\alpha, n)$ is also closed under all translations, and this representation is irreducible. The minimal left ideal $M(\alpha, n)$ is spanned by all functions $\varepsilon(\alpha)_{mn}$, where n is fixed. Symmetrically, any minimal right ideal $M(n, \alpha)$, spanned by all functions $\varepsilon(\alpha)_{nm}$ with a fixed n, is a representation space of some irreducible representation of H^+ algebra. Roughly speaking, $M(\alpha, n)$, $M(n, \alpha)$ are respectively columns and rows of the matrices $[\varepsilon(\alpha)_{ab}]$. And the elements of $M(\alpha, n)$ are "wave functions", "state vectors" of the "α-th type". Subspaces

$$M(\alpha, n_1), \quad M(\alpha, n_2), \qquad n_1 \neq n_2$$

are merely different representatives, just equivalent descriptions of physically the same situation. By convention we may simply fix $n = 1$. And then the linear shell of all subspaces $M(\alpha, 1)$ is the space of "state vectors".

2. Another scheme is one in which G is meant as the classical configuration space (take, e.g., $\mathrm{SO}(3, \mathbb{R})$ or $\mathrm{SU}(2)$ as the configuration space of rigid body). Then all functions on G are interpreted as wave functions, and $\varepsilon(\alpha)_{mk}$, $\varepsilon(\alpha)_{ml}$ for $k \neq l$ are different wave functions, different physical situations. One can easily construct Hamiltonians which predict "quantum transitions" between those subspaces. But also within such Schrödinger wave-mechanical framework, group-algebraic structures are physically relevant. Namely, the most important, geometrically distinguished operators are unitaries (36), (37) representing transformation groups motivated by G. They describe some physically significant unitary representations of G. And then the linear shells (41), (44), (49) and so on of these representations appear in a natural way. Roughly speaking, those linear shells are physically interpretable representations of the abstract group algebra over G. And they in a sense "parametrize" the algebra of operators acting on wave functions. If the function F in (41), (44), (49) is of the $L^1(G)$- or $L^1\,(G \times G)$-class, then the resulting operators are bounded. But it is just certain unbounded operators that are physically interesting. They describe important physical quantities. We obtain them from the group-algebraic scheme, formally admitting in (41), (44), (49) distributions instead of functions F. In differential theory, when G is a Lie group, some derivatives of Dirac delta are then used. But even some important bounded operators, e.g., the identity operator and G-translations, are expressed in terms of distributions, namely, Dirac deltas, as formally included into group algebra. Differential concepts are not used then.
   Hence, physically relevant and operationally interpretable quantities are to be sought first of all among elements of the group algebras (41), (44), (49), formally extended by admitting distributions.

3. Quantum Mechanics on Lie Groups and Methods of Group Algebras

The previous section was devoted to the H^+-algebras as a mathematical tool for describing quantum mechanics. It was shown there that the H^+-algebras form in a sense a beautiful scheme useful for the mathematical expression of quantum rules and for the very formulation of quantum ideas. This is due to their mathematical structure which in a sense unifies both the linear space and the non-commutative associative algebra with involution. The idea of using the H^{+}-algebras in such a context was formulated many years ago by W. M. Tulczyjew. It enables us both to investigate the basic ideas of quantum mechanics and, basing on the Clebsch-Gordan series, to formulate the rules of composing two quantum systems into a single one.

It turns out that the convolution group algebras, first of all ones over the compact and Abelian groups, are the most interesting examples from the point of view of quantum applications. In this section we will investigate that very particular choice of the Lie group and of the metric structure on it. We assume that the group underlying our investigation is either a compact semisimple Lie group or a group isomorphic with $\mathbb{R}^n$, or something between those two extreme special cases, e.g., the Cartesian product of the two mentioned situations. In certain problems we may decide to admit semisimple but not necessarily compact Lie group of transformations acting on some Abelian Lie group of translations. Nevertheless, to be as concrete as possible, we usually concentrate on the compact/semisimple Lie group of transformations acting on a linear space or just on a linear space interpreted as an $\mathbb{R}^n$-type additive Abelian Lie group.

When dealing with semisimple Lie groups of transformations acting in a linear space, we mainly use the Cartan-Killing metric tensor. In certain problems the metric tensors invariant under the group of all left or right regular translations are admitted. Obviously, they lead to the same volume-metric on G. Analytically, it is given by the square root of the determinant of the matrix of the metric tensor. Basing on the theory of representations of compact Lie groups, we construct the canonical complete system of states on the group. Discussed is also another complete system of states suited to the group of inner automorphisms acting on the group. It turns out that this alternative set of states is a very convenient tool of the analysis leading to the quasi-classical analysis.

3.1. Compact Lie Groups

Let us now discuss the very important situation when G is a compact Lie group. The special stress is laid on semisimple Lie groups or their central extension. We are particularly interested in problems concerning angular momentum, i.e., the group $\mathrm{SU}(2)$ or its quotient $\mathrm{SO}\,(3,\mathbb{R}) = \mathrm{SU}\,(2)\,/\mathbb{Z}_2$. Nevertheless, it is convenient to begin with remarks concerning the general situation.

The Lie algebra of G will be denoted by $\mathfrak{g}$. We assume G to be a linear group, i.e., a group of finite matrices, some subgroup of $\mathrm{GL}\,(N,\mathbb{R})$ or $\mathrm{GL}\,(N,\mathbb{C})$. This simplifies notation. Of course, any compact Lie group is linear. Lie algebras are meant in the matrix commutator sense. Let $(\ldots, e_a, \ldots)$ be some basis in $\mathfrak{g}$, the structure constants are meant in the following convention

$$[e_a, e_b] = e_a e_b - e_b e_a = e_k C^k{}_{ab}. \tag{68}$$

The **Killing metric tensor** on $\mathfrak{g}$, i.e., the Ad-invariant scalar product γ, is meant in the following convention

$$\gamma\,(u, v) = \mathrm{Tr}\,(\mathrm{ad}_u \mathrm{ad}_v) \tag{69}$$

where $\mathrm{ad}_u \in L\,(\mathfrak{g}) \simeq \mathfrak{g} \otimes \mathfrak{g}^*$ is given by the usual formula

$$\mathrm{ad}_u \cdot x = [u, x]\,.$$

Analytically, in terms of the basis e

$$\gamma_{ab} = C^k{}_{la} C^l{}_{kb}, \qquad \gamma = \gamma_{ab} e^a \otimes e^b \tag{70}$$

where $e^a \in \mathfrak{g}^*$ are elements of the dual basis, $\langle e^a, e_b \rangle = \delta^a{}_b$. If G is compact and semisimple, then γ is negatively definite and in an appropriate basis e, γ_{ab} is a negative multiple of δ_{ab}. Usually the basis is chosen in some convenient way motivated by various reasons, then it is customary to change the normalization of γ_{ab} replacing it just by $g_{ab} = \delta_{ab}$. The contravariant inverse of γ, $\gamma^{-1} \in \mathfrak{g} \otimes \mathfrak{g}$, is analytically given by

$$\gamma^{-1} = \gamma^{ab} e_a \otimes e_b, \qquad \gamma^{ac}\gamma_{cb} = \delta^a{}_b. \tag{71}$$

In the trivial central extension $G \times U\,(1)$ of G, the Killing tensor is degenerate and $\mathfrak{u}\,(1)$ is the degenerate direction of $\mathfrak{g} \times \mathfrak{u}\,(1)$. Then it is customary to use the metric tensor obtained as a direct combination of the Killing metric on $\mathfrak{g}$ and the invariant metric on $\mathfrak{u}\,(1)$. The latter is unique up to normalization. Sometimes one proceeds similarly when dealing with direct or semidirect products of semisimple groups and Abelian ones of arbitrary dimension, however, if that dimension is higher than one, the Abelian component of metric has a non-canonical arbitrariness.

Canonical coordinates of the first kind k^a are defined by the formula

$$g\left(k^1, \ldots, k^n\right) = \mathrm{e}\,(k^a e_a)\,, \qquad \dim G = n \tag{72}$$

(the summation convention is used on the right-hand side). This choice is often convenient, but also other ones are useful, e.g., canonical coordinates of the second kind

$$g\left[\xi^1, \ldots, \xi^n\right] = \mathrm{e}\left(\xi^1 e_1\right) \ldots \mathrm{e}\,(\xi^n e_n)$$

or something between, like Euler angles on $\mathrm{SO}\,(3, \mathbb{R})$ or $\mathrm{SU}\,(2)$. Often some generalized coordinates, "curvilinear" with respect to k^a or ξ^a, are better suited to particular problems. In any case, the choice of coordinates is a matter of convenience.

The differential structure of G offers some powerful tools of analysis. First of all, one uses differential operators generating transformations (36), (37). Generators of left and right regular translations are defined in the convention

$$\begin{aligned}\left(\mathcal{L}_a\psi\right)\left(g\left(\overline{k}\right)\right) &= \frac{\partial}{\partial x^a}\left(\psi\left(g\left(\overline{x}\right)g\left(\overline{k}\right)\right)\right)\Big|_{\overline{x}=0}\\ \left(\mathcal{R}_a\psi\right)\left(g\left(\overline{k}\right)\right) &= \frac{\partial}{\partial x^a}\left(\psi\left(g\left(\overline{k}\right)g\left(\overline{x}\right)\right)\right)\Big|_{\overline{x}=0}\end{aligned}$$

i.e., roughly, we have the following expansions for small values of the group parameters $\overline{\varepsilon}$

$$\begin{aligned}\psi\left(g\left(\overline{\varepsilon}\right)g\right) &\approx \psi\left(g\right)+\varepsilon^a\left(\mathcal{L}_a\psi\right)\left(g\right)\\ \psi\left(gg\left(\overline{\varepsilon}\right)\right) &\approx \psi\left(g\right)+\varepsilon^a\left(\mathcal{R}_a\psi\right)\left(g\right)\end{aligned}$$

valid to terms quadratic and higher order in $\overline{\varepsilon}$.

$\mathcal{L}_a$, $\mathcal{R}_a$ are respectively, basic right- and left-invariant vector fields on G. We represent them as follows

$$\mathcal{L}_a = \mathcal{L}^i{}_a\left(\overline{k}\right)\frac{\partial}{\partial k^i}, \qquad \mathcal{R}_a = \mathcal{R}^i{}_a\left(\overline{k}\right)\frac{\partial}{\partial k^i}\cdot \tag{73}$$

With this convention we have the following commutation rules

$$[\mathcal{L}_a,\mathcal{L}_b] = -C^k{}_{ab}\mathcal{L}_k, \qquad [\mathcal{R}_a,\mathcal{R}_b] = C^k{}_{ab}\mathcal{R}_k, \qquad [\mathcal{L}_a,\mathcal{R}_b]=0. \tag{74}$$

Similarly one defines differential operators $\mathcal{D}_a$ generating inner automorphisms

$$\left(\mathcal{A}_a\psi\right)\left(g\left(\overline{k}\right)\right) = \frac{\partial}{\partial x^a}\left(\psi\left(g\left(\overline{x}\right)g\left(\overline{k}\right)g\left(-\overline{x}\right)\right)\right)\Big|_{\overline{x}=0}$$

i.e., roughly

$$\psi\left(g(\overline{\varepsilon})gg(-\overline{\varepsilon})\right) \approx \psi(g)+\varepsilon^a\left(\mathcal{A}_a\psi\right)(g)$$

up to higher-order corrections in $\overline{\varepsilon}$. Here

$$\mathcal{A}_a = \mathcal{L}_a - \mathcal{R}_a$$

and we use the notation

$$\mathcal{A}_a = \mathcal{A}^i{}_a(\overline{k})\frac{\partial}{\partial k^i}\cdot$$

It is also clear that

$$[\mathcal{A}_a,\mathcal{A}_b] = -C^k{}_{ab}\mathcal{A}_k \tag{75}$$

and

$$[\mathcal{A}_a,\mathcal{L}_b] = -C^k{}_{ab}\mathcal{L}_k, \qquad [\mathcal{A}_a,\mathcal{R}_b] = C^k{}_{ab}\mathcal{R}_k. \tag{76}$$

The $\pm$ signs on the right-hand sides of (74), (75), (76) are essential. As mentioned, the translation operators (36), (37), (55) are unitary in $L^2(G)$ due to the very definition and properties of the Haar measure on G. Therefore, their generators $\mathcal{L}_a$,

$\mathcal{R}_a$, $\mathcal{A}_a$ are skew-symmetric in the corresponding dense subdomain of $L^2(G)$

$$\begin{aligned}\langle \mathcal{L}_a\psi|\varphi\rangle &= -\langle\psi|\mathcal{L}_a\varphi\rangle\\ \langle \mathcal{R}_a\psi|\varphi\rangle &= -\langle\psi|\mathcal{R}_a\varphi\rangle\\ \langle \mathcal{A}_a\psi|\varphi\rangle &= -\langle\psi|\mathcal{A}_a\varphi\rangle\,.\end{aligned}$$

By their very definition as differential operators, $\mathcal{L}_a$, $\mathcal{R}_a$, $\mathcal{A}_a$ are not globally defined on $L^2(G)$.

Let us quote the following formulas

$$\begin{aligned}L\left[g(\overline{k})^{-1}\right] = L\left[g(-\overline{k})\right] &= \mathrm{e}\left(k^a\mathcal{L}_a\right)\\ R\left[g(\overline{k})^{-1}\right] = R\left[g(-\overline{k})\right] &= \mathrm{e}\left(k^a\mathcal{R}_a\right)\\ A\left[g(\overline{k})^{-1}\right] = A\left[g(-\overline{k})\right] &= \mathrm{e}\left(k^a\mathcal{A}_a\right)\end{aligned}$$

which hold when their right-hand sides are well defined. Thus, in an appropriate dense subdomain, with the convergence meant in the sense of $L^2(G)$-norm. Certainly, the left-hand sides are well defined in action on the total linear space of all possible functions on G (with arbitrary target spaces, not necessarily $\mathbb{C}$).

The imaginary-unit multiples of $\mathcal{L}_a$, $\mathcal{R}_a$, $\mathcal{A}_a$ are formally Hermitian (symmetric). Because of the obvious physical reasons we introduce the formally Hermitian operators of $L[G]$-, $R[G]$- and $A[G]$-momenta, just the quantum versions of the corresponding classical momentum mappings

$$\mathbf{\Sigma}_a = \frac{\hbar}{\mathrm{i}}\mathcal{L}_{a,} \qquad \widehat{\mathbf{\Sigma}}_a = \frac{\hbar}{\mathrm{i}}\mathcal{R}_{a,} \qquad \mathbf{\Delta}_a = \frac{\hbar}{\mathrm{i}}\mathcal{A}_a = \mathbf{\Sigma}_a - \widehat{\mathbf{\Sigma}}_a. \tag{77}$$

As operators acting on $L^2(G)$-wave functions, they satisfy the obvious quantum Poisson brackets

$$\frac{1}{\mathrm{i}\hbar}\left[\mathbf{\Sigma}_a,\mathbf{\Sigma}_b\right] = \left\{\mathbf{\Sigma}_a,\mathbf{\Sigma}_b\right\}_Q = C^k{}_{ab}\mathbf{\Sigma}_k \tag{78}$$

$$\frac{1}{\mathrm{i}\hbar}\left[\widehat{\mathbf{\Sigma}}_a,\widehat{\mathbf{\Sigma}}_b\right] = \left\{\widehat{\mathbf{\Sigma}}_a,\widehat{\mathbf{\Sigma}}_b\right\}_Q = -C^k{}_{ab}\widehat{\mathbf{\Sigma}}_k \tag{79}$$

$$\frac{1}{\mathrm{i}\hbar}\left[\mathbf{\Sigma}_a,\widehat{\mathbf{\Sigma}}_b\right] = 0. \tag{80}$$

The corresponding classical counterparts are given by the phase-space functions

$$\Sigma_a = p_i\mathcal{L}^i{}_a, \qquad \widehat{\Sigma}_a = p_i\mathcal{R}^i{}_a, \qquad \Delta_a = p_i\Delta^i{}_a = \Sigma_a - \widehat{\Sigma}_a. \tag{81}$$

Their classical Poisson brackets are structurally identical with (78)–(80), i.e.,

$$\left\{\Sigma_a,\Sigma_b\right\} = C^k{}_{ab}\Sigma_k, \qquad \left\{\widehat{\Sigma}_a,\widehat{\Sigma}_b\right\} = -C^k{}_{ab}\widehat{\Sigma}_k, \qquad \left\{\Sigma_a,\widehat{\Sigma}_b\right\} = 0.$$

As mentioned, the regular translations and automorphisms (36), (37) and all operators of convolution (41), (44), (49) preserve separately all subspaces/minimal ideals $M(\alpha)$. This is also true for the operators $\mathcal{L}_a$, $\mathcal{R}_a$, Δ_a as generators of those group actions. Of course, their multiples Σ_a, $\widehat{\Sigma}_a$, Δ_a also preserve all ideals $M(\alpha)$. The basic right- and left-invariant differential forms on G will be denoted respectively by $\mathcal{L}^a$, $\mathcal{R}^a$. By definition they are assumed to be dual to $\mathcal{L}_a$, $\mathcal{R}_a$

$$\langle \mathcal{L}^a, \mathcal{L}_b \rangle = \langle \mathcal{R}^a, \mathcal{R}_b \rangle = \delta^a{}_b.$$

We shall use the standard analytical representation dual to (73)

$$\mathcal{L}^a = \mathcal{L}^a{}_i(\overline{k})\mathrm{d}k^i, \qquad \mathcal{R}^a = \mathcal{R}^a{}_i(\overline{k})\mathrm{d}k^i$$

where

$$\mathcal{L}^i{}_a \mathcal{L}^a{}_j = \mathcal{R}^i{}_a \mathcal{R}^a{}_j = \delta^i{}_j, \qquad \mathcal{L}^a{}_i \mathcal{L}^i{}_b = \mathcal{R}^a{}_i \mathcal{R}^i{}_b = \delta^a{}_b.$$

Then the following equations are satisfied, dual to (74)

$$\mathrm{d}\mathcal{L}^a = \frac{1}{2} C^a{}_{bd} \mathcal{L}^b \wedge \mathcal{L}^d, \qquad \mathrm{d}\mathcal{R}^a = -\frac{1}{2} C^a{}_{bd} \mathcal{R}^b \wedge \mathcal{R}^d.$$

Let us notice that

$$\mathcal{L}_a(g) = \left(\mathrm{Ad}_{g^{-1}}\right)^b{}_a \mathcal{R}_b(g) \tag{82}$$

where the matrices $\left[(\mathrm{Ad}_g)^b{}_a\right]$ are implicitly given by

$$\mathrm{Ad}_g e_a = g e_a g^{-1} = e_b \left(\mathrm{Ad}_g\right)^b{}_a.$$

Similarly, $(\mathrm{ad}_y)^b{}_a$ are given by

$$\mathrm{ad}_y e_a = [y, e_a] = e_b \left(\mathrm{ad}_y\right)^b{}_a$$

thus

$$\left(\mathrm{ad}_y\right)^b{}_a = C^b{}_{da} y^d = -C^b{}_{ad} y^d$$

and

$$\mathrm{Ad}_{\mathrm{e}(a)} = \mathrm{e}\left(\mathrm{ad}_a\right). \tag{83}$$

In finite dimensions all above expressions are well-defined. Dually to (82) we have

$$\mathcal{L}^a(g) = \left(\mathrm{Ad}_g\right)^a{}_b \mathcal{R}^b(g). \tag{84}$$

The above differential operators and differential forms are a very useful tool of analysis. When constructing important tensor fields and differential operators on G we need certain intrinsically constructed tensors on its Lie algebra $\mathfrak{g}$. We mean some tensors built of the structure constants $C^i{}_{jk}$ with the use of universal algebraic operations. The first of them is C itself, it is a mixed tensor once contravariant and twice covariant, $C \in \mathfrak{g} \otimes \mathfrak{g}^* \otimes \mathfrak{g}^*$, skew-symmetric in its lower indices. The next one is the Killing tensor $\gamma \in \mathfrak{g}^* \otimes \mathfrak{g}^*$ (69), (70) and its inverse tensor $\gamma^{-1} \in \mathfrak{g} \otimes \mathfrak{g}$. One can also construct the higher-order covariant tensors like

$$\gamma(3)_{ijk} = C^a{}_{bi} C^b{}_{cj} C^c{}_{ak} \tag{85}$$

and so on, e.g.,

$$\gamma(m)_{i_1 \cdots i_m} = C^a{}_{bi_1} C^b{}_{ci_2} \ldots C^k{}_{li_{m-1}} C^l{}_{ai_m} \tag{86}$$

all of them covariant and in general non-symmetric (unlike the Killing tensor $\gamma(2)_{ij} = \gamma_{ij}$). Let us also mention other tensors like

$$\gamma(1)_i = C^a{}_{ai}, \qquad \Gamma_{ij} = C(1)_k C^k{}_{ij} = -\Gamma_{ji}. \tag{87}$$

If G is semisimple, then the inverse tensor (71) does exist and one can construct the whole ZOO of γ-tensors by the Killing-shift of indices. And similarly when G is a trivial central extension of some semisimple group. The invariant metric tensor on the centre is unique up to normalization.

The Killing metric tensor on G is given by

$$g = \gamma_{ab} \mathcal{L}^a \otimes \mathcal{L}^b = \gamma_{ab} \mathcal{R}^a \otimes \mathcal{R}^b \tag{88}$$

i.e., analytically

$$g_{ij} = \gamma_{ab} \mathcal{L}^a{}_i \mathcal{L}^b{}_j = \gamma_{ab} \mathcal{R}^a{}_i \mathcal{R}^b{}_j.$$

It is invariant under right and left regular translations on G. Usually one changes its normalization in such a way that in certain practically useful coordinates, at the group identity g_{ij} coincides with the Kronecker δ_{ij}. In particular, if G is compact, then γ, g are negatively definite, so it is the natural to inverse their signs.

The most general right-invariant metric on G is given by

$$_r g = \varkappa_{ab} \mathcal{L}^a \otimes \mathcal{L}^b \tag{89}$$

where the matrix $[\varkappa_{ab}]$ is non-degenerate and constant. Similarly, for the left-invariant metrics we have

$$_l g = \varkappa_{ab} \mathcal{R}^a \otimes \mathcal{R}^b. \tag{90}$$

They become identical and doubly-invariant when $\varkappa_{ab} = \gamma_{ab}$. The corresponding inverse contravariant metrics are given by

$$\begin{aligned} g^{-1} &= \gamma^{ab} \mathcal{L}_a \otimes \mathcal{L}_b = \gamma^{ab} \mathcal{R}_a \otimes \mathcal{R}_b \\ g^{ij} &= \gamma^{ab} \mathcal{L}^i{}_a \mathcal{L}^j{}_b = \gamma^{ab} \mathcal{R}^i{}_a \mathcal{R}^j{}_b \end{aligned}$$

and similarly for the inverses of (89) and (90)

$$_r g^{-1} = \varkappa^{-1ab} \mathcal{L}_a \otimes \mathcal{L}_b, \qquad _l g^{-1} = \varkappa^{-1ab} \mathcal{R}_a \otimes \mathcal{R}_b.$$

The Laplace-Beltrami operator corresponding to the Killing metric (88) is given by

$$\Delta = \gamma^{ab} \mathcal{L}_a \mathcal{L}_b = \gamma^{ab} \mathcal{R}_a \mathcal{R}_b. \tag{91}$$

Quite similarly, for the right-invariant metric (89) and left-invariant metric (90) we would have respectively

$$_r \Delta = \varkappa^{ab} \mathcal{L}_a \mathcal{L}_b, \qquad _l \Delta = \varkappa^{ab} \mathcal{R}_a \mathcal{R}_b. \tag{92}$$

Note that if G is non-Abelian, these expressions are different when $\varkappa_{ab} \neq \gamma_{ab}$.

One can show that all these expressions coincide with the usual definition of the **Laplace-Beltrami operator** [3]

$$\Delta = g^{ab}\nabla_a\nabla_b \tag{93}$$

where ∇_a denotes the Levi-Civita affine connection induced by the corresponding vector tensors (88), (89), (90). Surely, this coincides with the analytical formula

$$\Delta\psi = \frac{1}{\sqrt{|g|}}\sum_{i,j}\frac{\partial}{\partial k^i}\left(\sqrt{|g|}g^{ij}\frac{\partial\psi}{\partial k^j}\right)$$

where again for g the expressions (88), (89), (90) are substituted, their contravariant inverses g^{ij} are used, and $|g|$ denotes the determinant of the matrix $[g_{ij}]$.

The Haar measure in G is identical with the n-form

$$\mathcal{L}^1 \wedge \ldots \wedge \mathcal{L}^n = \mathcal{R}^1 \wedge \ldots \wedge \mathcal{R}^n$$

in the sense that

$$\int f(g)\mathrm{d}g = \int f\mathcal{L}^1 \wedge \ldots \wedge \mathcal{L}^n = \int f\mathcal{R}^1 \wedge \ldots \wedge \mathcal{R}^n.$$

In this prescription it is implicitly assumed that the orientation of G is chosen in such a way that the integral of non-negative functions is non-negative. Analytically we have that

$$\int f(g)\mathrm{d}g = \int f\det[\mathcal{L}^a{}_i]\mathrm{d}k^1\ldots\mathrm{d}k^n = \int f\det[\mathcal{R}^a{}_i]\mathrm{d}k^1\ldots\mathrm{d}k^n.$$

This integration coincides (up to a constant factor) with the usual Riemann integration

$$\int f(h)\mathrm{d}h = \int f\sqrt{|g|}\mathrm{d}k^1\ldots\mathrm{d}k^n$$

where g denotes any of the metric tensors (88), (89), (90). The Laplace-Beltrami operators (91), (92), (93) are formally self-adjoint (symmetric) with respect to the usual scalar product in $L^2(G)$.

The properties (38)–(40) imply immediately that

$$\mathcal{L}_a(F*G) = (\mathcal{L}_aF)*G, \qquad \mathcal{R}_a(F*G) = F*(\mathcal{R}_aG). \tag{94}$$

Again we conclude that $\mathcal{L}_a$, $\mathcal{R}_a$ are not differentiations of the convolution algebra, although they are so for the pointwise product algebra. If F is constant on equivalence classes of adjoint elements, i.e., if it is a linear combination or series of idempotents $\varepsilon(\alpha)$ or characters

$$\chi(\alpha) = \frac{1}{n(\alpha)}\varepsilon(\alpha)$$

then

$$\mathcal{A}_aF = 0 \tag{95}$$

therefore

$$\mathcal{L}_a F = \mathcal{R}_a F.$$

In particular, it is so for the Dirac distribution δ which formally plays the role of the convolution unity. Let us stress that in differential manifolds the distributions are well defined. In any case, for any finite subset $I \subset \Omega$

$$\delta\left(I\right) = \sum_{\alpha \in I} \varepsilon(\alpha)$$

is the well-defined unity of the two-sided ideal

$$M(I) := \underset{\alpha \in I}{\otimes} M(\alpha).$$

If J is a family of finite subsets of Ω ordered by inclusion and such that

$$\bigcup_{I \in J} M(I) = \Omega$$

then δ is the distribution limit of the generalized sequence $J \ni I \rightarrow \delta(I)$.

Equations (94) imply that

$$\mathcal{L}_a F = \mathcal{L}_a\left(\delta * F\right) = \left(\mathcal{L}_a \delta\right) * F, \qquad \mathcal{R}_a F = \mathcal{R}_a\left(F * \delta\right) = F * \left(\mathcal{R}_a \delta\right)$$

for any differentiable function F. This reduces separately to the ideals $M(\alpha)$, where the action of operators $\mathcal{L}_a$, $\mathcal{R}_a$ reduces respectively to the left and right convolutions with $\mathcal{L}_a \varepsilon(\alpha)$, $\mathcal{R}_a \varepsilon(\alpha)$.

Let us quote some important and intuitive commutation relations in the convolution algebra

$$\begin{aligned} \left(\mathcal{L}_a \delta\right) * \left(\mathcal{L}_b \delta\right) - \left(\mathcal{L}_b \delta\right) * \left(\mathcal{L}_a \delta\right) &= -C^k{}_{ab}\left(\mathcal{L}_k \delta\right) \\ \left(\mathcal{R}_a \delta\right) * \left(\mathcal{R}_b \delta\right) - \left(\mathcal{R}_b \delta\right) * \left(\mathcal{R}_a \delta\right) &= -C^k{}_{ab}\left(\mathcal{R}_k \delta\right). \end{aligned}$$

This is of course the same relation written in two ways, because $\mathcal{L}_a \delta = \mathcal{R}_a \delta$.

Roughly speaking, the functions constant on manifolds of mutually adjoint elements are scalars of the group of inner automorphisms of G, they satisfy the conditions

$$\mathcal{A}_c F = 0, \qquad \mathcal{A}_c \delta = 0, \qquad \mathcal{A}_c \sum_{\alpha} c_\alpha \varepsilon(\alpha) = 0.$$

It is no longer the case with their $\mathcal{L}_r$-derivatives

$$\mathcal{L}_a F = \mathcal{R}_a F, \qquad \mathcal{L}_a \delta = \mathcal{R}_a \delta, \qquad \mathcal{L}_a \sum_{\alpha} c_\alpha \varepsilon(\alpha) = \mathcal{R}_a \sum_{\alpha} c_\alpha \varepsilon(\alpha). \tag{96}$$

Roughly speaking, they are vectors of the group of inner automorphisms, e.g., denoting

$$Q_a := \mathcal{L}_a \delta = \mathcal{R}_a \delta \tag{97}$$

we have

$$\mathcal{A}_a Q_b = -C^k{}_{ab} Q_k$$

and similarly, for all other quantities in (96) and their multiples by functions constant on equivalence classes. Similarly, we have higher-order tensors, e.g.

$$Q_{ab} = \mathcal{L}_a \mathcal{L}_b \delta = (\mathcal{L}_a \delta) * (\mathcal{L}_b \delta) = Q_a * Q_b. \tag{98}$$

They satisfy

$$\mathcal{A}_c Q_{ab} = -C^k{}_{ca} Q_{kb} - C^k{}_{cb} Q_{ak}$$

and so on, for example, for

$$Q_{abc} = \mathcal{L}_a \mathcal{L}_b \mathcal{L}_c \delta = Q_a * Q_b * Q_c \tag{99}$$

we have

$$\mathcal{A}_d Q_{abc} = -C^k{}_{da} Q_{kbc} - C^k{}_{db} Q_{akc} - C^k{}_{dc} Q_{abk}$$

etc.

Casimir $\mathcal{L}$-operators are polynomials of $\mathcal{L}_b$ with constant coefficients, commuting with all $\mathcal{L}_a$. They are expected to be polynomials of $\mathcal{L}_b$ with coefficients built intrinsically of structure constants C, like (70), (85), (86), (87) or rather their versions with γ-raised indices. The most important example is the Laplace-Beltrami operator (91). It is clear that

$$[\Delta, \mathcal{L}_a] = [\Delta, \mathcal{R}_a] = 0.$$

Other expected quantities of this type are

$$\gamma(m)^{i_1 \ldots i_m} \mathcal{L}_{i_1} \ldots \mathcal{L}_{i_m}$$

etc. The raising of indices is meant in the sense of the Killing tensor. In the group-algebraic representation, these Casimir objects are given by functions/distributions like

$$\begin{aligned} C(2) &= \gamma^{ij} (\mathcal{L}_i \delta) * (\mathcal{L}_j \delta) = \gamma^{ij} \mathcal{L}_i \mathcal{L}_j \delta \\ C(m) &= \gamma(m)^{i_1 \ldots i_m} (\mathcal{L}_{i_1} \delta) * \cdots * (\mathcal{L}_{i_m} \delta) = \gamma(m)^{i_1 \ldots i_m} \mathcal{L}_{i_1} \ldots \mathcal{L}_{i_m} \delta. \end{aligned}$$

They are expected to satisfy

$$C(m) * f - f * C(m) = 0$$

(central elements of the convolution algebra).

To avoid distributions, one can consider their "α-versions", built of elements of $M(\alpha)$

$$\begin{aligned} C(2, \alpha) &= \gamma^{ij} (\mathcal{L}_i \varepsilon(\alpha)) * (\mathcal{L}_j \varepsilon(\alpha)) \\ C(m, \alpha) &= \gamma(m)^{i_1 \ldots i_m} (\mathcal{L}_{i_1} \varepsilon(\alpha)) * \cdots * (\mathcal{L}_{i_m} \varepsilon(\alpha)). \end{aligned}$$

Similarly, for any fixed $\alpha \in \Omega$, the quantities (97), (98), (99), and so on become usual functions

$$\begin{aligned} Q_a(\alpha) &= \mathcal{L}_a \varepsilon(\alpha) = \mathcal{R}_a \varepsilon(\alpha) && (100)\\ Q_{ab}(\alpha) &= \mathcal{L}_a \mathcal{L}_b \varepsilon(\alpha) = (\mathcal{L}_a \varepsilon(\alpha)) * (\mathcal{L}_b \varepsilon(\alpha)) = Q_a(\alpha) * Q_b(\alpha) && (101)\\ Q_{abc}(\alpha) &= Q_a(\alpha) * Q_b(\alpha) * Q_c(\alpha) && (102)\\ \vdots\ \ & \ \ \vdots \\ Q_{ab\ldots r}(\alpha) &= Q_a(\alpha) * Q_b(\alpha) * \cdots * Q_r(\alpha) && (103) \end{aligned}$$

etc. Evidently, Q_a, Q_{ab}, Q_{abc}, etc., are distributions obtained as series (in the distribution sense of limit) of all the above Q-s. One important circumstance must be stressed: The quantities $Q_{ab\ldots r}$ are tensors under the action of automorphisms, however they are not irreducible tensors, because they are not symmetric if G is non-Abelian. To obtain irreducible tensors one must take their symmetric parts, skew-symmetric ones, and remove the γ-traces from the symmetric parts.

For any fixed α the tensors Q_a, Q_{ab}, etc., form some basis of $M(\alpha)$ alternative to $\varepsilon(\alpha)_{ij}$. Of course, when α is fixed, the order of tensors $Q(\alpha)$ terminates at some value, because $\dim M(\alpha) = n(\alpha)^2$ cannot be exceeded.

From some point of view one might suppose that the pointwise products of Q_a, e.g., Q_aQ_b, $Q_aQ_bQ_c$, etc., might be simpler and more convenient. And they are tensors of $\mathcal{A}_i$ as well. However, it is not the case, because $Q_a(\alpha)Q_b(\alpha)$, etc., are no longer elements of $M(\alpha)$. Nevertheless, they may be useful in a sense. They may become elements of $M(\alpha)$ when multiplied by appropriate scalars under inner automorphisms, i.e., multiplied by appropriate functions $f(\alpha)$ constant on classes of adjoint elements, thus, satisfying (95).

The matrices of irreducible representations $D(\alpha)$ will be represented (at least locally, in some neighbourhood of the group identity), as follows

$$D(\alpha)(g) = \mathrm{e}\left(k^a e(\alpha)_a\right), \qquad g\left(k^1, \ldots, k^n\right) = \mathrm{e}\left(k^a e_a\right) \tag{104}$$

where $e(\alpha)$ are $n(\alpha) \times n(\alpha)$ matrices which obey the commutation rules (68)

$$[e(\alpha)_a, e(\alpha)_b] = e(\alpha)_k C^k{}_{ab}.$$

If $D(\alpha)$ are unitary, that is always assumed here, then $e(\alpha)$ are anti-Hermitian, so we have that

$$D(\alpha)^+ = D(\alpha)^{-1}, \qquad e(\alpha)^+ = -e(\alpha).$$

In quantum-mechanical considerations the fundamental role is played by Hermitian matrices

$$\Sigma(\alpha)_a = \frac{\hbar}{\mathrm{i}}\, e(\alpha)_a = \Sigma(\alpha)^+_a$$

which obey the commutation rules analogous to (78)–(80)

$$\frac{1}{\mathrm{i}\hbar}\left[\Sigma(\alpha)_a, \Sigma(\alpha)_b\right] = C^k{}_{ab}\Sigma(\alpha)_k.$$

Then we have the favourite formulas of physicists

$$\begin{aligned} D(\alpha)\left(g\left(\overline{k}\right)\right) &= \mathrm{e}\left(\frac{\mathrm{i}}{\hbar}k^a\Sigma(\alpha)_a\right) \\ L\left(g(\overline{k})^{-1}\right) &= \mathrm{e}\left(\frac{\mathrm{i}}{\hbar}k^a\mathbf{\Sigma}_a\right) \\ R\left(g(\overline{k})^{-1}\right) &= \mathrm{e}\left(\frac{\mathrm{i}}{\hbar}k^a\widehat{\mathbf{\Sigma}}_a\right). \end{aligned} \tag{105}$$

Note that the last two formulas are meant in an appropriate function domain, if to be meaningful. The representation property and definition of operators $\mathcal{L}_a$, $\mathcal{R}_a$, $\mathcal{A}_a$ and their Hermitian counterparts $\mathbf{\Sigma}_a$, $\widehat{\mathbf{\Sigma}}_a$, $\mathbf{\Delta}_a$ imply that the matrix-valued functions $D(\alpha)$ on G (equivalently $\varepsilon(\alpha) = n(\alpha)D(\alpha)$) satisfy the following differential equations

$$\mathcal{L}_aD(\alpha) = e(\alpha)_aD(\alpha) \tag{106}$$

$$\mathcal{R}_aD(\alpha) = D(\alpha)e(\alpha)_a \tag{107}$$

$$\mathcal{A}_aD(\alpha) = e(\alpha)_aD(\alpha) - D(\alpha)e(\alpha)_a = [e(\alpha)_a, D(\alpha)] \tag{108}$$

or, in terms of "Hermitian" operators

$$\mathbf{\Sigma}_aD(\alpha) = \Sigma(\alpha)_aD(\alpha) \tag{109}$$

$$\widehat{\mathbf{\Sigma}}_aD(\alpha) = D(\alpha)\Sigma(\alpha)_a \tag{110}$$

$$\mathbf{\Delta}_aD(\alpha) = \left[\Sigma(\alpha)_a, D(\alpha)\right]. \tag{111}$$

Let $C(\mathcal{L})$, $C(\mathcal{R})$, $C(\mathcal{A})$ denote the mentioned Casimir operators. Let us remind that $C(\mathcal{L})$ commute with all $\mathcal{L}_a$-operators, $C(\mathcal{R})$ commute with all $\mathcal{R}_a$-operators, and $C(\mathcal{A})$ commute with all $\mathcal{A}_a$-operators. They are built in a polynomial way respectively of $\mathcal{L}$, $\mathcal{R}$, $\mathcal{A}$. Moreover, $C(\mathcal{L})$-Casimirs commute also with all $\mathcal{R}$- and $\mathcal{A}$-operators and $C(\mathcal{R})$-Casimirs also commute with $\mathcal{L}$- and $\mathcal{A}$-operators. This follows from the obvious fact that all $\mathcal{L}$-operators commute with all $\mathcal{R}$-operators. But attention: $C(\mathcal{A})$-Casimirs do not commute with all $\mathcal{L}$- and $\mathcal{R}$-operators. However, they do commute with $C(\mathcal{L})$- and $C(\mathcal{R})$-Casimirs. For physical reasons one uses often the $C(\mathbf{\Sigma})$-, $C\left(\widehat{\mathbf{\Sigma}}\right)$-, and $C(\mathbf{\Delta})$-Casimirs. They are built of $\mathbf{\Sigma}$-, $\widehat{\mathbf{\Sigma}}$- and $\mathbf{\Delta}$-operators just like $C(\mathcal{L})$, $C(\mathcal{R})$ and $C(\mathcal{A})$ are built of the indicated operators. Usually there are a few ones of each kind. If necessary, some additional label is introduced (e.g. polynomial degree, etc.).

The use of differential operators acting on the functions on G, in particular, the use of their associative products, enables one to avoid dealing with more abstract and non-intuitive notion of the enveloping algebra of $\mathfrak{g}$.

The most important Casimirs are γ-quadratic functions of $\mathcal{L}$, $\mathcal{R}$, $\mathcal{A}$

$$\begin{aligned} C(\mathcal{L},2) = C(\mathcal{R},2) \,&=\, \Delta = \gamma^{ab}\mathcal{L}_a\mathcal{L}_b = \gamma^{ab}\mathcal{R}_a\mathcal{R}_b \\ C(\mathcal{A},2) \,&=\, \gamma^{ab}\mathcal{A}_a\mathcal{A}_b. \end{aligned}$$

As mentioned, in addition to the obvious rules

$$[C(\mathcal{L},2),\mathcal{L}_a] = [C(\mathcal{L},2),\mathcal{R}_a] = [C(\mathcal{A},2),\mathcal{A}_a] = 0$$

we have also

$$[C(\mathcal{L},2),\mathcal{A}_a] = [C(\mathcal{L},2),C(\mathcal{A},2)] = 0. \tag{112}$$

The corresponding expressions for "Hermitian" operators will be denoted by

$$C\left(\mathbf{\Sigma},2\right) = C\left(\widehat{\mathbf{\Sigma}},2\right), \qquad C\left(\mathbf{\Delta},2\right), \qquad \text{etc.}$$

They are built according to the prescriptions for $C(\mathcal{L},2)$, $C(\mathcal{R},2)$, $C(\mathcal{A},2)$ with $\mathbf{\Sigma}$, $\widehat{\mathbf{\Sigma}}$, $\mathbf{\Delta}$ substituted respectively instead of $\mathcal{L}$, $\mathcal{R}$, $\mathcal{A}$, therefore, for quadratic Casimirs we have

$$C\left(\mathbf{\Sigma},2\right) = -\hbar^2 C(\mathcal{L},2),\; C\left(\widehat{\mathbf{\Sigma}},2\right) = -\hbar^2 C(\mathcal{R},2),\; C\left(\mathbf{\Delta},2\right) = -\hbar^2 C(\mathcal{A},2)$$

and similarly for other Casimirs.

When we fix some α and act with our Casimirs on functions $D(\alpha)$ ($\varepsilon(\alpha)$), they simply suffer the multiplication by scalars, just the eigenvalues of Casimirs. This follows from the Schur lemma, because $D(\alpha)$ are irreducible. Therefore, e.g., iterating appropriately (106)–(108), (109)–(111), we obtain

$$\begin{aligned} \gamma^{ab}\mathcal{L}_a\mathcal{L}_b D(\alpha) \,&=\, \gamma^{ab}\mathcal{R}_a\mathcal{R}_b D(\alpha) \\ &=\, \gamma^{ab} e(\alpha)_a e(\alpha)_b D(\alpha) = C(2,\alpha) D(\alpha) \end{aligned} \tag{113}$$

$$\gamma^{ab}\mathbf{\Sigma}_a\mathbf{\Sigma}_b D(\alpha) \,=\, \gamma^{ab}\widehat{\mathbf{\Sigma}}_a\widehat{\mathbf{\Sigma}}_b D(\alpha) = -\hbar^2 C(2,\alpha) D(\alpha) \tag{114}$$

where

$$\gamma^{ab} e(\alpha)_a e(\alpha)_b = C(2,\alpha)\mathrm{Id}_{n(\alpha)}$$

and $C(2,\alpha)$ are elements of the spectrum of Δ (91). These eigenvalues are $n(\alpha)^2$-fold degenerate. It was mentioned that although $\gamma^{ab}\mathcal{A}_a\mathcal{A}_b$ does not commute in general with $\mathcal{L}_a$, $\mathcal{R}_b$, nevertheless, it does commute with

$$\Delta = \gamma^{ab}\mathcal{L}_a\mathcal{L}_b = \gamma^{ab}\mathcal{R}_a\mathcal{R}_b.$$

However, $D(\alpha)_{ij}$ are not their common eigenfunctions. Indeed

$$\gamma^{ab}\mathcal{A}_a\mathcal{A}_b D(\alpha) = 2C(2,\alpha)D(\alpha) - 2\gamma^{ab} e(\alpha)_a D(\alpha) e(\alpha)_b \tag{115}$$

i.e.,

$$\gamma^{ab}\Delta_a\Delta_b D(\alpha) = -2C(2,\alpha)\hbar^2 D(\alpha) - 2\gamma^{ab}\Sigma(\alpha)_a D(\alpha)\Sigma(\alpha)_b. \tag{116}$$

Nevertheless, their common eigenfunctions do exist and are given by (100)–(103). For any fixed $\alpha \in \Omega$, the order of tensors (100)–(103) terminates at some fixed value.

It is seen that in the action on functions

$$\varepsilon(\alpha)_{ij} = n(\alpha)D(\alpha)_{ij}$$

our differential operators become algebraic. This is just the obvious counterpart and generalization of the well-known facts in Fourier analysis. Let us quote a few obvious and practically important formulas.

It was mentioned earlier about the Peter-Weyl expansion (30). Let us write it a bit more symbolically as

$$F = \sum_{\alpha\in\Omega} \operatorname{Tr}\left(F(\alpha)^T \varepsilon(\alpha)\right) = \sum_{\alpha\in\Omega} \operatorname{Tr}\left(F(\alpha)^T D(\alpha)\right) n(\alpha). \tag{117}$$

The general operations of group algebras are then represented in a suggestive way by the corresponding operations performed on the matrices $F(\alpha)$, cf. (32)–(33). Together with the formulas (106)–(108), (109)–(111), (113)–(114), (115), (116) this implies that the action of differential operators may be expressed in the following way by the corresponding algebraic operations on the representing matrices $F(\alpha)$

$$\mathcal{L}_a, \mathbf{\Sigma}_a : \quad F(\alpha) \mapsto F(\alpha)e(\alpha)_a, \qquad F(\alpha) \mapsto F(\alpha)\Sigma(\alpha)_a \tag{118}$$

$$\mathcal{R}_a, \widehat{\mathbf{\Sigma}}_a : \quad F(\alpha) \mapsto R(\alpha)_a F(\alpha), \qquad F(\alpha) \mapsto \Sigma(\alpha)_a F(\alpha) \tag{119}$$

$$\mathcal{A}_a, \mathbf{\Delta}_a : \quad F(\alpha) \mapsto \left[F(\alpha), e(\alpha)_a\right], \quad F(\alpha) \mapsto \left[F(\alpha), \Sigma(\alpha)_a\right]. \tag{120}$$

Therefore, the action of

$$\gamma^{ab}\mathcal{L}_a\mathcal{L}_b = \gamma^{ab}\mathcal{R}_a\mathcal{R}_b = \Delta$$

is represented by multiplication of matrices $F(\alpha)$ by $C(2,\alpha)$, and similarly for other Casimirs.

Let us mention that for some purposes the convention of transposed $F(\alpha)$-matrices might be more convenient, namely

$$F = \sum_{\alpha\in\Omega} \operatorname{Tr}\left(F(\alpha)\varepsilon(\alpha)\right) = \sum_{\alpha\in\Omega} \operatorname{Tr}\left(F(\alpha)D(\alpha)\right) n(\alpha). \tag{121}$$

A disadvantage is that then $F * G$ is not represented by the system of $F(\alpha)G(\alpha)$ but $G(\alpha)F(\alpha)$. But, and this is an aesthetic advantage, the matrix transposition is avoided, namely, the $\mathcal{L}_a/\mathbf{\Sigma}_a$ act respectively as follows

$$\mathcal{L}_a, \mathbf{\Sigma}_a: \ F(\alpha) \mapsto F(\alpha)e(\alpha)_a, \qquad F(\alpha) \mapsto F(\alpha)\Sigma(\alpha)_a \tag{122}$$

$$\mathcal{R}_a, \widehat{\mathbf{\Sigma}}_a: \ F(\alpha) \mapsto R(\alpha)_a F(\alpha), \qquad F(\alpha) \mapsto \Sigma(\alpha)_a F(\alpha) \tag{123}$$

$$\mathcal{A}_a, \mathbf{\Delta}_a: \ F(\alpha) \mapsto [F(\alpha), e(\alpha)_a], \quad F(\alpha) \mapsto [F(\alpha), \Sigma(\alpha)_a]. \tag{124}$$

But again, a disadvantage is that the left/right differential generators are represented algebraically by the right/left matrix multiplication, thus, conversely. Of course, all this is a merely matter of convention.

If we use the convention (117), then the functions (97), (98), (99), etc., i.e.,

$${}^{l}Q_{ab\ldots k} = \mathcal{L}_a\mathcal{L}_b \ldots \mathcal{L}_k\delta = (\mathcal{L}_a\delta) * \cdots * (\mathcal{L}_k\delta) = Q_a * Q_b * \cdots * Q_k \tag{125}$$

are represented by matrices

$${}^{l}\widehat{Q}(\alpha)_{ab\ldots k} = e(\alpha)_a{}^T e(\alpha)_b{}^T \ldots e(\alpha)_k{}^T. \tag{126}$$

And similarly, the functions

$$\begin{aligned} {}^{r}Q_{ab\ldots k} &= \mathcal{R}_a\mathcal{R}_b \ldots \mathcal{R}_k\delta = \mathcal{L}_k \ldots \mathcal{L}_b\mathcal{L}_a\delta = (\mathcal{R}_a\delta) * (\mathcal{R}_b\delta) * \ldots * (\mathcal{R}_k\delta) \\ &= (\mathcal{L}_k\delta) * \ldots * (\mathcal{L}_b\delta) * (\mathcal{L}_a\delta) = Q_k * \ldots * Q_b * Q_a = {}^{l}Q_{k\ldots ba} \end{aligned}$$

are represented by matrices

$${}^{r}\widehat{Q}(\alpha) = e(\alpha)_k{}^T \ldots e(\alpha)_b{}^T e(\alpha)_a{}^T. \tag{127}$$

If we use the convention (121), then instead of (126), (127) we obtain respectively

$$\begin{aligned} {}^{l}\widehat{Q}(\alpha)_{ab\ldots k} &= e(\alpha)_k \ldots e(\alpha)_b e(\alpha)_a \\ {}^{r}\widehat{Q}(\alpha)_{ab\ldots k} &= e(\alpha)_a e(\alpha)_b \ldots e(\alpha)_k. \end{aligned} \tag{128}$$

The Hermitian version of Q_a, representing a physical observable, is obtained by replacing the operators $\mathcal{L}_a$, $\mathcal{R}_a$ by (77), i.e., by

$$\mathbf{\Sigma}_a = \frac{\hbar}{\mathrm{i}}\mathcal{L}_a, \qquad \widehat{\mathbf{\Sigma}}_a = \frac{\hbar}{\mathrm{i}}\mathcal{R}_a.$$

They are given by

$$\Sigma_a = \frac{\hbar}{\mathrm{i}}Q_a \tag{129}$$

whereas Q_a themselves are anti-Hermitian.

Let us observe that if G is non-Abelian (and here we concentrate mainly on semi-simple ones), then in general the functions

$${}^{l}Q_{ab\ldots k}, \qquad {}^{r}Q_{ab\ldots k}$$

and the representing matrices

$${}^{l}\widehat{Q}(\alpha)_{ab\ldots k}, \qquad {}^{r}\widehat{Q}(\alpha)_{ab\ldots k}$$

fail to be anti-Hermitian. Therefore, the corresponding monomials of $\Sigma(\alpha)$ and $\Sigma(\alpha)^T$ are not Hermitian. But their symmetrizations

$$\Sigma(\alpha)_{(a} \ldots \Sigma(\alpha)_{k)}, \qquad \Sigma(\alpha)^T_{(a} \ldots \Sigma(\alpha)^T_{k)} \tag{130}$$

are Hermitian and so are the functions

$$\Sigma_{(a\ldots k)} = \left(\frac{\hbar}{\mathrm{i}}\right)^p \left(\mathcal{L}_{(a}\delta\right) * \cdots * \left(\mathcal{L}_{k)}\delta\right) = \left(\frac{\hbar}{\mathrm{i}}\right)^p \left(\mathcal{R}_{(a}\delta\right) * \cdots * \left(\mathcal{R}_{k)}\delta\right) \tag{131}$$

where p is the order of tensors (the number of convolution factors). Note that (130) are matrices of (131) when the conventions (121), (117) are used, respectively.

In realistic dynamical models Hamiltonians are usually given by simple algebraic functions of the above Hermitian elements of group algebras. As a rule, those Hamiltonians or their important terms are low-order polynomials. In special cases of high symmetry they are built according to the Casimir prescriptions.

3.2. Abelian Lie Groups

Let us finish with some remarks concerning Abelian Lie groups. The only (up to isomorphism) connected Abelian groups are

$$\mathbb{R}^n, \qquad T^n = U(1)^n = \mathbb{R}^n/\mathbb{Z}^n$$

and their Cartesian products $\mathbb{R}^n \times T^m$, i.e., linear spaces, tori and cylinders. The group operation in $\mathbb{R}^n$ is meant as the addition of vectors (null vector being the neutral element). In T^n it is meant as the quotient action obtained when $\mathbb{R}^n$ is divided by the "crystallographic" lattice $\mathbb{Z}^n \subset \mathbb{R}^n$.

Some conflicts between the above notational conventions and various customs from the classical Fourier analysis appear, so one must be careful with an automatic use of traditional formulas.

It is perhaps convenient to write down some formulas concerning $\mathbb{R}^n$ in the language of abstract vector space. So, let V be a finite-dimensional linear space, and V^* be its dual. We put $n = \dim V = \dim V^*$. We consider them as Abelian additive Lie groups. So, $G = V$ with the "$+$" composition rule, $\widehat{G}$ is isomorphic with V^*. And the particular choice of this isomorphism is a matter of convention. G being non-compact, there is no standard of normalization. If V is endowed with some fixed metric tensor $\gamma \in V^* \otimes V^*$, as it usually is in physical applications, then, of course, the standard of Lebesgue measure is fixed

$$\int f(x)\mathrm{d}\mu(x) = \int f e^1 \wedge \ldots \wedge e^n$$

where $(\ldots, e^a, \ldots)$ is an arbitrary orthonormal co-basis in V^*

$$g = \delta_{ij} e^i \otimes e^j.$$

In arbitrary coordinates, including curvilinear ones, we have

$$\int f(x)\mathrm{d}\mu(x) = \int f(x)\sqrt{\det\left[g_{ij}\right]}\,\mathrm{d}x^1 \dots \mathrm{d}x^n.$$

The dual linear space V^* parametrizes the dual group $\widehat{V}$ with the help of the standard covering homomorphism of the (additive) $\mathbb{R}$ onto (multiplicative) $U(1)$

$$\mathbb{R} \ni \varphi \mapsto \mathrm{e}(\mathrm{i}\varphi) \in U(1)$$

so, $\chi\left(\underline{k}\right) \in \widehat{V}$ is given by

$$\left\langle \chi\left(\underline{k}\right), \overline{x}\right\rangle = \mathrm{e}\left(\mathrm{i}\left\langle \underline{k}, \overline{x}\right\rangle\right)$$

where $\left\langle \underline{k}, \overline{x}\right\rangle$ is the evaluation of $\underline{k} \in V^*$ on $\overline{x} \in V$. Analytically

$$\left\langle \underline{k}, \overline{x}\right\rangle = k_a x^a.$$

Using the language of quantum momentum $\underline{p} = \hbar\underline{k}$, one writes also

$$\left\langle \chi\left[\underline{p}\right], \overline{x}\right\rangle = \mathrm{e}\left(\frac{\mathrm{i}}{\hbar}\left\langle \underline{p}, \overline{x}\right\rangle\right) = \mathrm{e}\left(\frac{\mathrm{i}}{\hbar}\,p_a x^a\right).$$

The corresponding conventions of Fourier analysis, particularly popular in quantum mechanics, are as follows

$$\begin{aligned} f\left(\overline{x}\right) &= \frac{1}{(2\pi)^n}\int \widehat{f}\left(\underline{k}\right)\mathrm{e}\left(\mathrm{i}\left\langle \underline{k}, \overline{x}\right\rangle\right)\mathrm{d}_n k \\ &= \frac{1}{(2\pi\hbar)^n}\int \widehat{f}\left(\underline{p}\right)\mathrm{e}\left(\frac{\mathrm{i}}{\hbar}\left\langle \underline{p}, \overline{x}\right\rangle\right)\mathrm{d}_n p \end{aligned} \tag{132}$$

$$\widehat{f}\left(\underline{k}\right) = \widehat{f}\left[\underline{p}\right] = \int f\left(\overline{x}\right)\mathrm{e}\left(-\frac{\mathrm{i}}{\hbar}\left\langle \underline{p}, \overline{x}\right\rangle\right)\mathrm{d}_n x. \tag{133}$$

The convolution on V is meant in the usual convention

$$\left(A * B\right)\left(\overline{x}\right) = \int A(\overline{y})B(\overline{x} - \overline{y})\mathrm{d}\overline{y}.$$

We have then the following rules

$$\begin{aligned} \chi\left(\underline{k}\right) * \chi\left(\underline{l}\right) &= (2\pi)^n\delta\left(\underline{k} - \underline{l}\right)\chi\left(\underline{k}\right) = (2\pi)^n\delta\left(\underline{k} - \underline{l}\right)\chi\left(\underline{l}\right) \\ \left(\chi\left(\underline{k}\right), \chi\left(\underline{l}\right)\right) &= (2\pi)^n\delta\left(\underline{k} - \underline{l}\right) \\ \chi\left[\underline{p}\right] * \chi\left[\underline{p}'\right] &= (2\pi\hbar)^n\delta\left(\underline{p} - \underline{p}'\right)\chi\left[\underline{p}\right] = (2\pi\hbar)^n\delta\left(\underline{p} - \underline{p}'\right)\chi\left[\underline{p}'\right] \\ \left(\chi\left[\underline{p}\right], \chi\left[\underline{p}'\right]\right) &= (2\pi\hbar)^n\delta\left(\underline{p} - \underline{p}'\right) \end{aligned}$$

rather unpleasant ones, because of the $(2\pi)^n$- and $(2\pi\hbar)^n$-factors. But this has to do with the use of traditional symbols of analysis. If we remember that it is not

$\mathrm{d}_n k$, or $\mathrm{d}_n p$, but rather

$$\frac{\mathrm{d}_n k}{(2\pi)^n}, \qquad \frac{\mathrm{d}_n p}{(2\pi\hbar)^n}$$

that is a measure Fourier-synchronized with $\mathrm{d}_n x$, that it is just

$$(2\pi)^n \delta\left(\underline{k} - \underline{l}\right) \qquad \text{or} \qquad (2\pi\hbar)^n \delta\left(\underline{p} - \underline{p}'\right)$$

that is to be interpreted as a "true Dirac delta", let us say

$$\Delta\left(\underline{k} - \underline{k}'\right), \qquad \Delta\left(\underline{p} - \underline{p}'\right)$$

respectively in the spaces of wave co-vectors and linear momenta.

There are various conventions concerning Fourier transforms and synchronization of measures on G, $\widehat{G}$, it is even stated in the book by Loomis [9], that it is "an interesting and non-trivial problem".

In classical analysis one often prefers the "symmetric" convention

$$A\left(\overline{x}\right) = \frac{1}{(2\pi)^{n/2}} \int \widehat{A}\left(\underline{k}\right) \mathrm{e}\left(\mathrm{i}\left\langle \underline{k}, \overline{x}\right\rangle\right) \mathrm{d}_n \underline{k}$$
$$\widehat{A}\left(\underline{k}\right) = \frac{1}{(2\pi)^{n/2}} \int A\left(\overline{x}\right) \mathrm{e}\left(-\mathrm{i}\left\langle \underline{k}, \overline{x}\right\rangle\right) \mathrm{d}_n \overline{x}.$$

An additional advantage of this convention is that the iteration of Fourier transformation results in the inversion (total reflection) of the original function, with respect to the origin

$$\widehat{\widehat{A}}\left(x\right) = A(-x).$$

And, roughly speaking, Gauss function is invariant under Fourier transformation. More precisely, we have

$$\mathcal{G}(\overline{x}) = \mathrm{e}\left(-\frac{1}{2}\, \overline{x} \cdot \overline{x}\right), \qquad \widehat{\mathcal{G}}(\underline{k}) = \mathrm{e}\left(-\frac{1}{2}\, \underline{k} \cdot \underline{k}\right)$$

where the scalar product in V is meant in the sense of metric $g \in V^* \otimes V^*$, and in V^* — under its contravariant inverse $g^{-1} \in V \otimes V$

$$\overline{x} \cdot \overline{x} = g(x, x) = g_{ij} x^i x^j, \qquad \underline{k} \cdot \underline{k} = \widehat{g}(\underline{k}, \underline{k}) = g^{ij} x_i x_j.$$

If we identify $V = \mathbb{R}^n = V^*$, then the Gauss function is literally invariant under the Fourier transformation.

The counterparts of Clebsch-Gordon series, i.e.,

$$\varepsilon(\alpha)_{ab} \varepsilon(\varrho)_{rs} = \sum_{\varkappa, k, l} \frac{n(\alpha) n(\varrho)}{n(\varkappa)} \left(\alpha \varrho a r | \varkappa k\right) \left(\alpha \varrho b s | \varkappa l\right) \varepsilon(\varkappa)_{kl} \tag{134}$$

and

$$U\left(u\left(\alpha\right)_a \otimes v\left(\varrho\right)_r\right) = \sum_{\varkappa, k} \left(\alpha \varrho a r | \varkappa k\right) w\left(\varkappa\right)_k \tag{135}$$

are very simple now, because

$$\begin{aligned}
\chi(\underline{k})\chi(\underline{l}) &= \chi(\underline{k}+\underline{l}) \\
\chi\left[\underline{p}\right]\chi\left[\underline{p}'\right] &= \chi\left[\underline{p}+\underline{p}'\right] \\
\chi(\underline{k})\chi(\underline{l}) &= \int \delta(\underline{k}+\underline{l}-\underline{m})\chi(\underline{m})\mathrm{d}_n\underline{m} \\
\chi\left[\underline{p}\right]\chi\left[\underline{p}'\right] &= \int \delta(\underline{p}+\underline{p}'-\underline{\pi})\chi\left[\underline{\pi}\right]\mathrm{d}_n\underline{\pi}.
\end{aligned}$$

Let us now fix some symbols concerning the compact case $T^n = U(1)^n$. Just like $\mathbb{R}^n$ is an analytical model of any n-dimensional linear space over reals, T^n is parametrized by the system of angles $\left(\varphi^1, \ldots, \varphi^n\right)$ taken modulo 2π, or uniquely, by the system of unimodular complex numbers

$$\left(\zeta^1, \ldots, \zeta^n\right), \qquad \zeta^a = \mathrm{e}\left(\mathrm{i}\varphi^a\right).$$

Sometimes the convention "modulo 1" is accepted instead "modulo 2π", i.e., one puts

$$\zeta^a = \mathrm{e}\left(2\pi\mathrm{i}\xi^a\right).$$

This is often used when T^n is realized as a quotient of V modulo the "crystallographic lattice" generated freely by some fixed basis $(\ldots, e_a, \ldots)$ in V. Of course, that discrete translation group is isomorphic with $\mathbb{Z}^n$. The parametrization modulo 2π is more popular in theory of Fourier series. Torus is compact and it is natural to take the Haar measure normalized to unity, as usual. If the multiple Fourier series on T^n are meant in the convention

$$f(\overline{\varphi}) = \sum_{\underline{m}\in\mathbb{Z}^n} \widehat{f}(\underline{m})\mathrm{e}\left(\mathrm{i}\underline{m}\cdot\overline{\varphi}\right)$$

then the inverse formula for coefficients $\widehat{f}$ reads

$$\widehat{f}(\underline{m}) = \frac{1}{(2\pi)^n}\int f(\overline{\varphi})\mathrm{e}\left(-\mathrm{i}\underline{m}\cdot\overline{\varphi}\right)\mathrm{d}_n\overline{\varphi}.$$

Concerning notation, analytical meaning of the expressions above is as follows

$$\underline{m} = (m_1, \ldots, m_n) \in \mathbb{Z}^n, \qquad \overline{\varphi} = \left(\varphi^1, \ldots, \varphi^n\right)^T$$

contractions in exponents are given by

$$\underline{m}\cdot\overline{\varphi} = m_a\varphi^a = m_1\varphi^1 + \cdots + m_n\varphi^n$$

and the range of variables φ^a in the integration element

$$\mathrm{d}_n\overline{\varphi} = \mathrm{d}\varphi^1\ldots\mathrm{d}\varphi^n$$

is given by $[0, 2\pi]$.

It is seen that the occurrence of factors $(2\pi)^{-n}$ is reciprocal to that in Fourier analysis on $\mathbb{R}^n$. This spoils the formal analogy, but suits the convention that the

Haar volume of compact groups equals the unity. To save the analogy, we would have to replace (132)–(133) by

$$\begin{aligned} f(\overline{x}) &= \int \widehat{f}(\underline{k})\mathrm{e}\left(\mathrm{i}\left\langle \underline{k}, \overline{x}\right\rangle\right) \mathrm{d}_n\underline{k} \\ \widehat{f}(\underline{k}) &= \frac{1}{(2\pi)^n} \int f(\overline{x})\mathrm{e}\left(-\mathrm{i}\left\langle \underline{k}, \overline{x}\right\rangle\right) \mathrm{d}_n\overline{x} \end{aligned}$$

which, by the way, is sometimes used indeed, however, it is incompatible with some other customs of physicists and their taste.

Characters on T^n are labelled by multi-indices $\underline{m} \in \mathbb{Z}^n$

$$\langle \chi(\underline{m}), \zeta(\overline{\varphi})\rangle = \left(\zeta^1\right)^{m_1} \ldots (\zeta^n)^{m_n} = \mathrm{e}\left(\mathrm{i}\underline{m} \cdot \overline{\varphi}\right).$$

The idempotence and independence property is literally satisfied, because T^n is compact and $\mathbb{Z}^n$ is discrete

$$\begin{aligned} \chi(\underline{m}) * \chi(\underline{l}) &= \delta_{\underline{ml}}\chi(\underline{m}) = \delta_{\underline{ml}}\chi(\underline{l}) \\ \chi(\underline{m})\chi(\underline{l}) &= \chi(\underline{m} + \underline{l}) \\ (\chi(\underline{m}), \chi(\underline{l})) &= \delta_{\underline{ml}} \end{aligned} \tag{136}$$

where the multi-index Kronecker symbol $\delta_{\underline{ml}}$ vanishes if $\underline{m} \neq \underline{l}$ (i.e., at least one component of $\underline{m}$ differs from the corresponding component of $\underline{l}$), and $\delta_{\underline{ml}} = 1$ when $\underline{m} = \underline{l}$. In other words

$$\delta_{\underline{ml}} = \delta_{m_1 l_1} \ldots \delta_{m_n l_n}.$$

Concerning the "Clebsch-Gordon" rule (136), its representation in terms of (134), (135) reads

$$\chi(\underline{m})\chi(\underline{l}) = \sum_{\underline{\pi} \in \mathbb{Z}^n} (\underline{m}\,\underline{l}|\underline{\pi})\,(\underline{m}\,\underline{l}|\underline{\pi})\,\chi(\underline{\pi})$$

where

$$(\underline{m}\,\underline{l}|\underline{\pi}) = \delta_{\underline{m}+\underline{l},\underline{\pi}} = (\underline{m}\,\underline{l}|\underline{\pi})^2.$$

Let us notice that in the non-compact case $G = \mathbb{R}^n$, the counterpart of (134), i.e., the right-hand side of (135), fails because the square of Dirac-delta is not well defined.

Note that if we take as an arena of our physics the discrete group $\mathbb{Z}^n$, then its dual group T^n is compact and continuous. Again the mentioned problems with squared delta-distribution appear.

3.3. Byproducts of Group Structure

For certain reasons, first of all ones concerning quasiclassical analysis, it is interesting to discuss certain byproducts of the group structure in G. It is well known that the Lie algebra $\mathfrak{g}$ of G encodes a great amount of information about the global structure of G, although, of course, not the total information. This is due to the very analytic structure of Lie groups. Making use of exponential mapping of $\mathfrak{g}$ into G (not "onto" in general) one can "pull back" some structures of G and some physics in G to its tangent space $\mathfrak{g} = T_e G$. But now, $\mathfrak{g}$ as a finite-dimensional linear space is an Abelian Lie group under addition of its elements. Therefore, we can consider some physics, using the group algebra of $\mathfrak{g}$ as an additive group of vectors. But of course this would be completely non-physical and non-interesting if we did not take into account the Lie-algebraic structure of $\mathfrak{g}$. This structure leads to certain additional structures and relationships in the group algebra of $\mathfrak{g}$. Namely, it is well know that the co-algebra $\mathfrak{g}^*$, i.e., the algebraic dual space of $\mathfrak{g}$, carries the canonical Poisson structure. Namely, Poisson bracket of differentiable functions A, B on $\mathfrak{g}^*$ is analytically given by

$$\{A, B\} := \sigma_k C^k{}_{lm} \frac{\partial A}{\partial \sigma_l} \frac{\partial B}{\partial \sigma_m} \tag{137}$$

where σ_k are linear coordinates in $\mathfrak{g}^*$ and $C^k{}_{lm}$ are structure constants with respect to these coordinates, or more precisely, with respect to the dual linear coordinates in $\mathfrak{g}$. Notice that, being linear functions on $\mathfrak{g}^*$, i.e., elements of the second dual $\mathfrak{g}^{**}$, functions σ_k are canonically identical with some basis vectors e_k in $\mathfrak{g}$ and

$$[e_l, e_m] = e_k C^k{}_{lm}.$$

We might simply use the symbols σ_k instead of e_k in this formula, however, this might be perhaps a bit confusing, although essentially true.

It is obvious that the expression (137) is correct, i.e., coordinate-independent. It is well known that it may be formulated without any use of coordinates. Namely, take differentials $\mathrm{d}A_\sigma$, $\mathrm{d}B_\sigma$ at the point $\sigma \in \mathfrak{g}^*$. Being linear functions on $\mathfrak{g}^* \simeq T_\sigma \mathfrak{g}^*$, they are canonically identical with some elements of $\mathfrak{g}$. We take their bracket/commutator $[\mathrm{d}A_\sigma, \mathrm{d}B_\sigma] \in \mathfrak{g}$ and evaluate the one-form $\sigma \in \mathfrak{g}^*$ on this vector, $\langle \sigma, [\mathrm{d}A_\sigma, \mathrm{d}B_\sigma] \rangle$. One obtains the prescription assigning a number to any point $\sigma \in \mathfrak{g}^*$. The resulting function is just the value of $\{A, B\}$ at σ

$$\{A, B\}(\sigma) = \langle \sigma, [\mathrm{d}A_\sigma, \mathrm{d}B_\sigma] \rangle \,. \tag{138}$$

The skew-symmetry is obvious and the Jacobi identity follows from the identity satisfied by structure constants, thus, finally from the Jacobi identity in Lie algebra.

It is worth to note that (137) and (138) are defined only for differentiable functions. The associative algebra of smooth functions $C^\infty(\mathfrak{g}^*)$ in the sense of pointwise product becomes simultaneously an infinite-dimensional Lie algebra under Poisson bracket. The two structures are compatible in the sense that the Poisson-bracket ad-operation is a differentiation of the associative algebra

$$\mathrm{ad}_C(AB) = \{C, AB\} = A\{C, B\} + \{C, A\}B = (\mathrm{ad}_C A)\, B + A\, (\mathrm{ad}_C B)\,.$$

The both structures may be transported from the function space over $\mathfrak{g}^*$ into function space over $\mathfrak{g}$ by means of the Fourier transform. The pointwise product in $\mathfrak{g}^*$ becomes the convolution in $\mathfrak{g}$. All relationships are preserved. The new Poisson bracket in $\mathfrak{g}$ is a differentiation of the Abelian convolution.

Let us denote the corresponding Poisson bracket in $\mathfrak{g}$ by $[,]$. More precisely, if F, G are functions on $\mathfrak{g}$ Fourier-expressed as

$$\begin{aligned} F(\overline{\omega}) &= \frac{1}{(2\pi\hbar)^n} \int \widehat{F}(\underline{\sigma})\mathrm{e}\left(\frac{\mathrm{i}}{\hbar}\,\underline{\sigma}\cdot\overline{\omega}\right)\mathrm{d}_n\underline{\sigma} \\ G(\overline{\omega}) &= \frac{1}{(2\pi\hbar)^n} \int \widehat{G}(\underline{\sigma})\mathrm{e}\left(\frac{\mathrm{i}}{\hbar}\,\underline{\sigma}\cdot\overline{\omega}\right)\mathrm{d}_n\underline{\sigma} \end{aligned}$$

then their bracket is defined as

$$[F, G]\,(\overline{\omega}) = \frac{1}{(2\pi\hbar)^n} \int \{\widehat{F}, \widehat{G}\}(\underline{\sigma})\mathrm{e}\left(\frac{\mathrm{i}}{\hbar}\,\underline{\sigma}\cdot\overline{\omega}\right)\mathrm{d}_n\underline{\sigma}.$$

One can show that

$$[F, G] = \frac{1}{\mathrm{i}\hbar}\,(\mathcal{A}_a F) * (\omega^a G) = \frac{1}{\mathrm{i}\hbar}\mathcal{A}_a\,(F * \omega^a G)\,. \tag{139}$$

Concerning the last formula, let us notice that

$$\mathcal{A}_a\,(f * g) = (\mathcal{A}_a f) * g + f * (\mathcal{A}_a g)$$

but it may be also shown that for any G

$$\mathcal{A}_a\,(\omega^a G) = 0.$$

This explains why only one term appears in the middle expression in (139). Another, equivalent expression for $[F, G]$ is

$$[F, G] = -\frac{1}{\mathrm{i}\hbar}\mathcal{A}_a\,((\omega^a F) * G) = -\frac{1}{\mathrm{i}\hbar}\,(\omega^a F) * (\mathcal{A}_a G)\,. \tag{140}$$

Therefore, the more symmetric formula for $[F, G]$ would be

$$\begin{aligned} [F, G] &= \frac{1}{\mathrm{i}\hbar}\,((\mathcal{A}_a F) * (\omega^a G) - (\omega^a F) * (\mathcal{A}_a G)) \\ &= \frac{1}{\mathrm{i}\hbar}\,((\mathcal{A}_a F) * (\omega^a G) - (\mathcal{A}_a G) * (\omega^a F))\,. \end{aligned} \tag{141}$$

Let us stress here some subtle point concerning the relationship between symbols k^a, ω^a. Roughly speaking, they denote almost the same, however, some delicate difference in their meaning should be noted. In (72) the canonical coordinates k^a are analytically used as coefficients at the basic elements e_a of the Lie algebra $\mathfrak{g}$.

Being used as a parametrization of $\mathfrak{g}$, they are functions on the group manifold G, in general in a local sense. The exponential mapping e of $\mathfrak{g}$ into G establishes a correspondence between k^a and ω^a, namely, $\omega^a = k^a \circ \mathrm{e}$, when carefully taking domains into account. One must remember however that strictly speaking, k^a as functions on G are defined locally and the range of their values is not identical with $\mathbb{R}^n$. Unlike this, ω^a are global linear coordinates on the linear space $\mathfrak{g}$. Interpretation of functions on G in terms of functions on $\mathfrak{g}$ is also local. As a rule, the global identification fails, even because of simple topological reasons. The point is, however, that in the quasiclassical limit these obstacles become inessential. In this limit we deal with "large quantum numbers", i.e., with "quickly oscillating" functions. One performs some truncation or cut-off procedure, namely, the total group algebra over G is replaced by its subalgebra composed of ideals $M(\alpha)$ the generating units $\varepsilon(\alpha)$ of which have the number of nods above some fixed value. The higher is the truncation threshold, the more is the essential behaviour of admissible functions concentrated in a small neighbourhood of the group unity e. The admissible functions on G practically vanish far away from e, and "do not feel" the topology of G. They may be in a good approximation represented by functions on $\mathfrak{g}$, thus, on a linear space. More precisely, it is so for functions superposed in a quasiclassical way of the basic quickly oscillating functions $\varepsilon(\alpha)_{ij}$. By that we mean that the combination coefficients $C(\alpha)_{ij}$ are concentrated in a "wide range" of the label α and are "slowly varying" within that range. To be more (even if roughly) rigorous with such statements, one must specify what is meant when we say that the labels α, β are nearby. Simply we mean then that the numbers of nodes of $\varepsilon(\alpha)$, $\varepsilon(\beta)$ are nearby (roughly speaking, the corresponding quantum numbers are nearby). Functions on G constructed according to such prescription may be reasonably represented by functions on the Lie algebra $\mathfrak{g}$. Operations in the group algebra of G may be approximated by certain operations in the group algebra of $\mathfrak{g}$, where, just as above, $\mathfrak{g}$ is interpreted as an Abelian additive Lie group. Continuous Fourier expansion approximates in a satisfactory way the discrete Peter-Weyl expansion on the compact group G. Expanding in the convolution formulas the group multiplication rule in Taylor series and retaining the lowest-order terms, we obtain some asymptotic approximate formulas, namely

$$F \underset{G}{*} H \approx F \underset{\mathfrak{g}}{*} H + \frac{\mathrm{i}\hbar}{2}[F, H] \tag{142}$$

where $[F, H]$ is just (140), (141) and the symbols $\underset{G}{*}$, $\underset{\mathfrak{g}}{*}$ denote respectively convolutions in the sense of G and $\mathfrak{g}$ (as an additive group). The use of the same symbols F, H on the left and right sides of (142) is rough, however, the meaning is obvious: just the "identification" in terms of the exponential map. In the lowest order of approximation, the quantum Poisson bracket is expressed as follows

$$\{F, H\}_{\mathrm{q}} = \frac{1}{\mathrm{i}\hbar}\left(F \underset{G}{*} H - H \underset{G}{*} F\right) \approx [F, H]. \tag{143}$$

We can notice that (142) and (143) is a counterpart of the well-known quasiclassical expansion of star products, first of all, the Weyl-Moyal product.

Some more details will be presented when discussing the physically important special case $G = \mathrm{SU}(2)$ or $G = \mathrm{SO}(3, \mathbb{R})$, i.e., quantum description of angular momentum.

4. Group Algebra $\mathfrak{su}(2)$, Quantum Angular Momentum and Quasiclassical Asymptotics

In the previous sections of this paper we have investigated some general problems of the formulation of quantum mechanics based on the H^+-algebras. In particular, we reviewed their important subclass, namely, the associative convolution algebras of functions on locally compact topological groups, first of all, Lie groups. In the final section below we concentrate on the main subject of this paper, namely, on the theory of systems with quantum angular momentum. It is not decided what is the nature of this angular momentum; it may be either orbital or spin or some their superposition. The nature of the system as that of angular momentum is specified by using the convolution algebra of once integrable functions on the group $\mathrm{SU}(2)$.

We begin with the usual expressions involving the Pauli matrices and canonical coordinates of the first kind on $\mathrm{SU}(2)$, i.e., components of the rotation vector, admitting the doubled range of the angle of rotation. We also mention about the projective parametrization based on the so-called vector of finite rotations, when during the multiplication of matrices some purely algebraic rule of computation of parameters is used. Operators $\mathcal{L}_a$, $\mathcal{R}_a$, $\mathcal{A}_a$, introduced previously, i.e., generators of the left and right multiplicative argument-wise action of $\mathrm{SU}(2)$ and its quotient $\mathrm{SO}(3, \mathbb{R})$, are below explicitly expressed in terms of partial differentiation with respect to the group coordinates on $\mathrm{SU}(2)$. Some algebraic formulas for the action of those operators on our configuration space functions and the resulting differential equations satisfied by the unitary irreducible matrix elements $D(j)_{mn}$ and magnetic multipoles $Q^p{}_{kl}$ are below discussed.

As it is well known, there are various objections against the usage of the $\hbar \to 0$ limit transition from the quantum to classical mechanics. In our $\mathrm{SU}(2)$-approach to the theory of angular momentum this problem is particularly essential because, as a matter of fact, apparently there is no use of the Planck constant at all. The method of quasiclassical analysis we suggest below is based on some other limit transition. Namely, instead of the $\hbar \to 0$ procedure, we perform the procedure of eliminating the low quantum numbers in the group algebra on $\mathrm{SU}(2)$. So, the problem is to fix some value of L and to investigate only the sub-algebra of $L(\mathrm{SU}(2))$ obtained as the direct sum of all two-sided ideals with the value of the angular momentum $j > L$. And then we perform the limit transition with $L \to \infty$.

After some manipulations one obtains in this "classical limit" a Poisson system in the Lie co-algebra of $\mathrm{SU}(2)$. In the special case of the usual dipole model, one obtains the traditional classical equations of motion. If the model is more complicated, one obtains models with Hamiltonians containing, e.g., higher-order multipole magnetic moments. It is interesting that when we perform the quasiclassical analysis in this sense, then some classically strange maxima/minima of functions on $\mathrm{SU}(2)$ for $k = 2\pi$ approximately cancel each other for the neighbouring values of large scalar angular moments $j, j+1$.

4.1. Lie Algebra of $\mathrm{SU}(2)$ and $\mathrm{SO}(3,\mathbb{R})$

Theory of angular momentum is based on the group $\mathrm{SU}(2)$ and its quotient, i.e., $\mathrm{SO}(3,\mathbb{R}) = \mathrm{SU}(2)/\mathbb{Z}_2$ (see, e.g. [14, 15]). The two-element centre and maximal normal divisor $\mathbb{Z}_2$ of the simply-connected group $\mathrm{SU}(2)$ is given by

$$\mathbb{Z}_2 = \{I_2, -I_2\}$$

where I_2 is the 2×2 unit matrix.

Let σ_a, $a = 1, 2, 3$ denote Pauli matrices in the following convention

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad \sigma_2 = \begin{bmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{bmatrix}, \qquad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

They are basic traceless Hermitian 2×2-matrices. The Lie algebra of $\mathrm{SU}(2)$, $\mathfrak{su}(2)$, consists of anti-Hermitian traceless matrices. The basic ones are chosen as

$$e_a := \frac{1}{2\mathrm{i}} \sigma_a. \tag{144}$$

The corresponding structure constants are given by the Ricci symbol, more precisely

$$[e_a, e_b] = \varepsilon^c{}_{ab} e_c$$

where ε_{abc} is just the totally antisymmetric Ricci symbol, $\varepsilon_{123} = 1$, and the raising/lowering of indices is meant here in the sense of the "Kronecker delta" δ_{ab} as the standard metric of $\mathbb{R}^3$. So, this shift of indices is here analytically a purely "cosmetic" procedure, however we use it to follow the standard convention.

We know that $\mathrm{SU}(2)$ is the universal $2:1$ covering group of $\mathrm{SO}(3,\mathbb{R})$, the proper orthogonal group in $\mathbb{R}^3$. The projection epimorphism

$$\mathrm{SU}(2) \ni u \mapsto R = \mathrm{p}(u) \in \mathrm{SO}(3,\mathbb{R})$$

is given by

$$u e_b u^{-1} = u e_b u^{+} = e_a R^a{}_b. \tag{145}$$

With respect to the basis (144) the Killing metric γ has the components

$$\gamma_{ab} = -2\delta_{ab}.$$

Its negative definiteness is due to the compactness of the simple algebra/group $\mathfrak{su}(2)/\mathrm{SU}(2)$. For practical purposes one eliminates the factor (-2) and takes the metric

$$\Gamma_{ab} = -\frac{1}{2}\gamma_{ab} = \delta_{ab}. \tag{146}$$

In terms of the canonical coordinates of the first kind

$$u(\overline{k}) = \mathrm{e}\left(k^a e_a\right) = \cos\frac{k}{2} I_2 - \frac{\mathrm{i}}{k}\sin\frac{k}{2} k^a \sigma_a \tag{147}$$

where k denotes the Euclidean length of the vector $\overline{k} \in \mathbb{R}^3$

$$k = \sqrt{\overline{k}\cdot\overline{k}} = \sqrt{\delta_{ab}k^a k^b}.$$

Its range is $[0, 2\pi]$ and the range of the unit vector (versor) $\overline{n} := \overline{k}/k$ is the total unit sphere $S^2(0,1) \subset \mathbb{R}^3$. This coordinate system is singular at $k = 0$, $k = 2\pi$, where

$$u(0\overline{n}) = I_2, \qquad u(2\pi\overline{n}) = -I_2$$

for any $\overline{n} \in S^2(0,1)$. Of course, the formula (147) remains meaningful for $k > 2\pi$, however, the "former" elements of $\mathrm{SU}(2)$ are then repeated.

Sometimes one denotes

$$\sigma_0 = I_2, \qquad e_0 = \frac{1}{2}I_2.$$

Then (147) may be written down as follows

$$u = \xi^\mu(\overline{k})\,(2e_\mu) \tag{148}$$

where the summation convention is meant over $\mu = 0, 1, 2, 3$

$$\left(\xi^0\right)^2 + \left(\xi^1\right)^2 + \left(\xi^2\right)^2 + \left(\xi^3\right)^2 = 1 \tag{149}$$

and this formula together with the structure of parametrization (147), (148) tells us that $\mathrm{SU}(2)$ is the unit sphere $S^3(0,1)$ in $\mathbb{R}^4$. Roughly speaking, $k = 0$ is the "north pole" and $k = 2\pi$ is the corresponding "south pole".

This "pseudo-relativistic" notation is rather misleading. The point is that the matrices σ_μ, e_μ above are used to represent linear mappings in $\mathbb{C}^2$, i.e., mixed tensors in $\mathbb{C}^2$. In the relativistic theory of spinors, e.g., in Lagrangians for (anti)neutrino fields, σ_μ are used as matrices of sesqulinear Hermitian forms, thus, twice covariant tensors on $\mathbb{C}^2$. The space of such forms carries an intrinsic conformal-Minkowskian structure (Minkowskian up to the normalization of the scalar product). And then σ_μ form a Lorentz-ruled multiplet. This is seen in the standard procedure of using $\mathrm{SL}(2,\mathbb{C})$ as the universal covering of the restricted Lorentz group $\mathrm{SO}^{\uparrow}(1,3)$, namely

$$a\sigma_{\mu}a^{+} = \sigma_{\nu}\Lambda^{\nu}{}_{\mu}$$

describes the covering assignment

$$\mathrm{SL}(2,\mathbb{C}) \ni a \mapsto \Lambda \in \mathrm{SO}^{\uparrow}(1,3).$$

The four-dimensional quantity $\left(\xi^0,\xi^1,\xi^2,\xi^3\right)$ in (148), (149) may be also interpreted in terms of the group $\mathrm{SO}(4,\mathbb{R})$ and its covering group, however, this interpretation is relatively complicated and must not be confused with the relativistic aspect of the quadruplet of σ_{μ}-matrices as analytical representants of sesquilinear forms.

The Lie algebra of $\mathrm{SO}(3,\mathbb{R})$, $\mathfrak{so}(3,\mathbb{R})$, consists of 3×3 skew-symmetric matrices with real entries. The standard choice of basis of $\mathfrak{so}(3,\mathbb{R})$, adapted to (144) and to the procedure (145), is given by matrices E_a, $a=1,2,3$, with entries

$$\left(E_a\right)^{b}{}_{c} := -\varepsilon_{a}{}^{b}{}_{c}$$

where again ε_{abc} is the totally antisymmetric Ricci symbol, and indices are "cosmetically" shifted with the help of the Kronecker symbol. Then, of course

$$[E_a, E_b] = \varepsilon^{c}{}_{ab}E_c.$$

In spite of having isomorphic Lie algebras, the groups $\mathrm{SU}(2)$ and $\mathrm{SO}(3,\mathbb{R}) \simeq \mathrm{SU}(2)/\mathbb{Z}_2$ are globally different. The main topological distinction is that $\mathrm{SU}(2)$ is simply connected and $\mathrm{SO}(3,\mathbb{R})$ is doubly connected.

Using canonical coordinates of the first kind, we have in analogy to (147) the formula

$$R(\overline{k}) = \mathrm{e}\left(k^a E_a\right).$$

Because of the obvious reasons, known from elementary geometry and mechanics, $\overline{k}$ is referred to as the rotation vector, $k=\sqrt{\overline{k}\cdot\overline{k}}$ is the rotation angle, and the unit vector (versor)

$$\overline{n} = \frac{\overline{k}}{k}$$

is the oriented rotation axis. We use the all standard concepts and symbols of the vector calculus in $\mathbb{R}^3$, in particular, scalar products $\overline{a}\cdot\overline{b}$ and vector products $\overline{a}\times\overline{b}$. The rotation angle k runs over the range $[0,\pi]$ and the antipodal points on the sphere $S^2(0,\pi)\subset\mathbb{R}^3$ are identified, they describe the same rotation

$$R\left(\pi\overline{n}\right) = R\left(-\pi\overline{n}\right). \tag{150}$$

Therefore, this sphere, taken modulo the antipodal identification, is the manifold of non-trivial square roots of the identity I_3 in $\mathrm{SO}(3,\mathbb{R})$. It is seen in this picture that $\mathrm{SO}(3,\mathbb{R})$ is doubly connected, because any curve in the ball $K^2(0,1)\subset\mathbb{R}^3$ joining two antipodal points on the boundary $S^2(0,1)$ is closed under this identification, i.e., it is a loop, but it cannot be continuously contracted into a single point.

It is worth to note that formally the values $k > \pi$ are admitted, however, they correspond to rotations by $k < \pi$, taken earlier into account. By abuse of language, in $\mathrm{SU}(2)$ the quantities $\overline{k}$, k are also referred to as the rotation vector and rotation angle. But one must "rotate" by 4π to go back to the same situation, not by 2π. The matrix of $R(\overline{k})$ is given by

$$R\left(\overline{k}\right)^a{}_b = \cos k \delta^a{}_b + \frac{1}{k^2}\left(1 - \cos k\right) k^a k_b + \frac{1}{k} \sin k \varepsilon^a{}_{bc} k^c$$

i.e.,

$$R\left(\overline{k}\right)\overline{x} = \cos k \overline{x} + \frac{1 - \cos k}{k^2}\left(\overline{k}\cdot\overline{x}\right)\overline{k} + \frac{\sin k}{k}\overline{k}\times\overline{x}$$

or, symbolically

$$R(\overline{k})\cdot\overline{x} = \overline{x} + \overline{k}\times\overline{x} + \frac{1}{2!}\overline{k}\times\left(\overline{k}\times\overline{x}\right) + \cdots + \frac{1}{n!}\overline{k}\times\left(\overline{k}\times\cdots\times\left(\overline{k}\times\overline{x}\right)\cdots\right) + \ldots$$

Let us distinguish between two ways of viewing, representing geometry of $\mathrm{SU}(2)$ and $\mathrm{SO}(3,\mathbb{R})$ in terms of some subsets in $\mathbb{R}^3$ as the space of rotation vectors $\overline{k}$ or, alternatively, in terms of closed submanifolds and their quotients in $\mathbb{R}^4$.

As seen from (149), $\mathrm{SU}(2)$ is a unit sphere $S^3(0,1) \subset \mathbb{R}^4$, $\mathrm{SO}(3,\mathbb{R})$ is obtained by the antipodal identification. Then $\mathrm{SO}(3,\mathbb{R})$ is doubly connected because the curves on $S^3(0,1)$ joining antipodal points project to the quotient manifold onto closed loops non-contractible to points in a continuous way. In $\mathbb{R}^3$ the group $\mathrm{SU}(2)$ is represented by the ball $K^2(0,2\pi)$ and the whole shell $S^2(0,2\pi)$ represents the single point $-I_2 \in \mathrm{SU}(2)$. Then $\mathrm{SO}(3,\mathbb{R})$ is pictured as the ball $K^2(0,\pi) \subset \mathbb{R}^3$ with the antipodal identification of points on the shell $S^2(0,\pi)$, cf. (150). This exhibits the identification of $\mathrm{SO}(3,\mathbb{R})$ with the projective space $\mathbb{RP}^3$. The antipodally identified points on $S^2(0,\pi)$ represent the improper points at infinity in $\mathbb{R}^3$.

For certain reasons, both practical and deeply geometrical, it is convenient to use also another parametrization of $\mathrm{SO}(3,\mathbb{R})$, using so-called vector of finite rotation

$$\overline{\varkappa} = \frac{2}{k}\,\mathrm{tg}\,\frac{k}{2}\,\overline{k}. \tag{151}$$

One can note that in the neighbourhood of group identity, when $\overline{k} \approx 0$, $\overline{\varkappa}$ differs from $\overline{k}$ by higher-order quantity. The practical advantage of $\overline{\varkappa}$ is that the composition rule and the action of rotations are described by very simple and purely algebraic expressions

$$R\left(\overline{\varkappa}_1\right) R\left(\overline{\varkappa}_2\right) = R\left(\overline{\varkappa}\right)$$

where

$$\overline{\varkappa} = \left(1 - \frac{1}{4}\overline{\varkappa}_1 \cdot \overline{\varkappa}_2\right)^{-1} \left(\overline{\varkappa}_1 + \overline{\varkappa}_2 + \frac{1}{2}\overline{\varkappa}_1 \times \overline{\varkappa}_2\right)$$

$$R\left[\overline{\varkappa}\right]\overline{x} = \overline{x} + \left(1 + \frac{1}{4}\overline{\varkappa}^2\right)^{-1} \overline{\varkappa} \times \left(\overline{x} + \frac{1}{2}\overline{\varkappa} \times \overline{x}\right).$$

An important property of this parametrization is that it describes the projective mapping of $\mathrm{SO}(3,\mathbb{R})$ onto the projective space $\mathbb{RP}^3$. The one-parameter subgroups and their cosets in $\mathrm{SO}(3,\mathbb{R})$ are mapped onto straight-lines in $\mathbb{R}^3$. The manifold of π-rotations (non-trivial square roots of identity) is mapped onto the set of improper points in $\mathbb{RP}^3$, i.e., it "blows up" to infinity.

The homomorphism (145) of $\mathrm{SU}(2)$ onto $\mathrm{SO}(3,\mathbb{R})$, $u \mapsto R(u)$, may be alternatively described in terms of inner automorphisms of $\mathrm{SU}(2)$ and the rotation-vector parametrization

$$uv(\overline{k})u^{-1} = v\left(R(u)\overline{k}\right), \qquad u \in \mathrm{SU}(2). \tag{152}$$

Roughly speaking, inner automorphisms in $\mathrm{SU}(2)$ result in rotation of the rotation vector. The same holds in $\mathrm{SO}(3,\mathbb{R})$

$$OR(\overline{k})O^{-1} = R\left(O\overline{k}\right), \qquad O \in \mathrm{SO}(3,\mathbb{R}).$$

Therefore, inner automorphisms preserve the length k of the rotation vector $\overline{k}$, and the classes of conjugate elements are characterized by the fixed values of the rotation angle (but all possible oriented rotation axes $\overline{n}$). This means that in the above description they are represented by spheres $S^2(0,k) \subset \mathbb{R}^3$ in the space of rotation vectors. There are two one-element singular equivalence classes in $\mathrm{SU}(2)$, namely $\{I_2\}$, $\{-I_2\}$ corresponding respectively to $k=0$, $k=2\pi$. Of course, in $\mathrm{SO}(3,\mathbb{R})$ there is only one singular class $\{I_3\}$. More precisely, in $\mathrm{SO}(3,\mathbb{R})$ the class $k=\pi$ is not the sphere, but rather its antipodal quotient, so-called elliptic space. The idempotents $\varepsilon(\alpha)$/characters $\chi(\alpha) = \varepsilon(\alpha)/n(\alpha)$ and all central functions of the group algebras of $\mathrm{SU}(2)$ and $\mathrm{SO}(3,\mathbb{R})$ are constant on the spheres $S^2(0,k)$, i.e., depend on $\overline{k}$ only through the rotation angle k. In many problems it is convenient to parametrize $\mathrm{SU}(2)$ and $\mathrm{SO}(3,\mathbb{R})$ with the help of spherical variables k, θ, φ in the space $\mathbb{R}^3$ of the rotation vector $\overline{k}$. Historically the most popular parametrization is that based on the **Euler angles** (φ,ϑ,ψ). It is given by

$$u[\varphi,\vartheta,\psi] = u(0,0,\varphi)u(0,\vartheta,0)u(0,0,\psi) \tag{153}$$

$$R[\varphi,\vartheta,\psi] = R(0,0,\varphi)R(0,\vartheta,0)R(0,0,\psi). \tag{154}$$

Historically φ, ϑ, ψ are referred to respectively as the precession angle, nutation angle and the rotation angle.

Sometimes one uses $u(\vartheta,0,0)$, $R(\vartheta,0,0)$ instead $u(0,\vartheta,0)$, $R(0,\vartheta,0)$ in (153), (154). The only thing which matters here is that one uses the product of three elements which belong to two one-parameter subgroups. The Euler angles are practically important in gyroscopic problems. Canonical parametrization of the second kind

$$u(\alpha,\beta,\gamma) = u(\alpha,0,0)u(0,\beta,0)u(0,0,\gamma)$$

are not very popular. One must say, however, that many formulas have the same form in variables (φ,ϑ,ψ) and (α,β,γ).

4.2. Irreducible Unitary Representations

It is well known that in $\mathrm{SU}(2)$ irreducible unitary representations, or rather their equivalence classes, are labelled by non-negative integers and half-integers,

$$\alpha = j = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$$

i.e.,

$$\Omega = \{0\} \bigcup \frac{\mathbb{N}}{2}$$

where $\mathbb{N}$ denotes the set of naturals (positive integers). And

$$n(\alpha) = n(j) = 2j + 1.$$

On $\mathrm{SO}(3,\mathbb{R})$ one uses integers only

$$\alpha = j = 0,1,2,\ldots, \qquad \Omega = \{0\} \bigcup \mathbb{N}.$$

For any $\alpha = j$, there is only one irreducible representation of dimension

$$n(\alpha) = n(j) = 2j + 1$$

i.e., only one up to equivalence. It is not the case for many practically important groups, e.g., for $\mathrm{SU}(3)$ or for the non-compact group $\mathrm{SL}(2,\mathbb{C})$.

Historically, the irreducible representations of $\mathrm{SU}(2)$, $\mathrm{SO}(3,\mathbb{R})$, $\mathrm{SL}(2,\mathbb{C})$, and $\mathrm{SO}^{\uparrow}(1,3)$ were found in two alternative ways

i) algebraic one, based on taking the tensor products of fundamental representation (by itself),

ii) differential one, based on solving differential equations like (106)–(111), (113), (107).

The left and right generators $\mathcal{L}_a$, $\mathcal{R}_a$, i.e., respectively the basic right- and left-invariant vector fields, are analytically given by

$$\mathcal{L}_a = \frac{k}{2}\mathrm{ctg}\frac{k}{2}\frac{\partial}{\partial k^a} + \left(1 - \frac{k}{2}\mathrm{ctg}\frac{k}{2}\right)\frac{k_a}{k}\frac{k^b}{k}\frac{\partial}{\partial k^b} + \frac{1}{2}\varepsilon_{ab}{}^{c}k^b\frac{\partial}{\partial k^c} \tag{155}$$

$$\mathcal{R}_a = \frac{k}{2}\mathrm{ctg}\frac{k}{2}\frac{\partial}{\partial k^a} + \left(1 - \frac{k}{2}\mathrm{ctg}\frac{k}{2}\right)\frac{k_a}{k}\frac{k^b}{k}\frac{\partial}{\partial k^b} - \frac{1}{2}\varepsilon_{ab}{}^{c}k^b\frac{\partial}{\partial k^c} \tag{156}$$

and therefore

$$\mathcal{A}_a = \mathcal{L}_a - \mathcal{R}_a = \varepsilon_{ab}{}^c k^b \frac{\partial}{\partial k^c}.$$

In terms of explicitly written components

$$\begin{aligned}
\mathcal{L}^i{}_a &= \frac{k}{2}\mathrm{ctg}\frac{k}{2}\delta^i{}_a + \left(1 - \frac{k}{2}\mathrm{ctg}\frac{k}{2}\right)\frac{k_a}{k}\frac{k^i}{k} + \frac{1}{2}\varepsilon_{ab}{}^i k^b \\
\mathcal{R}^i{}_a &= \frac{k}{2}\mathrm{ctg}\frac{k}{2}\delta^i{}_a + \left(1 - \frac{k}{2}\mathrm{ctg}\frac{k}{2}\right)\frac{k_a}{k}\frac{k^i}{k} - \frac{1}{2}\varepsilon_{ab}{}^i k^b \\
\mathcal{A}^i{}_a &= \varepsilon_{ab}{}^i k^b.
\end{aligned}$$

The shift of indices is meant here in the Kronecker-delta sense.

The corresponding Cartan one-forms are given by

$$\begin{aligned}
\mathcal{L}^a &= \frac{\sin k}{k}\mathrm{d}k^a + \left(1 - \frac{\sin k}{k}\right)\frac{k^a}{k}\frac{k_b}{k}\mathrm{d}k^b + \frac{2}{k^2}\sin^2\frac{k}{2}\varepsilon^a{}_{bc}k^b\mathrm{d}k^c \\
\mathcal{R}^a &= \frac{\sin k}{k}\mathrm{d}k^a + \left(1 - \frac{\sin k}{k}\right)\frac{k^a}{k}\frac{k_b}{k}\mathrm{d}k^b - \frac{2}{k^2}\sin^2\frac{k}{2}\varepsilon^a{}_{bc}k^b\mathrm{d}k^c
\end{aligned}$$

i.e., in terms of the components

$$\mathcal{L}^a{}_i = \frac{\sin k}{k}\delta^a{}_i + \left(1 - \frac{\sin k}{k}\right)\frac{k^a}{k}\frac{k_i}{k} + \frac{2}{k^2}\sin^2\frac{k}{2}\varepsilon^a{}_{bi}k^b \tag{157}$$

$$\mathcal{R}^a{}_i = \frac{\sin k}{k}\delta^a{}_i + \left(1 - \frac{\sin k}{k}\right)\frac{k^a}{k}\frac{k_i}{k} - \frac{2}{k^2}\sin^2\frac{k}{2}\varepsilon^a{}_{bi}k^b. \tag{158}$$

The central functions on $\mathrm{SU}(2)$ and on $\mathrm{SO}(3,\mathbb{R})$, in particular the idempotents $\varepsilon(j)$/characters $\chi(j)$ satisfy the obvious differential equations

$$\mathcal{A}_a f = 0, \qquad \text{i.e.,} \qquad \mathcal{L}_a f = \mathcal{R}_a f, \qquad a = 1, 2, 3.$$

Then the analytical formulas (155)–(158) are formally valid both on $\mathrm{SU}(2)$ and $\mathrm{SO}(3,\mathbb{R})$, and in general the calculus on $\mathrm{SU}(2)$ is simpler than that on $\mathrm{SO}(3,\mathbb{R})$. It is convenient to rewrite the formulas (155)-(158) so as to express them explicitly in terms of the angular and radial differential operations in the space of rotation vectors $\overline{k}$. After simple calculations one obtains

$$\begin{aligned}
\mathcal{L}_a &= n_a\frac{\partial}{\partial k} - \frac{1}{2}\mathrm{ctg}\frac{k}{2}\varepsilon_{abc}n^b\mathcal{A}^c + \frac{1}{2}\mathcal{A}_a \\
\mathcal{R}_a &= n_a\frac{\partial}{\partial k} - \frac{1}{2}\mathrm{ctg}\frac{k}{2}\varepsilon_{abc}n^b\mathcal{A}^c - \frac{1}{2}\mathcal{A}_a \\
\mathcal{L}_a &= n^a\mathrm{d}k + 2\sin^2\frac{k}{2}\varepsilon^a{}_{bc}n^b\mathrm{d}n^c + \sin k\mathrm{d}n^a \\
\mathcal{R}_a &= n^a\mathrm{d}k - 2\sin^2\frac{k}{2}\varepsilon^a{}_{bc}n^b\mathrm{d}n^c + \sin k\mathrm{d}n^a.
\end{aligned}$$

Using the $\mathbb{R}^3$-vector notation, including also the vectors with operator components, we can denote briefly, without using indices and labels

$$\overline{\mathcal{L}} = \overline{n}\frac{\partial}{\partial k} - \frac{1}{2}\mathrm{ctg}\frac{k}{2}\overline{n} \times \overline{\mathcal{A}} + \frac{1}{2}\overline{\mathcal{A}} \tag{159}$$

$$\overline{\mathcal{R}} = \overline{n}\frac{\partial}{\partial k} - \frac{1}{2}\mathrm{ctg}\frac{k}{2}\overline{n} \times \overline{\mathcal{A}} - \frac{1}{2}\overline{\mathcal{A}} \tag{160}$$

$$\underline{\mathcal{L}} = \overline{n}\mathrm{d}k + 2\sin^2\frac{k}{2}\overline{n} \times \mathrm{d}\overline{n} + \sin k\mathrm{d}\overline{n} \tag{161}$$

$$\underline{\mathcal{R}} = \overline{n}\mathrm{d}k - 2\sin^2\frac{k}{2}\overline{n} \times \mathrm{d}\overline{n} + \sin k\mathrm{d}\overline{n} \tag{162}$$

$$\overline{\mathcal{A}} = \overline{k} \times \overline{\nabla} \tag{163}$$

where $\overline{\nabla}$ denotes the Euclidean gradient operator.

Let us note the following interesting and suggestive duality relations

$$\langle \mathrm{d}k, \mathcal{A}_a \rangle = \mathcal{A}_a k = 0, \qquad \left\langle \mathrm{d}k, \frac{\partial}{\partial k} \right\rangle = 1$$

$$\langle \mathrm{d}n_a, \mathcal{A}_b \rangle = \mathcal{A}_b n_a = \varepsilon_{abc} n^c, \qquad \left\langle \mathrm{d}n_a, \frac{\partial}{\partial k} \right\rangle = \frac{\partial n_a}{\partial k} = 0.$$

In $\mathbb{R}^3$, considered as an Abelian group under addition of vectors, the right-invariant fields coincide with the left-invariant ones, and when using spherical variables we have then

$$\overline{\mathcal{L}} = \overline{\mathcal{R}} = \overline{\nabla} = \overline{n}\frac{\partial}{\partial r} - \frac{1}{r}\overline{n} \times \overline{\mathcal{A}}(\overline{r}) \tag{164}$$

$$\underline{\mathcal{L}} = \underline{\mathcal{R}} = \mathrm{d}\overline{r} = \overline{n}\mathrm{d}r + r\mathrm{d}\overline{n}. \tag{165}$$

This is in agreement with the formulas (159)–(163), namely, in a small neighbourhood of the group identity $I_2 \in \mathrm{SU}(2)$, i.e., for $\overline{k} \approx \overline{0}$, expressions (159)–(163) up to higher-order terms in $\overline{k}$, one obtains

$$\overline{\mathcal{L}} \approx \overline{\mathcal{R}} \approx \overline{\nabla}_{\overline{k}} = \overline{n}\frac{\partial}{\partial k} - \frac{1}{k}\overline{n} \times \overline{\mathcal{A}}$$

$$\underline{\mathcal{L}} \approx \underline{\mathcal{R}} \approx \mathrm{d}\overline{k} = \overline{n}\mathrm{d}k + k\mathrm{d}\overline{n}.$$

The quantities $\overline{n}$, $\mathrm{d}\overline{n}$, $\overline{\mathcal{A}}$ are non-sensitive to the asymptotics $k \to 0$, because they are purely angular (θ, φ) variables, independent of k.

$\mathrm{SU}(2)$ is the sphere $S^3(0,1)$ in $\mathbb{R}^4$. Taking the sphere of radius R, $S^3(0,R) \subset \mathbb{R}^4$, and performing the limit transition $R \to \infty$, one obtains also the relationships (164), (165) as an asymptotic limit.

The Killing metric tensor with the modified normalization (146) is given by

$$g_{ij} = \frac{4}{k^2}\sin^2\frac{k}{2}\delta_{ij} + \left(1 - \frac{4}{k^2}\sin^2\frac{k}{2}\right)\frac{k^i}{k}\frac{k^j}{k} \tag{166}$$

and its contravariant inverse by

$$g^{ij} = \frac{k^2}{4\sin^2 k/2}\delta^{ij} + \left(1 - \frac{k^2}{4\sin^2 k/2}\right)\frac{k^i}{k}\frac{k^j}{k}.$$

The corresponding metric element may be concisely written as

$$\mathrm{d}s^2 = \mathrm{d}k^2 + 4\sin^2\frac{k}{2}\left(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\varphi^2\right) = \mathrm{d}k^2 + 4\sin^2\frac{k}{2}\mathrm{d}\overline{n}\cdot\mathrm{d}\overline{n} \tag{167}$$

or, in a more sophisticated way

$$g = \mathrm{d}k\otimes\mathrm{d}k + 4\sin^2\frac{k}{2}\delta_{AB}\mathrm{d}n^A\otimes\mathrm{d}n^A \tag{168}$$

and, similarly, for the inverse tensor

$$g^{-1} = \frac{\partial}{\partial k}\otimes\frac{\partial}{\partial k} + \frac{1}{4\sin^2 k/2}\delta^{AB}\mathcal{A}_A\otimes\mathcal{A}_B.$$

According to the standard procedure, the volume element on the Riemannian manifold is given by

$$\mathrm{d}\mu\left(\overline{k}\right) = \sqrt{|g|}\mathrm{d}^3\overline{k} = \sqrt{\det\left[g_{ij}\left(\overline{k}\right)\right]}\mathrm{d}^3\overline{k}.$$

It is easy to see that for our normalization of the metric tensor

$$\mathrm{d}\mu\left(\overline{k}\right) = 4\sin^2\frac{k}{2}\sin\theta\mathrm{d}k\mathrm{d}\theta\mathrm{d}\varphi = \frac{4\sin^2 k/2}{k^2}\,\mathrm{d}^3\overline{k} \tag{169}$$

where $\mathrm{d}^3\overline{k}$ is the usual volume element in $\mathbb{R}^3$ as the space of rotation vectors $\overline{k}$. This volume element is identical with that given by (169), (170), (171). The reason is that all these expressions are translationally-invariant and the Haar measure is unique. We assume here that G is unimodular. In fact, we mean only the compact semisimple groups and their products with Abelian groups (clearly, in the latter case it is not the Killing tensor that is meant in the Abelian factor). Nevertheless, any metric meant there is also assumed translationally-invariant, and so the total Riemann measure also coincides with (169), (170), (171).

Let us mention that when the Euler angles $(\varphi, \vartheta, \psi)$ are used as a parametrization, then the Riemann metric is given by

$$\mathrm{d}s^2 = \mathrm{d}\vartheta^2 + \mathrm{d}\varphi^2 + 2\cos\vartheta\mathrm{d}\varphi\mathrm{d}\psi + \mathrm{d}\psi^2. \tag{170}$$

The measure element is then expressed

$$\mathrm{d}\mu\left(\varphi, \vartheta, \psi\right) = \sin\vartheta\mathrm{d}\vartheta\mathrm{d}\varphi\mathrm{d}\psi. \tag{171}$$

The metric element expression (170) may be diagonalized by introducing the new "angles"

$$\alpha = \varphi + \psi, \qquad \beta = \varphi - \psi$$

however, this representation rather is not used practically.

Let us remind that on $\mathrm{SU}(2)$ the range of Euler angles is $[0,4\pi]$ for φ, ψ, and $[0,2\pi]$ for ϑ, on $\mathrm{SO}(3,\mathbb{R})$, it is respectively $[0,2\pi]$ and $[0,\pi]$.

In some of earlier formulas we used the convention of the Haar measure on compact groups normalized to unity, $\mu(G)=1$. When normalized in this way, it will be denoted as μ_1. The label "(1)" will be omitted when the normalization is clear from the context or when there is no danger of confusion.

After elementary integrations we find that on $\mathrm{SU}(2)$ the element of normalized measure is given by

$$\mathrm{d}\mu_{(1)} = \frac{1}{4\pi^2}\sin^2\frac{k}{2}\sin\theta \mathrm{d}k\mathrm{d}\theta\mathrm{d}\varphi = \frac{\sin^2 k/2}{4\pi^2 k^2}\,\mathrm{d}^3\overline{k}.$$

If we used the normalization (169), the "volume" of $\mathrm{SU}(2)$ would be $16\pi^2$. With the same normalization, the volume of $\mathrm{SO}(3,\mathbb{R})$ would be $8\pi^2$. It is intuitively clear: $\mathrm{SU}(2)$ is "twice larger" than $\mathrm{SO}(3,\mathbb{R})$. So, we would have

$$\mathrm{d}\mu_{(1)\mathrm{SO}(3,\mathbb{R})} = \frac{1}{2\pi^2}\sin^2\frac{k}{2}\sin\theta \mathrm{d}k\mathrm{d}\theta\mathrm{d}\varphi = \frac{\sin^2 k/2}{2\pi^2 k^2}\mathrm{d}^3\overline{k}.$$

However, as mentioned, all formulas will be meant in the covering group sense $\mathrm{SU}(2)$.

The metric tensor (166), (167) is conformally flat. It is seen when we introduce some new variables $\overline{\varrho}$ instead of $\overline{k}$, namely

$$\varrho = |\overline{\varrho}| = a\mathrm{tg}\frac{k}{4}, \qquad \frac{\overline{\varrho}}{\varrho} = \frac{\overline{k}}{k} = \overline{n} \tag{172}$$

where a denotes some positive constant. Then (167) becomes

$$\begin{aligned}\mathrm{d}s^2 &= \frac{16a^2}{(a^2+\varrho^2)^2}\left(\mathrm{d}\varrho^2 + \varrho^2\left(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\varphi^2\right)\right)\\ &= \frac{16a^2}{(a^2+\varrho^2)^2}\left(\mathrm{d}\varrho^2 + \varrho^2\mathrm{d}\overline{n}\cdot\mathrm{d}\overline{n}\right)\end{aligned}$$

or, using again the "sophisticated" form (168)

$$g = \frac{16a^2}{(a^2+\varrho^2)^2}\left(\mathrm{d}\varrho\otimes\mathrm{d}\varrho + \varrho^2\delta_{ab}\mathrm{d}n^a\otimes\mathrm{d}n^b\right).$$

Apparently, (172) is a conformal mapping of $\mathrm{SU}(2)$ onto $\mathbb{R}^3$ with its usual Euclidean metric. The ball $K^2(0,2\pi)$ "blows up" to the total $\mathbb{R}^3$ and the sphere $S^2(0,2\pi)$ "blows up" to infinity. In other words, $\mathrm{SU}(2)$ is identified with the one-point compactification of $\mathbb{R}^3$ and the element $-I_2\in\mathrm{SU}(2)$ becomes just the compactifying point. The ball $K^2(0,\pi)$ corresponding to the manifold of $\mathrm{SO}(3,\mathbb{R})$ and its boundary sphere $S^2(0,\pi)$ (non-trivial square-root of identity) become respectively $K^2(0,a)$ and $S^2(0,a)$. If we put $a=\pi$, they are mapped onto themselves.

From the conformal point of view the particular choice of the constant a does not matter.

The projective mapping (151) of $\mathrm{SO}(3,\mathbb{R})$ onto $\mathbb{RP}^3$ maps geodetics of (167) onto straight lines in $\mathbb{R}^3$. However, it is neither isometry nor the conformal transformation, instead we have that

$$\mathrm{d}s^2 = \frac{4}{4+\varkappa^2}\left(\frac{4}{4+\varkappa^2}\mathrm{d}\varkappa^2 + \varkappa^2\left(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\varphi^2\right)\right).$$

On $\mathrm{SU}(2)$ the formulas (82), (84) take on the following form

$$\mathcal{L}^a(u) = R(u)^a{}_b\mathcal{R}^b(u), \qquad \mathcal{L}_a(u) = \mathcal{R}_b(u)R(u)^{-1b}{}_a$$

where the dependence

$$\mathrm{SU}(2) \ni u \mapsto R(u) \in \mathrm{SO}(3,\mathbb{R})$$

is given by (145), (152). In orthonormal coordinates this is the same formula, because inverses of orthogonal matrices coincide with their transposes. The corresponding symmetric operators $\mathbf{\Sigma}_a$, $\widehat{\mathbf{\Sigma}}_a$, denoted respectively by

$$\mathbf{S}_a = \frac{\hbar}{\mathrm{i}}\mathcal{L}_a, \qquad \widehat{\mathbf{S}}_a = \frac{\hbar}{\mathrm{i}}\mathcal{R}_a \tag{173}$$

are interrelated by the same formula

$$\mathbf{S}_a(u) = \widehat{\mathbf{S}}_b(u)R(u)^{-1b}{}_a.$$

When interpreted in terms of action on the wave functions on $\mathrm{SU}(2)$, they are operators of rotational angular momentum (spin) respectively in the spatial and co-moving representations. The corresponding operators of hyperspin

$$\mathbf{\Delta}_a = \frac{\hbar}{\mathrm{i}}\mathcal{A}_a = \frac{\hbar}{\mathrm{i}}\varepsilon_{ab}{}^c k^b\frac{\partial}{\partial k^c}$$

are given by

$$\mathbf{\Delta}_a = \mathbf{S}_a - \widehat{\mathbf{S}}_a, \qquad \mathcal{A}_a = \mathcal{L}_a - \mathcal{R}_a.$$

The term "hyper" is used because this quantity tells us "how much" the spatial components of spin exceed the corresponding laboratory ones. The operators $\mathcal{A}_a$ generate rotations of the rotation vector and this is just the meaning of "hyper".

According to (81), the corresponding classical quantities are given by

$$S_a = p_j\mathcal{L}^j{}_a, \qquad \widehat{S}_a = p_j\mathcal{R}^j{}_a = S_bR^b{}_a$$
$$\Delta_a = p_j\Delta^j{}_a = S_a - \widehat{S}_a = \varepsilon_{ab}{}^ck^bp_c$$

where p_j denote canonical momenta conjugate to k^j or rather to the corresponding generalized velocities $\mathrm{d}k^j/\mathrm{d}t$.

Evaluating differential forms on vector tangent to trajectories in the configuration spaces $\mathrm{SU}(2)$, $\mathrm{SO}(3,\mathbb{R})$, we obtain the following quantities

$$\omega^a = \mathcal{L}^a{}_i\left(\overline{k}\right)\frac{\mathrm{d}k^i}{\mathrm{d}t}, \qquad \widehat{\omega}^a = \mathcal{R}^a{}_i\left(\overline{k}\right)\frac{\mathrm{d}k^i}{\mathrm{d}t}, \qquad \omega^a\left(u,\dot{u}\right) = R(u)^a{}_b\widehat{\omega}^a\left(u,\dot{u}\right).$$

They are respectively spatial (ω^a) and co-moving $(\widehat{\omega}^a)$ components of angular velocity. They are non-holonomic, i.e., fail to be time derivatives of any generalized coordinates. The following duality relations are satisfied

$$s_a\omega^a = \widehat{s}_a\widehat{\omega}^a = p_i\frac{\mathrm{d}k^i}{\mathrm{d}t}.$$

Let us quote the obvious commutators and Poisson brackets

$$\begin{aligned}
&[\mathcal{L}_a,\mathcal{L}_b] = -\,\varepsilon_{ab}{}^c\mathcal{L}_c, && [\mathcal{R}_a,\mathcal{R}_b] = \varepsilon_{ab}{}^c\mathcal{R}_c, && [\mathcal{L}_a,\mathcal{R}_b] = 0\\
&[\mathcal{A}_a,\mathcal{L}_b] = -\,\varepsilon_{ab}{}^c\mathcal{L}_c, && [\mathcal{A}_a,\mathcal{R}_b] = -\,\varepsilon_{ab}{}^c\mathcal{R}_c, && [\mathcal{A}_a,\mathcal{A}_b] = -\,\varepsilon_{ab}{}^c\mathcal{A}_c\\
&\frac{1}{\mathrm{i}\hbar}[\mathbf{S}_a,\mathbf{S}_b] = \varepsilon_{ab}{}^c\mathbf{S}_c, && \frac{1}{\mathrm{i}\hbar}[\widehat{\mathbf{S}}_a,\widehat{\mathbf{S}}_b] = -\,\varepsilon_{ab}{}^c\widehat{\mathbf{S}}_c, && \frac{1}{\mathrm{i}\hbar}[\mathbf{S}_a,\widehat{\mathbf{S}}_b] = 0\\
&\frac{1}{\mathrm{i}\hbar}[\mathbf{\Delta}_a,\mathbf{S}_b] = \varepsilon_{ab}{}^c\mathbf{S}_c, && \frac{1}{\mathrm{i}\hbar}[\mathbf{\Delta}_a,\widehat{\mathbf{S}}_b] = \varepsilon_{ab}{}^c\widehat{\mathbf{S}}_c, && \frac{1}{\mathrm{i}\hbar}[\mathbf{\Delta}_a,\mathbf{\Delta}_b] = -\,\varepsilon_{ab}{}^c\mathbf{\Delta}_c\\
&\{S_a,S_b\} = \varepsilon_{ab}{}^cS_c, && \{\widehat{S}_a,\widehat{S}_b\} = -\,\varepsilon_{ab}{}^c\widehat{S}_c, && \{S_a,\widehat{S}_b\} = 0\\
&\{\Delta_a,S_b\} = \varepsilon_{ab}{}^cS_c, && \{\Delta_a,\widehat{S}_b\} = \varepsilon_{ab}{}^c\widehat{S}_c, && \{\Delta_a,\Delta_b\} = -\,\varepsilon_{ab}{}^c\Delta_c.
\end{aligned}$$

In the enveloping algebras built over Lie algebras of $\mathcal{L}$- and $\mathcal{R}$-operators there exists only one Casimir invariant, namely, the second-order one

$$C(\mathcal{L},2) = C(\mathcal{R},2) = \Delta = \delta^{ab}\mathcal{L}_a\mathcal{L}_b = \delta^{ab}\mathcal{R}_a\mathcal{R}_b. \tag{174}$$

In physical expressions like various kinetic energies and so on, one uses their $(-\hbar^2)$-multiplies

$$\mathbf{S}^2 := C(S,2) = C\left(\widehat{S},2\right) = -\hbar^2\Delta.$$

There is also only one Casimir in the associative algebra generated by the Lie algebra of $\mathcal{A}$-operators

$$\mathcal{A}^2 := C(\mathcal{A},2) = \delta^{ab}\mathcal{A}_a\mathcal{A}_b, \qquad \Delta^2 := -\hbar^2\mathcal{A}^2.$$

After some easy calculations one obtains for (174)

$$\Delta = \frac{\partial^2}{\partial k^2} + \mathrm{ctg}\frac{k}{2}\frac{\partial}{\partial k} + \frac{1}{4\sin^2 k/2}A^2.$$

For obvious reasons, when expressed by the spherical angular variables (θ,φ) in the space of rotation vector $\overline{k}$, Δ^2 has identical form with the operator of the squared magnitude of orbital angular momentum. Its spectrum consists of non-negative numbers $\hbar^2 l(l+1)$, where l denotes non-negative integers, $l = 0,1,2,\ldots$.

As we saw in (112), $\Delta^2 = -\hbar^2 \mathcal{A}^2$ does commute with the Laplace-Beltrami Casimir

$$\mathbf{S}^2 = -\hbar^2 \delta^{ab} \mathcal{L}_a \mathcal{L}_b = -\hbar^2 \delta^{ab} \mathcal{R}_a \mathcal{R}_b$$

so they have common wave functions. Spectrum of the Laplace-Beltrami operator consists of non-negative numbers $\hbar^2 j(j+1)$, where j runs over non-negative half-integers and integers

$$j = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots, \qquad \text{i.e.,} \qquad j \in \{0\} \bigcup \frac{\mathbb{N}}{2}$$

where $\mathbb{N}$ denotes the set of naturals and j is just the label of irreducible unitary representations of $\mathrm{SU}(2)$. When j is fixed, then l runs over the range $l = 0, 1, \ldots, 2j$ for the possible common eigenfunctions of $\mathbf{S}^2$ and Δ^2. According to (106)–(111), (113), (114), we have that

$$\mathbf{S}^2 D(j) = \hbar^2 j(j+1) D(j), \qquad \mathbf{S}^2 \varepsilon(j) = \hbar^2 j(j+1) \varepsilon(j)$$

i.e., all matrix elements of the j-th irreducible unitary representation, or, equivalently, all elements of the minimal two-sided ideal $M(j)$, are eigenfunctions of

$$\mathbf{S}^2 = \delta^{ab} \mathbf{S}_a \mathbf{S}_b = \delta^{ab} \widehat{\mathbf{S}}_a \widehat{\mathbf{S}}_b = -\hbar^2 \Delta[g]$$

with eigenvalues $\hbar^2 j(j+1)$.

Further, we have the following algebraization of operators

$$\mathbf{S}_a = \frac{\hbar}{\mathrm{i}} \mathcal{L}_a, \qquad \widehat{\mathbf{S}}_a = \frac{\hbar}{\mathrm{i}} \mathcal{R}_a$$

in this representation

$$S_a D(j) = S(j)_a D(j), \qquad \widehat{S}_a D(j) = D(j) S(j)_a \tag{175}$$

$$\Delta_a D(j) = [S(j)_a, D(j)] \tag{176}$$

and similarly for elements of the canonical basis, because, as we saw, there is a proportionality

$$\varepsilon(j)_{km} = (2j+1) D(j)_{km}.$$

Here $S(j)_a$ are standard $(2j+1) \times (2j+1)$ Hermitian matrices of the j-labelled angular momentum. According to (104), (105) we have that

$$D(j)\left(u\left(\overline{k}\right)\right) = \mathrm{e}\left(\frac{\mathrm{i}}{\hbar} k^a S(j)_a\right).$$

This algebraization of differential operators is very convenient because the matrices of angular momentum are standard. Therefore, (117)–(120), or, alternatively, (121)–(124), may be used, where the label α to be replaced by j and the symbols $\Sigma(\alpha)_a$ by $S(j)_a$.

Representations $D(j)$ are irreducible, so, by definition

$$\delta^{ab}S(j)_a S(j)_b = \sum_a S(j)^2{}_a = \hbar^2 j(j+1)\mathrm{Id}_{(2j+1)}.$$

The only Abelian Lie subgroups of $\mathrm{SU}(2)$, $\mathrm{SO}(3,\mathbb{R})$ are one-dimensional, just the one-parameter subgroups. Therefore, one can choose only one $\mathcal{L}$-type operator and only one $\mathcal{R}$-type operator to form, together with $-\hbar^2\Delta[g]$, the complete system of eigenequations for the functions $\varepsilon(j)_{kl}/D(j)_{kl}$. Traditionally one chooses for $S(j)_a$ such a representation that $S(j)_3$ are diagonal. Then, of course, one should choose the operators $\mathcal{L}_3$, $\mathcal{R}_3$, or in terms of observables $\mathbf{S}_3$, $\widehat{\mathbf{S}}_3$. This is certainly the matter of convention. One could as well take any versor $\overline{n} \in \mathbb{R}^3$ and operators $n^a\mathcal{L}_a$, $n^a\mathcal{R}_a$ (or $n^a\mathbf{S}_a$, $n^a\widehat{\mathbf{S}}_a$), assuming only that $n^a S(j)_a$ is diagonal for any j. When we fix the quantum number j, then the eigenvalues of $\mathbf{S}_3$, $\widehat{\mathbf{S}}_3$ have the form $\hbar m$, where $m = -j, -j+1, \ldots, j-1, j$, jumping by one. Therefore, the matrix labels of $D(j)_{mk}$, $\varepsilon(j)_{mk}$ are not taken as $1, \ldots, 2j+1$, but rather as $-j, -j+1, \ldots, j-1, j$. The matrices $S(j)_3$ are then chosen as

$$\begin{aligned} S(j)_3 &= \mathrm{diag}(-\hbar j, -\hbar(j-1), \ldots, \hbar(j-1), \hbar j) \\ &= \hbar\,\mathrm{diag}(-j, -(j-1), \ldots, (j-1), j). \end{aligned}$$

Therefore, the basic functions

$$D(j)_{mk}, \qquad \varepsilon(j)_{mk} = (2j+1)D(j)_{mk}$$

are defined by the following maximal system of compatible eigenequations

$$\mathbf{S}^2 D(j)_{mk} = \hbar^2 j(j+1) D(j)_{mk} \tag{177}$$

$$\mathbf{S}_3 D(j)_{mk} = m\hbar D(j)_{mk} \tag{178}$$

$$\widehat{\mathbf{S}}_3 D(j)_{mk} = k\hbar D(j)_{mk}. \tag{179}$$

The solution is unique up to normalization and this one is fixed by the first and third equations in (175), (176) with

$$n(\alpha) = n(j) = 2j+1.$$

Quite independently on the representation theory, the functions $D(j)_{mk}$ as solutions of (177)–(179) were found as basic stationary states of the free symmetric top, i.e., one with the following Hamiltonian (kinetic energy)

$$\mathbf{H} = \frac{1}{2I}\left(\widehat{\mathbf{S}}_1\right)^2 + \frac{1}{2I}\left(\widehat{\mathbf{S}}_2\right)^2 + \frac{1}{2K}\left(\widehat{\mathbf{S}}_3\right)^2.$$

The corresponding energy levels (eigenvalues of energy) are given by

$$E_{j,k} = \frac{1}{2I}\hbar^2 j(j+1) + \left(\frac{1}{2K} - \frac{1}{2I}\right)\hbar^2 k^2.$$

Certainly, they are $2(2j+1)$-fold degenerate, i,e, they do not depend on m at all and they do not distinguish the sign of k. If the top is spherical, $K = I$, they are $(2j+1)^2$-fold degenerate. When the top is completely asymmetric, the energy levels are $(2j+1)$-fold degenerate (independence on the spatial quantum number m).

Matrix elements $D(j)_{mk}$ of irreducible unitary representations, i.e., equivalently, elements of the canonical basis

$$\varepsilon(j)_{mk} = (2j+1)D(j)_{mk}$$

are common solutions of the system of eigenequations (177)–(179).

There is also another complete system of commuting operators, namely, $\mathbf{S}^2$, $\mathbf{\Delta}^2$, $\mathbf{\Delta}_3$. Of course, taking the third component is but just a custom, we could take as well $n^a\mathbf{\Delta}_a$, where $\overline{n}$ is an arbitrary unit vector in $\mathbb{R}^3$. Any common eigenfunction of Δ^2, Δ_3 has the following form

$$\psi\left(\overline{k}\right) = \psi\left(k, \theta, \varphi\right) = f(k)Y_{lm}\left(\overline{n}\left(\theta, \varphi\right)\right)$$

where f is an arbitrary function of the "rotation angle" $k = |\overline{k}|$, Y_{lm} is the standard symbol of spherical functions, and $\overline{n}$ is the unit vector of the oriented rotation axis. The eigenvalues are respectively given by $\hbar^2 l(l+1)$, where $l \in \{0\} \bigcup \mathbb{N}$ is an arbitrary non-negative integer, and $m\hbar$, where m runs over the range $m = -l, -l+1, \ldots, l-1, l$, jumping by one. The well-known system of eigenequations is satisfied

$$\Delta^2\psi = \hbar^2 l(l+1)\psi, \qquad \Delta_3\psi = \hbar m\psi. \tag{180}$$

The function f is arbitrary, because it is transparent for the action of Δ^2, Δ_3. The space of solutions of (180) is infinite-dimensional and this infinity is due to the arbitrariness of f. Roughly speaking, for any fixed value of l, such a system of functions represents an irreducible tensor of the group of automorphisms (152). The value $l = 0$ corresponds to scalars, i.e., functions constant on classes of adjoint elements. They are linear combinations or rather series of idempotents/characters $\varepsilon(j)/\chi(j)$. Similarly, all higher-order tensors may be combined from their orthogonal projections onto ideals $M(j)$. Those projections are common eigenfunctions

$$Q\{j\}_{lm} = f_{jl}(k)Y^l{}_m\left(\overline{n}\right)$$

of $\mathbf{S}^2$, $\mathbf{\Delta}^2$, $\mathbf{\Delta}_3$, therefore, the "radial" functions f_{jl} satisfy the following reduced eigenequation

$$\frac{\mathrm{d}^2 f_{jl}}{\mathrm{d}k^2} + \mathrm{ctg}\frac{k}{2}\frac{\mathrm{d}f_{jl}}{\mathrm{d}k} + \left(j(j+1) - \frac{l(l+1)}{4\sin^2 k/2}\right)f_{jl} = 0. \tag{181}$$

When j is fixed, then l runs over the range

$$l = 0, 1, \ldots, 2j-1, 2j$$

i.e., integers from 0 to $2j$. It turn, any l-level is $(2j+1)$-fold degenerate, thus, for any fixed j, the number of independent functions $Q\{j\}_{lm}$ equals

$$\sum_{l=0}^{2j}(2l+1)=(2j+1)^2$$

just as expected, because $\dim M(j)=(2j+1)^2$.

This is an alternative choice of basis, or rather of orthonormal complete system in $L^2\left(\mathrm{SU}(2)\right)$, tensorially ruled by irreducible representations of $\mathrm{SO}(3,\mathbb{R})$ as the automorphism group of $\mathrm{SU}(2)$.

The corresponding finite transformation rule reads

$$\begin{aligned} Q\{j\}_{lm}\left(gu\left(\overline{k}\right)g^{-1}\right) &= Q\{j\}_{lm}\left(u\left(R(g)\right)\overline{k}\right)=Q\{j\}_{lm}\left(k,R(g)\overline{n}\right)\\ &=\sum_{n}Q\{j\}_{ln}\left(k,\overline{n}\right)D(l)_{nm}\left(R(g)\right). \end{aligned}$$

Infinitesimally this is expressed as

$$\Delta_a Q\{j\}_{lm}=\sum_n Q\{j\}_{ln}S(l)_{anm}.$$

In terms of the convolution commutator

$$\left[\frac{\hbar}{\mathrm{i}}\mathcal{L}_a\delta,Q\{j\}_{lm}\right]=\left[\frac{\hbar}{\mathrm{i}}\mathcal{L}_a\varepsilon(j),Q\{j\}_{lm}\right]=\sum_n Q\{j\}_{ln}S(l)_{anm}.$$

Of course, the convolution commutator is meant in the sense

$$[f,g]=f*g-g*f. \tag{182}$$

The use of spherical functions ${Y^l}_m\left(\overline{n}\right)$ in (180) expresses explicitly the fact that for a fixed l we are dealing with an irreducible object of the group of inner automorphisms. This is so-to-speak a non-redundant description of such objects, with all its advantages and disadvantages. The obvious disadvantage is that the tensorial structure is hidden. The point is that ${Y^l}_m\left(\overline{n}\right)$ are independent quantities extracted from the l-th tensorial power of the unit versor $\overline{n}$, $\underset{l}{\otimes}\overline{n}$. Analytically such a symmetric tensor is given by the system of components

$$n^{a_1}\ldots n^{a_l}. \tag{183}$$

The transformation rule under $R\in\mathrm{SO}(3,\mathbb{R})$

$$(R\overline{n})^{a_1}\ldots(R\overline{n})^{a_l}={R^{a_1}}_{b_1}\ldots{R^{a_l}}_{b_l}n^{b_1}\ldots n^{b_l}$$

is evidently tensorial and preserves the symmetry, however, it is reducible, because orthogonal transformations preserve all trace operations. Irreducible objects are obtained from (183) by eliminating all traces, e.g.,

$$\mathcal{Y}(1)^a = n^a \tag{184}$$

$$\mathcal{Y}(2)^{ab} = n^a n^b - \frac{1}{3}\delta^{ab} \tag{185}$$

$$\mathcal{Y}(3)^{abc} = n^a n^b n^c - \frac{1}{5}\left(n^a\delta^{bc} + n^b\delta^{ca} + n^c\delta^{ab}\right) \tag{186}$$

$$\begin{aligned}\mathcal{Y}(4)^{abcd} &= n^a n^b n^c n^d - \frac{1}{7}\Big(n^a n^b \delta^{cd} + n^a n^c \delta^{bd} + n^a n^d \delta^{bc} + n^b n^c \delta^{ad} \\ &\quad + n^b n^d \delta^{ac} + n^c n^d \delta^{ab}\Big) + \frac{1}{35}\left(\delta^{ab}\delta^{cd} + \delta^{ac}\delta^{bd} + \delta^{ad}\delta^{bc}\right)\end{aligned} \tag{187}$$

and so on. The logic of those tensors is that they are algebraically built of n^a, δ^{bc}, are completely symmetric and traceless in any pair of indices (trace meant as a contraction with an appropriate δ_{ab}).

Any $\mathcal{Y}(l)$ has only $(2l{+}1)$ independent components, which are linear combinations of $Y^l{}_m$, $m = -l, \ldots, l$. Therefore, the representation is very redundant, however, the tensorial structure is explicitly visible. Instead of functions $Q\{j\}_{lm}$ one can use tensorial objects

$$Q\{j,l\}^{a_1\ldots a_l} = f_{jl}(k)\mathcal{Y}(l)\left(\overline{n}\right)^{a_1\ldots a_l}. \tag{188}$$

Infinitesimally, the tensorial character of quantities $Q\{j,l\}$ is represented by the following relationship

$$\mathcal{A}_b Q\{j,l\}^{a_1\ldots a_l} = [\mathcal{L}_b\delta, Q\{j,l\}^{a_1\ldots a_l}] = -\sum_c \varepsilon_b{}^{a_c}{}_d Q\{j,l\}^{a_1\ldots a_{c-1} d a_{c+1}\ldots a_l}$$

for example

$$\mathcal{A}_b Q\{j,2\}^{km} = \left[\mathcal{L}_b\delta, Q\{j,2\}^{km}\right] = -\varepsilon_b{}^k{}_d Q\{j,2\}^{dm} - \varepsilon_b{}^m{}_d Q\{j,2\}^{kd}$$

and so on. Surely, $\mathcal{L}_b\delta$ in these equations may be replaced by $\mathcal{R}_b\delta$ and both may be replaced by $\mathcal{L}_a\varepsilon(j) = \mathcal{R}_a\varepsilon(j)$. Irreducibility implies that

$$\begin{aligned}\delta^{ab}\mathcal{A}_a\mathcal{A}_b Q\{j,l\}^{a_1\ldots a_l} &= \delta^{ab}\left[\mathcal{L}_a\delta, \left[\mathcal{L}_b\delta, Q\{j,l\}^{a_1\ldots a_l}\right]\right] \\ &= \delta^{ab}\left[\mathcal{L}_a\varepsilon(j), \left[\mathcal{L}_b\varepsilon(j), Q\{j,l\}^{a_1\ldots a_l}\right]\right] \\ &= -l(l+a)Q\{j,l\}^{a_1\ldots a_l}\end{aligned}$$

with the (182)-meaning of the convolution commutator.

4.3. Some Problems Concerning Irreducible Tensors of Automorphism Group

There are some subtle points concerning irreducible tensors of the automorphism group, which were partially mentioned earlier in Section 2 devoted to general Lie groups. Namely, the tensorial quantities (131) were introduced there. They were obtained as convolution monomials of

$$Q_a = \mathcal{L}_a\delta = \mathcal{R}_a\delta \qquad \text{or} \qquad \Sigma_a = \frac{\hbar}{\mathrm{i}}Q_a$$

or rather as symmetrizations of these monomials. The symmetrizations of monomials built of Σ_a are Hermitian in the sense of group algebra, just as Σ_a themselves. However, in general they are not irreducible tensors, just the traces (in the sense of Killing metric tensor) must be subtracted. The symmetrized monomials are represented in the Peter-Weyl sense by matrices (130) alternatively, depending on whether the convention (117) or (121) is used. And an important point is of course that the monomials (125) are different from the pointwise products $Q_aQ_b\dots Q_k$. In particular, the pointwise products $Q(\alpha)_aQ(\alpha)_b\dots Q(\alpha)_k$ of $M(\alpha)$-projections do not belong to $M(\alpha)$, whereas the convolutions $Q(\alpha)_a * Q(\alpha)_b * \cdots * Q(\alpha)_k$ certainly do.

Let us specialize the problem to SU(2). The distribution $\Sigma_a = (\hbar/\mathrm{i})Q_a$, physically corresponding to the angular momentum, is suggestively expressed by the operators (173)

$$\Sigma_a = \mathbf{S}_a\delta = \widehat{\mathbf{S}}_a\delta = \frac{\hbar}{\mathrm{i}}\mathcal{L}_a\delta = \frac{\hbar}{\mathrm{i}}\mathcal{R}_a\delta \tag{189}$$

and its projections onto ideals $M(j)$ are given by

$$\Sigma(j)_a = \frac{\hbar}{\mathrm{i}}\mathcal{L}_a\varepsilon(j) = \frac{\hbar}{\mathrm{i}}\mathcal{R}_a\varepsilon(j). \tag{190}$$

The above expression (189) is a series built of (190) with all possible values of $j = 0, 1/2, 1, 3/2, \dots$ and the limit is meant in the distribution sense. But of course $\Sigma(j)_a$ themselves are well-defined smooth functions and

$$\Sigma(j)_a = \frac{\mathrm{d}\varepsilon(j)}{\mathrm{d}k}\frac{k_a}{k} = (2j+1)\frac{\mathrm{d}\chi(j)}{\mathrm{d}k}n_a$$

because the idempotents $\varepsilon(j)$/characters $\chi(j)$ depend only on k. The Peter-Weyl coefficients of Σ_a are given by the usual $(2j+1) \times (2j+1)$ matrices $S(j)_a$ of angular momentum or by their transposes $S(j)^T_a$, depending on which one of conventions (121) or (117) is used.

The higher-order Hermitian $\mathrm{SO}(3,\mathbb{R})$-tensors are again given by (131) and the corresponding Peter-Weyl matrices (130) will be denoted by

$$\begin{aligned} S(j,l)_{a_1\dots a_l} &= S(j)_{(a_1}\dots S(j)_{a_l)} \\ S(j,l)^T_{a_1\dots a_l} &= S(j)^T_{(a_1}\dots S(j)^T_{a_l)}. \end{aligned}$$

They are tensorial and symmetric, nevertheless, just like (183), they are still reducible. To obtain irreducible objects, one must eliminate traces, in analogy to

(184)–(187). The corresponding traceless parts of (matrix-valued) tensors $S(j,l)$, $S(j,l)^T$ will be denoted by

$$S^\circ(j,l) = \mathrm{Traceless}\,(S(j,l))\,, \qquad S^\circ(j,l)^T = \mathrm{Traceless}\left(S(j,l)^T\right). \quad (191)$$

Let us observe that the very literal analogy with (184)–(187) is, nevertheless, misleading, because in (180), (183), (184)–(187) we are dealing with the pointwise products

$$n^a n^b \ldots n^r \qquad \text{or} \qquad k^a k^b \ldots k^r.$$

Because of this the shape factor $f_{jl}(k)$ in (180), (188) must be introduced and subject to the “radial” Schrödinger-type equation. Unlike this, there is no problem of “radial” equation when one deals with functions $\Sigma(j,l)$ on $\mathrm{SU}(2)$ with the Peter-Weyl coefficients (191). Namely, for any fixed half/integer j and any $l \leq 2j$, the following functions on $\mathrm{SU}(2)$

$$T(j,l)_{a_1\ldots a_l} = \mathrm{Tr}\left(S^\circ(j,l)_{a_1\ldots a_l}\widehat{\varepsilon}(j)\right) = \mathrm{Tr}\left(S^\circ(j,l)_{a_1\ldots a_l}D(j)(2j+1)\right) \quad (192)$$

are eigenfunctions of $\mathbf{S}^2 = -\hbar^2\mathbf{\Delta}$ with the eigenvalue $\hbar^2 j(j+1)$, thus, they are elements of $M(j)$ and simultaneously are the eigenfunctions of $\mathbf{\Delta}^2 = \delta^{ab}\mathbf{\Delta}_a\mathbf{\Delta}_b$ with the eigenvalue $\hbar^2 l(l+1)$. Any element of $M(j)$ may be uniquely expanded as follows

$$F = \sum_{l=0}^{2j} P(l)^{a_1\ldots a_l} T(j,l)_{a_1\ldots a_l} \quad (193)$$

where the tensor $P(l)$ is totally symmetric and traceless.

Its Peter-Weyl matrix of coefficients $\widehat{F}$ in the convention (121) has the following form

$$\widehat{F} = \sum_{l=0}^{2j} P(l)^{a_1\ldots a_l}\; S^\circ(j,l)_{a_1\ldots a_l}. \quad (194)$$

Evidently, the function (193) is Hermitian in the sense of group algebra if and only if the coefficients $P(l)^{a_1\cdots a_l}$ are real, because all the matrices $S^\circ(j,l)_{a_1\ldots a_l}$ combined in (194) are Hermitian.

The above representation is tensorially symmetric, however, informationally redundant. In non-redundant description, based on spherical functions Y_{lm}, we have instead of (193) the representation

$$F = \sum_{l=0}^{2j}\sum_{m=-l}^{l} P_{lm} Q\{j\}_{lm} \quad (195)$$

where the functions $Q\{j\}_{lm}$ are given by (180).

The obvious properties of spherical functions, i.e.,

$$Y^l{}_m(-\overline{n}) = (-1)^l Y_{lm}(\overline{n}), \qquad \overline{Y}^l{}_m = Y^l{}_{-m}$$

imply that F is Hermitian in the sense of group algebra over $\mathrm{SU}(2)$ if and only if

$$\overline{P}_{lm} = (-1)^l P_{l(-m)}. \tag{196}$$

The Hermitian elements of the group algebra of $\mathrm{SU}(2)$ given by (192) are assumed to represent some important physical quantities. They have very suggestive tensorial structure and for $l = 1$ they represent the angular momentum. Because of this there is a natural temptation to interpret them physically in terms of magnetic multipole momenta [30]. Although in tensorial representation their system is redundant, it is convenient to expand with respect to them the density operators. The corresponding coefficients $P(l)^{a_1 \cdots a_l}$ are directly related to the expectation values of multipoles, and it is reasonable to interpret them physically as magnetic polarizations of the corresponding order [30]. It is clear that the physical situations, characterized by the fixed label j of the Casimir invariant, possess multipole momenta and polarizations of the orders $l = 0, 1, \ldots, 2j$. The algebraically non-redundant description of these objects is based on (195)–(196).

4.4. Quasiclassical Asymptotic of "Large Quantum Numbers"

Let us now discuss the quasiclassical limit. By this we mean the limit of "large quantum numbers" in equations like (180), (181) and others. An important aspect of this asymptotics is that the corresponding basic solutions are superposed with coefficients which are "slowly varying" functions of their arguments in some "wide" range of their values and practically vanishing outside this range. It is important that the range is simultaneously "wide" in the sense "much wider than one", but at the same time concentrated about some "large" mean value. This enables one to perform approximate "continuization" of discrete labels/(quantum numbers) and to replace their summation by integration.

For $l = 0$ the substitution of

$$f_{j0} = A \frac{\chi_{j0}}{\sin k/2}, \qquad A = \mathrm{const}$$

to (181) leads immediately to the following result

$$f_{j0} = A \frac{\sin(2j+1)k/2}{\sin k/2}.$$

But

$$\varepsilon(j)(0) = (2j+1)^2$$

thus, $A = 2j + 1$, and finally

$$\varepsilon(j)(k) = (2j+1) \frac{\sin(2j+1)k/2}{\sin k/2}, \quad \delta(k) = \sum_{j=0}^{\infty} (2j+1) \frac{\sin(2j+1)k/2}{\sin k/2}. \tag{197}$$

One can easily show that

$$f_{j,l+1} = \left(\frac{\mathrm{d}}{\mathrm{d}k} - \frac{l}{2}\mathrm{ctg}\frac{k}{2} \right) f_{jl} \tag{198}$$

and therefore, iterating this recurrence formula one obtains the explicit formula for the multipole basis (180)

$$f_{jl} = \prod_{n=l-1}^{0} \left(\frac{\mathrm{d}}{\mathrm{d}k} - \frac{n}{2}\mathrm{ctg}\frac{k}{2} \right) \varepsilon(j). \tag{199}$$

Let us discuss the asymptotic expansions of such expressions in a domain $[0, a]$ where $a < 2\pi$. One can show that for continuous functions f on $[0, a]$ the following holds

$$\begin{aligned} &\lim_{j\to\infty} \int_0^a f(k) \left(\frac{\sin(2j+1)k/2}{\sin k/2} - \frac{\sin(2j+1)k/2}{k/2} \right) \mathrm{d}k \\ &\qquad = \lim_{j\to\infty} \int_0^a f(k) \frac{k/2 - \sin k/2}{(k/2)\sin k/2} \sin(2j+1)\frac{k}{2}\mathrm{d}k = 0. \end{aligned} \tag{200}$$

Incidentally, this statement is true for more general "sufficiently regular" functions f, i.e., they need not be continuous. The equation (200) means that in the integral mean-value sense in $[0, a]$ the functions $\varepsilon(j)$ with "sufficiently large" j-s may be asymptotically replaced by

$$(2j+1)\frac{\sin(2j+1)k/2}{k/2}. \tag{201}$$

And for "sufficiently large" values of j the functions (201) are essentially concentrated about $k = 0$.

Therefore, for any $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that for any $n > n_0$ the following holds

$$\left| \int_0^a f(k)\frac{\sin nk/2}{k/2}\mathrm{d}k - \int_0^\infty f(k)\frac{\sin nk/2}{k/2}\mathrm{d}k \right| < \varepsilon$$

and

$$\lim_{n\to\infty} \int_0^a f(k)\frac{\sin nk/2}{k/2}\mathrm{d}k = \pi f(0)$$

i.e.,

$$\mathbb{N} \ni n \mapsto \frac{1}{\pi}\frac{\sin nk/2}{k/2}$$

is a "Dirac-delta sequence".

The functions $\varepsilon(j) = (2j+1)\chi(j)$ are concentrated about $k = 0$ and have there the maxima $(2j+1)^2$. At $k = 2\pi$ they have the extrema $\pm(2j+1)^2$ depending on

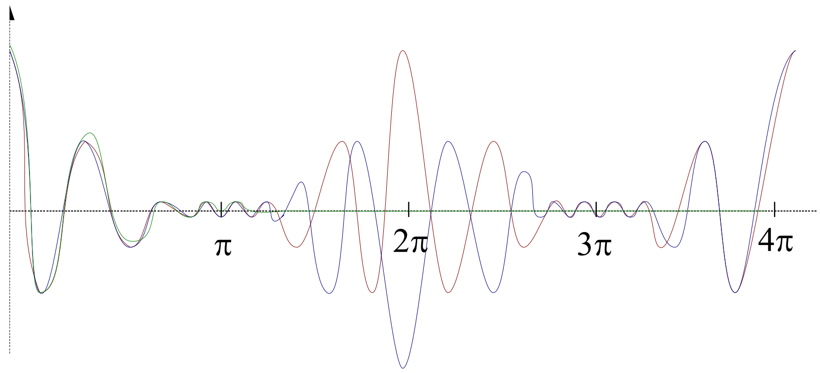

Figure 1. Asymptotic behaviour of functions $\varepsilon(j)$.

whether j is respectively integer $(+)$ or half-integer $(-)$. For $j \to \infty$, $\varepsilon(j)$ may be replaced by

$$\varepsilon^{\circ}(j) = (2j+1)\frac{2}{k}\sin\frac{(2j+1)k}{2} \tag{202}$$

in any interval $[0, a]$, $a < 2\pi$. But it is also seen that $\varepsilon(j)$ may be replaced by

$$\varepsilon^{2\pi}(j) = \pm(2j+1)\frac{2}{2\pi - k}\sin\frac{(2j+1)k}{2} \tag{203}$$

in any interval $[a, 2\pi]$, $a > 0$. The signs $+/-$ appear respectively for integer/half-integer values of j. Therefore, globally, in the total $\mathrm{SU}(2)$-range $k \in [0, 2\pi]$, we have the following asymptotics for $j \to \infty$

$$\varepsilon(j) \approx (2j+1)\sin\frac{(2j+1)k}{2}\left(\frac{2}{k} + (-1)^{2j}\frac{2}{2\pi - k}\right). \tag{204}$$

The oscillating, sign-changing extremum at $k = 2\pi$ is a purely quantum, spinorial effect. Such an effect does not appear on $\mathrm{SO}(3, \mathbb{R})$, when the range of k is given by $[0, \pi] \subset \mathbb{R}$. However, when the functions $\varepsilon(j)$ are superposed with slowly-varying coefficients concentrated at large values of j, then the subsequent peaks approximately cancel each other. Nevertheless, for any fixed j, it does not matter how large one, we have the asymptotic formula (204) with both peaks. We shall write it symbolically

$$\varepsilon(j) \approx \varepsilon^{\circ}(j) + \varepsilon^{2\pi}(j) \tag{205}$$

where $\varepsilon_0(j)$, $\varepsilon_{2\pi}(j)$ are concentrated respectively about $k=0$ and $k=2\pi$. The same is true for all other "radial" functions appearing in the multipole expansion (180).

Approximate equation for f_{jl} about $k=0$ and for large values of j has the following form

$$\frac{\mathrm{d}^2}{\mathrm{d}k^2}f^\circ_{jl}+\frac{2}{k}\frac{\mathrm{d}}{\mathrm{d}k}f^\circ_{jl}+\left(j(j+1)-\frac{l(l+1)}{k^2}\right)f^\circ_{jl}=0. \tag{206}$$

For $\varepsilon^\circ(j)=f^\circ_{j0}$ one re-obtains the known expression

$$\varepsilon^\circ(j)=(2j+1)\frac{\sin(2j+1)k/2}{k/2}.$$

One can easily show that

$$\begin{aligned} f^\circ_{j,l+1} &= \left(\frac{\mathrm{d}}{\mathrm{d}k}-\frac{l}{k}\right)f^\circ_{jl} \\ f^\circ_{jl} &= \left(\prod_{n=l-1}^{0}\left(\frac{\mathrm{d}}{\mathrm{d}k}-\frac{n}{k}\right)\right)\varepsilon^\circ(j) \end{aligned}$$

in a complete analogy to (198), (199).

Another often used approximation for large j is

$$j(j+1)\mapsto\left(j+\frac{1}{2}\right)^2.$$

Then the differential equation (206) becomes approximately

$$\frac{\mathrm{d}^2}{\mathrm{d}k^2}f^\circ_{jl}+\frac{2}{k}\frac{\mathrm{d}}{\mathrm{d}k}f^\circ_{jl}+\left(\left(j+\frac{1}{2}\right)^2-\frac{l\,(l+1)}{k^2}\right)f^\circ_{jl}=0.$$

Again one can show that the approximate solutions of rigorous equations for the large values of j have the following form

$$f_{jl}=f^\circ_{jl}+f^{2\pi}_{jl}$$

where

$$\begin{aligned} f^\circ_{jl} &= \left(\prod_{n=l-1}^{0}\left(\frac{\mathrm{d}}{\mathrm{d}k}-\frac{n}{k}\right)\right)\varepsilon^\circ\,(j) \\ f^{2\pi}_{jl} &= \left(\prod_{n=l-1}^{0}\left(\frac{\mathrm{d}}{\mathrm{d}k}-\frac{n}{2\pi-k}\right)\right)\varepsilon^{2\pi}\,(j)\approx -\,f^{2\pi}_{j+\frac{1}{2},l}. \end{aligned}$$

One can note that $\varepsilon(j)$ in (197) has the profound geometric interpretation of the generated unit of $M(j)$ and $\chi(j)=(1/(2j+1))\varepsilon(j)$ is the character of the j-th irreducible unitary representation of $\mathrm{SU}(2)$. And seemingly, one might have an impression that the asymptotic counterpart (202) is something "accidental", non-interpretable in geometric terms. However, as a matter of fact, it is an important object of the Fourier analysis on $\mathbb{R}^3 \simeq \mathfrak{su}(2) \simeq \mathfrak{so}(3, \mathbb{R})$.

Indeed, it may be easily shown that the Fourier representation of the Dirac delta distribution on $\mathbb{R}^3$ (as the Fourier transform of unity)

$$\delta\left(\overline{\omega}\right) = \frac{1}{\left(2\pi\right)^3} \int \mathrm{e}\left(\mathrm{i}\underline{\varkappa}\overline{\omega}\right) \mathrm{d}^3\underline{\varkappa}$$

after performing the integration over angels becomes

$$\delta\left(\overline{\omega}\right) = \frac{1}{\left(2\pi\right)^3} \int_0^{2\pi} \mathrm{d}\varphi \int_0^{\infty} \mathrm{d}\varkappa\varkappa^2 \int_0^{\pi} \mathrm{d}\vartheta \cos\vartheta \mathrm{e}\left(\mathrm{i}\varkappa\omega\cos\vartheta\right)$$

where $(\varkappa, \vartheta, \varphi)$ are spherical variables in the space $\mathbb{R}^3$ of vectors $\underline{\varkappa}$, adapted to the direction of $\overline{\omega}$ as the "z-axis direction". After the substitution $x = \cos\vartheta \in [1, -1]$ and $\varkappa = \zeta/2$ and elementary integrations, one obtains the following formula

$$\delta\left(\overline{\omega}\right) = \frac{1}{16\pi^2} \int_0^{\infty} \mathrm{d}\zeta \frac{\zeta \sin\zeta\omega/2}{\omega/2}.$$

Under substituting $\zeta = (2j+1)$ it is turned into

$$\delta\left(\overline{\omega}\right) = \frac{1}{8\pi^2} \int_{-1/2}^{\infty} \mathrm{d}j\, (2j+1) \frac{\sin\left(2j+1\right)\omega/2}{\omega/2} = \int \mathrm{d}j\, \varepsilon_{\text{class}}(j)(\omega). \tag{207}$$

Expression

$$\varepsilon_{\text{class}}(j)(\omega) = (2j+1) \frac{\sin\left(2j+1\right)\omega/2}{\omega/2} \tag{208}$$

is an obvious counterpart of (197) and of its expression for the Dirac distribution on $\mathrm{SU}(2)$, and the all other analogies are easily readable. They are not merely formal analogies, the point is that they are really true asymptotic approximations and geometric counterparts. Discrete summation over the "quantum number" j is now replaced by the integration over the continuous label j corresponding to the non-compactness of $\mathbb{R}^3 \simeq \mathfrak{su}(2)$ and well suited to the "classical" nature of expressions.

The superposed functions (208) play in the commutative group algebra $\mathfrak{su}(2) \approx \mathbb{R}^3$ the role of generating units of ideals $M(j)$ composed of functions with the fixed "square of linear momentum"

$$\left(j+1/2\right)^2 \hbar^2 \approx j\left(j+1\right)\hbar^2.$$

The last approximate "equality" corresponding to the "large" values of j. This ideal is not minimal. The minimal ones just correspond to the single exponents with the wave vectors $\underline{\varkappa}$, i.e., "linear momenta" $\hbar\underline{\varkappa}$. In $\varepsilon_{\text{class}}(j)$ superposed are (with equal "coefficients") all exponents $\mathrm{e}\left(\mathrm{i}\left(j+1/2\right)\overline{n}\cdot\overline{\omega}\right)$, where $\overline{n}$ runs over the manifold $S^2(0,1) \subset \mathbb{R}^3$ of all unit vectors. The ideals $M(j)$ are minimal ones invariant under the group $\mathrm{SO}(3, \mathbb{R})$ of outer automorphisms of $\mathrm{SU}(2)$. Those outer automorphisms are not only algebraic automorphisms of $\mathbb{R}^3$ as an Abelian additive group. In addition they preserve the standard Euclidean metric in $\mathbb{R}^3$. This metric just corresponds up to multiplicative constant factor to the Killing metric of $\mathfrak{su}(2) \approx \mathbb{R}^3$. It worth to note that the above terms like "group algebra" are now used rather in a metaphoric sense, because we are dealing with continuous spectrum and are outside of $L^2\left(\mathbb{R}^3\right)$ and $L^1\left(\mathbb{R}^3\right)$. Everything may be rigorously formulated in terms of rigged Hilbert spaces and direct integrals of Hilbert spaces, however, there is no place for that here.

The above limit transition and asymptotics are meant in the sense of truncation procedure in the rigorous group algebra of $\mathrm{SU}(2)$.

Quasiclassical limit is based on the truncation procedure of the group algebra of $\mathrm{SU}(2)$. Namely, we take the subalgebra consisting of all ideals $M(j)$ with $j \geq j_0$ for some fixed j_0

$$M\left(j \geq j_0\right) := \bigoplus_{j \geq j_0} M(j).$$

As mentioned, for large values of j, the generated units $\varepsilon(j) \in M(j)$ are essentially concentrated about $k = 0$, $k = 2\pi$

$$\varepsilon(j) \approx \varepsilon^{\circ}(j) + \varepsilon^{2\pi}(j)$$

cf. (202), (203), (204), (205), and the following holds

$$\varepsilon^{\circ}(j)(0) = (2j+1)^2\,, \qquad \varepsilon^{2\pi}(j)(2\pi) = (-1)^{2j}\,(2j+1)^2\,.$$

The larger truncation threshold j_0, the better the generated unit of $M\left(j \geq j_0\right)$

$$\varepsilon\left(j \geq j_0\right) := \sum_{j=j_0}^{\infty} \varepsilon(j) \tag{209}$$

is approximated by

$$\varepsilon_{\text{class}}\left(j \geq j_0\right) := \int_{j_0}^{\infty} \mathrm{d}j\; \varepsilon_{\text{class}}(j) \tag{210}$$

where $\varepsilon_{\text{class}}(j)$ is given by (208). Of course, the convergence of series (209) and integral (210) is meant in the distribution sense.

Projections of functions A, B on $\mathrm{SU}(2)$ onto the truncated ideal $M\left(j \geq j_0\right)$ will be denoted by

$$\widetilde{A} = A\left(j \geq j_0\right), \qquad \widetilde{B} = B\left(j \geq j_0\right).$$

The abbreviations $\widetilde{A}$, $\widetilde{B}$ are used when there is no danger of confusion.

For physically relevant functions A, B, the Peter-Weyl series expansions of $\widetilde{A}$, $\widetilde{B}$ may be reasonably approximated by continuous integral representations like (207), (210).

Let us go back to quantum states-densities represented in terms of non-redundant multipole expansions as follows

$$\varrho = \sum_{j=j_0}^{\infty} \sum_{l=0}^{2j} \sum_{m=-l}^{l} P(j)_{lm} Q\{j\}_{lm} \tag{211}$$

$$Q\{j\}_{lm}\left(\overline{k}\right) = f_{jk}(k) Y_{lm}\left(\frac{\overline{k}}{k}\right) \tag{212}$$

where the expansion coefficients $P(j)_{lm}$ may be roughly interpreted as magnetic multipole moments.

Quasiclassical states are represented by expressions (211), (212), where

- j_0 is "large"
- $P(j)_{lm}$ as functions of j are concentrated in some ranges
$$\left[\overline{j} - \Delta j/2, \overline{j} + \Delta j/2\right], \quad \overline{j} \gg \Delta j \gg 1$$
- within this range, $P(j)_{lm}$ are slowly varying functions of j
$$|P(j+1/2)_{lm} - P(j)_{lm}| \ll |P(j)_{lm}|.$$

Algebraic operations of group algebra on $\mathrm{SU}(2)$ attain some very peculiar representation in quasiclassical limit in the above sense. So, let us write down the convolution formula for "truncated" functions

$$\begin{aligned}&(A(j \geq j_0) * B(j \geq j_0))\left(u\left(\overline{k}\right)\right)\\ &\qquad = \int A(j \geq j_0)\left(u\left(\overline{l}\right)\right) B(j \geq j_0)\left(u\left(-\overline{l}\right) u\left(\overline{k}\right)\right) \frac{4\sin^2 l/2}{l^2} \frac{\mathrm{d}^3\overline{l}}{16\pi^2}.\end{aligned}$$

The terms concentrated about $k = 2\pi$, as it was seen, approximately cancel each other. One can assume that the integrated functions are essentially concentrated in a close neighbourhood of the unity in $\mathrm{SU}(2)$, i.e., the null of $\mathfrak{su}(2)$. There, in the lowest order of approximation, we have

$$u\left(\overline{l}\right) u\left(\overline{k}\right) \approx u\left(\overline{l} + \overline{k} + \frac{1}{2}\, \overline{l} \times \overline{k}\right).$$

Performing the corresponding Taylor expansions in our integral formulas and making use of the earlier mentioned relationship between the variables $\overline{k}$ and $\overline{\varpi}$, we finally obtain

$$A(j \geq j_0) *_{\mathrm{SU}(2)} B(j \geq j_0) \approx A(j \geq j_0) *_{\mathbb{R}^3} B(j \geq j_0) \tag{213}$$

where the convolution symbols on the left- and right-hand sides are meant in the non-commutative $\mathrm{SU}(2)$- and commutative $\mathbb{R}^3 \simeq \mathfrak{su}(2) \simeq \mathfrak{so}(3,\mathbb{R})$-senses, respectively. Surely, (213) is meant in the sense of lowest-order approximation, the terms with higher-order derivatives are neglected.

Similarly, for the quantum Poisson bracket we obtain the familiar expression

$$\left[\widetilde{A},\widetilde{B}\right]=\frac{1}{\mathrm{i}\hbar}\left(\widetilde{A}*_{\mathrm{SU}(2)}\widetilde{B}-\widetilde{B}*_{\mathrm{SU}(2)}\widetilde{A}\right)\approx\frac{1}{\mathrm{i}\hbar}\left(\left(\mathcal{A}_a\widetilde{A}\right)*_{\mathbb{R}^3}\left(\omega^a\widetilde{B}\right)\right). \quad (214)$$

Here again we mean the lowest-order approximation, when the higher-derivatives terms following from the Taylor expansions are neglected. As usual, $\mathcal{A}_a$ is the generator of inner automorphisms in $\mathrm{SU}(2)$, i.e., equivalently, of Killing rotations in $\mathfrak{su}(2)\approx\mathbb{R}^3$

$$\mathcal{A}_a=\varepsilon_{ab}{}^{c}\omega^b\frac{\partial}{\partial\omega^c}.$$

So, in terms of Fourier representants $\widehat{\widetilde{A}}(\underline{\sigma})$, we have

$$\{\sigma_i,\sigma_j\}=\sigma_k\varepsilon^k{}_{ij}, \qquad \left\{\widehat{\widetilde{A}},\widehat{\widetilde{B}}\right\}=\sigma_k\varepsilon^k{}_{ij}\frac{\partial\widehat{\widetilde{A}}}{\partial\sigma_i}\frac{\partial\widehat{\widetilde{B}}}{\partial\sigma_j}.$$

In particular, for the evolution of density $\widetilde{\varrho}$ we obtain

$$\frac{\partial\widetilde{\varrho}}{\partial t}=\left[\widetilde{H},\widetilde{\varrho}\right]_{\mathbb{R}^3}, \qquad \frac{\partial\widehat{\widetilde{\varrho}}}{\partial t}=\left[\widehat{\widetilde{H}},\widehat{\widetilde{\varrho}}\right]$$

where H denotes the Hamiltonian. Taking appropriate Hamiltonians one obtains classical asymptotics of various dynamical models of the evolution of quantum angular momentum, or rather systems of quantum angular momenta. This includes complicated interactions between magnetic multipoles as described above.

4.5. Final Comments Concerning Quasiclassical Limit

Let finish with some comments concerning quasiclassical formulas which may be helpful when operating with some geometrically and physically important quantities.

First of all, let us observe that (213) is a merely zeroth-order approximation. The first-order approximation is given by

$$\widetilde{A}*_{\mathrm{SU}(2)}\widetilde{B}\approx\widetilde{A}*_{\mathbb{R}^3}\widetilde{B}+\frac{1}{2}\left[\widetilde{A},\widetilde{B}\right]_{\mathbb{R}^3} \quad (215)$$

where, let us remind $\left[\widetilde{A},\widetilde{B}\right]_{\mathbb{R}^3}$ is the extreme right-hand side of (214). The second term is the lowest-order approximation to the $\mathrm{SU}(2)$-convolution commutator. It is well known that the commutator, or more precisely quantum Poisson bracket, describes infinitesimal transformations, in particular symmetries of quantum states (as described by density operators). It is well known that the operator eigenequation for density operators

$$\mathbf{A}\varrho=a\varrho$$

implies that the operators $\mathbf{A}$, ϱ do commute, thus, their quantum Poisson bracket vanishes

$$\{\mathbf{A}, \varrho\}_Q = \frac{1}{\mathrm{i}\hbar}(\mathbf{A}\varrho - \varrho\mathbf{A}) = 0.$$

It is assumed here that $\mathbf{A}$ represents a physical quantity, thus, it is self-adjoint, $\mathbf{A}^+ = \mathbf{A}$. Therefore, the concept of eigenstate, in particular that of pure state (one satisfying a maximal possible system of compatible eigenconditions), unifies in a very peculiar way two logically distinct concepts: information and symmetry. Information aspect is that the physical quantity $\mathbf{A}$ has a sharply defined value on the state ϱ, there is no statistical spread of measurement results. Symmetry aspect is that ϱ is invariant under the one-parameter group of unitaries, i.e., quantum automorphisms, generated by $\mathbf{A}$. On the quantum level, symmetry properties are implied by information properties, because the quantum Poisson bracket is algebraically built of the associative product. This is no longer the case in quasiclassical limit and on the classical level, where the Poisson bracket and (commutative) associative product are algebraically independent on each other. But information and symmetry are qualitatively different things, therefore, on the quasiclassical level, the two first terms of the expansion for the non-commutative associative product should be taken into account when discussing classical concepts corresponding to eigenequations. Otherwise the physical interpretation of eigenconditions would be damaged.

Let us remind, following (177)–(179), that differential equations satisfied by the functions $\varepsilon(j)_{mk}$ may be written in the following form

$$\begin{aligned}
\mathbf{S}^2\, \varepsilon(j)_{mk} &= j(j+1)\hbar^2\, \varepsilon(j)_{mk}\\
\mathbf{S}_3\, \varepsilon(j)_{mk} &= m\hbar\, \varepsilon(j)_{mk}\\
\widehat{\mathbf{S}}_3\, \varepsilon(j)_{mk} &= k\hbar\, \varepsilon(j)_{mk}
\end{aligned}$$

with the known spectra of quantum numbers j, m, k. Rewriting these equations in terms of $\mathrm{SU}(2)$-convolutions we obtain

$$\Sigma^2 * \varepsilon(j)_{mk} = \varepsilon(j)_{mk} * \Sigma^2 = j(j+1)\hbar^2\, \varepsilon(j)_{mk} \tag{216}$$

$$\Sigma_3 * \varepsilon(j)_{mk} = m\hbar\, \varepsilon(j)_{mk} \tag{217}$$

$$\varepsilon(j)_{mk} * \Sigma_3 = k\hbar\, \varepsilon(j)_{mk} \tag{218}$$

where Σ_a are given by (189) and Σ^2 denotes the convolution-squared vector Σ_a

$$\Sigma^2 = \Sigma_1 * \Sigma_1 + \Sigma_2 * \Sigma_2 + \Sigma_3 * \Sigma_3.$$

To obtain the quasiclassical counterparts of (216)–(218) we must use the asymptotic formulas (215). It is more convenient to express them in terms of Fourier transforms, which were defined as functions on the Lie co-algebra $(\mathfrak{su}(2))^* \simeq \mathbb{R}^3$.

So, we shall use the coordinates σ_i introduced above and the functions $\widehat{\varepsilon}(j)(\underline{\sigma})$ such that

$$\varepsilon(j)_{mn}(\overline{\varkappa}) = \frac{1}{(2\pi\hbar)^3}\int \widehat{\varepsilon}(j)_{mn}(\underline{\sigma})\mathrm{e}\left(\frac{\mathrm{i}}{\hbar}\underline{\sigma}\overline{\varkappa}\right)\mathrm{d}^3\underline{\sigma}. \tag{219}$$

The left-hand sides of (219) are functions on the Lie algebra $\mathfrak{su}(2) \simeq \mathbb{R}^3$ used to represent the approximate expressions for the elements of canonical basis (matrix elements of irreducible UNIREPS) as functions on $\mathrm{SU}(2)$. The system (216)–(218) is expressed in terms of these Fourier transforms as follows

$$\sigma^2\,\widehat{\varepsilon}(j)_{mn}(\underline{\sigma}) = j(j+1)\hbar^2\,\widehat{\varepsilon}(j)_{mn}(\underline{\sigma}) \tag{220}$$

$$\sigma_3\,\widehat{\varepsilon}(j)_{mn}(\underline{\sigma}) + \frac{1}{2}\{\sigma_3,\widehat{\varepsilon}(j)_{mn}(\underline{\sigma})\} = m\hbar\,\widehat{\varepsilon}(j)_{mn}(\underline{\sigma}) \tag{221}$$

$$\sigma_3\,\widehat{\varepsilon}(j)_{mn}(\underline{\sigma}) - \frac{1}{2}\{\sigma_3,\widehat{\varepsilon}(j)_{mn}(\underline{\sigma})\} = n\hbar\,\widehat{\varepsilon}(j)_{mn}(\underline{\sigma}). \tag{222}$$

The last two equations imply that

$$\{\sigma_3,\widehat{\varepsilon}(j)_{mn}(\underline{\sigma})\} = (m-n)\hbar\,\widehat{\varepsilon}(j)_{mn}(\underline{\sigma}).$$

It is convenient to use the polar angle φ in the plane $\sigma_3 = 0$ of variables σ_1, σ_2 in $\mathfrak{su}(2) \simeq \mathbb{R}^3$

$$\operatorname{tg}\varphi = \frac{\sigma_2}{\sigma_1}\cdot$$

Instead of cylindrical variables σ_3, $\varrho = \sqrt{{\sigma_1}^2 + {\sigma_2}^2}$, φ in the Lie co-algebra $(\mathfrak{su}(2))^* \simeq \mathbb{R}^3$, we shall use the modified, spherically-cylindrical coordinates

$$\sigma = \sqrt{{\sigma_1}^2 + {\sigma_2}^2 + {\sigma_3}^2}, \qquad \sigma_3, \qquad \varphi = \operatorname{arctg}\frac{\sigma_2}{\sigma_1}\cdot$$

They coincide with the canonical Darboux coordinates in $(\mathfrak{su}(2))^*$ as a Poisson manifold. Their Poisson brackets have the following form

$$\{\varphi,\sigma_3\} = 1, \qquad \{\sigma,\varphi\} = 0, \qquad \{\sigma,\sigma_3\} = 0.$$

In particular, $\sigma^2 = \underline{\sigma}\cdot\underline{\sigma}$ is the basic Casimir invariant. Its value surfaces $\sigma = \mathrm{const}$ are canonically two-dimensional symplectic manifolds. The exceptional "s-state" value $\sigma = 0$ is the singular co-adjoint orbit of dimension zero, just the origin of coordinates.

In these coordinates the following holds

$$\{\sigma_3, f(\underline{\sigma})\} = \frac{\partial}{\partial\varphi}f(\sigma,\sigma_3,\varphi)$$

therefore, the system (220)–(222) is solved as follows

$$\begin{aligned}\widehat{\varepsilon}(j)_{mn} &= N(j)\delta\left(\sigma^2 - \hbar^2 j(j+1)\right)\delta\left(\sigma_3 - \hbar\frac{m+n}{2}\right)\mathrm{e}\left(\mathrm{i}(m-n)\varphi\right)\\ &= \frac{N(j)}{2\hbar\sqrt{j(j+1)}}\delta\left(\sigma - \hbar\sqrt{j(j+1)}\right)\delta\left(\sigma_3 - \hbar\frac{m+n}{2}\right)\mathrm{e}\left(\mathrm{i}(m-n)\varphi\right)\end{aligned} \tag{223}$$

where $N(j)$ is a j-dependent normalization factor. It is defined by the demand that

$$\varepsilon(j)_{mn}|_{\overline{\varkappa}=0} = (2j+1)\delta_{mn}. \tag{224}$$

As already mentioned above, in quasiclassical situations the quantum-mysterious $j(j+1)$ is not very essential and may be replaced by $(j+1/2)^2$ or just by j^2.

Let us observe an interesting analogy with some formulas from the Weyl-Wigner-Moyal formalism for quantum systems with classical analogy. The "basis" of the wave function space consisting of non-normalizable, or rather "Dirac-δ-normalized", states $|\underline{\pi}\rangle$ of fixed linear momentum $\underline{\pi}$ implies the following H^+-algebra "basis" in the space of phase-space functions (including the Moyal quasi-probability distributions)

$$\varrho_{\underline{\pi}_1,\underline{\pi}_2}\left(\overline{q},\underline{p}\right) = \delta\left(p - \frac{1}{2}\left(\underline{\pi}_1 + \underline{\pi}_2\right)\right)\mathrm{e}\left(\frac{\mathrm{i}}{\hbar}\left(\underline{\pi}_1 - \underline{\pi}_2\right)\overline{q}\right).$$

There is an obvious analogy with the term

$$\delta\left(\sigma_3 - \frac{1}{2}\left(\mu_1 + \mu_2\right)\right)\mathrm{e}\left(\frac{\mathrm{i}}{\hbar}\left(\mu_1 - \mu_2\right)\varphi\right)$$

in (223), if we put $\mu_1 = \hbar m$, $\mu_2 = \hbar n$. This analogy is not accidental. However, there is no place here for more details.

Equation (223) and normalization condition (224) imply finally that

$$\begin{aligned}\widehat{\varepsilon}(j)_{mn} \approx 16\pi^2\hbar^4\left(j+\frac{1}{2}\right)^2\delta\left(\sigma^2 - \hbar^2\left(j+\frac{1}{2}\right)^2\right)\\ \delta\left(\sigma_3 - \hbar\frac{m+n}{2}\right)\mathrm{e}\left(\mathrm{i}(m-n)\varphi\right)\end{aligned}$$

therefore

$$\begin{aligned}\widehat{\mathcal{D}}(j)_{mn} \approx 8\pi^2\hbar^4\left(j+\frac{1}{2}\right)\delta\left(\sigma^2 - \hbar^2\left(j+\frac{1}{2}\right)^2\right)\\ \delta\left(\sigma_3 - \hbar\frac{m+n}{2}\right)\mathrm{e}\left(\mathrm{i}(m-n)\varphi\right).\end{aligned}$$

In the above formulas we mean the same as previously asymptotic "indifference" concerning $j(j+1)$, $(j+1/2)^2$, j^2 for large values of j.

Warning: It must be stressed that the above functions are not literally meant asymptotic expressions for $\varepsilon(j)_{mn}$, $\mathcal{D}(j)_{mn}$ for "large" values of j. They may be used instead of rigorous $\varepsilon(j)_{mn}$, $\mathcal{D}(j)_{mn}$ when superposing them with coefficients "slowly varying" as functions of j. And the very important point: The discrete quantum number j may be then formally admitted to be a continuous variable and the summation with "slowly-varying" coefficients may be approximated by integration. In this way the compactness of $\mathrm{SU}(2)$ is seemingly "lost". This procedure is well known in practical applications of Fourier analysis, where often Fourier series may be approximated by Fourier transforms.

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