# INDUCED GAUGE STRUCTURE IN QUANTUM THEORY FROM THE VIEWPOINT OF THE CONFINING POTENTIAL APPROACH

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**Abstract**. Main features of the induced gauge field in the confining potential approach to constrained system are summarized. The relation of this approach to the coset space approach is mentioned. Properties of the induced gauge field coming from W-Z-W term in $SU(3)$ Skyrme–Witten model is examined.

## 1. Introduction

It has been recognized, through construction of the quantum theory on curved manifolds, that some gauge structure emerges. This problem has been investigated along the following two approaches; the first one is an approach on the quantum theory on coset space [1, 2], and the other is an approach of investigating a physical system which is confined by some potential to a curved $n$-dimensional manifold $\mathbf{M}^n$ embedded in a $p$-dimensional Euclidean space $\mathbb{R}^p$ (with $p \geq n + 2$) [3, 6]. The latter is called the confining potential approach, abbreviated as CPA.

In the coset space approach the gauge structure which emerges on the coset manifold $G/H$ is called the $H$-connection [2]. The gauge field generated on the sphere $\mathbb{S}^{p-1}$ found by Ohnuki and Kitakado [1], as pointed out by Lévay *et al.* [2], is nothing less than a kind of the $H$-connection, because $\mathbb{S}^{p-1} \simeq SO(p)/SO(p-1)$. In CPA, the induced gauge structure comes from the normal fundamental form, called the $N$-connection [5, 6]. We can find relation

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between these two kinds of the gauge fields by employing (generalized) Gauss mapping [4, 6, 7].

In the following, we first summarize the main features of CPA in Sect. 2, and in Sect. 3, from the viewpoint of the induced gauge structure, the properties of the gauge field emerging in the $SU(3)$ Skyrme model in the collective-coordinate approximation is examined in accordance with the quantum-mechanical treatment given in Ref. [8]. Section 4 is devoted to summarizing remarks.

## 2. Main Features of CPA

The basic relations of CPA are given as follows. In order to describe a particle moving in a neighborhood $\Xi(\mathbf{M}^n)$ of $\mathbf{M}^n$ embedded in $\mathbb{R}^p$, we introduce on $\mathbf{M}^n$ a set of curvilinear coordinates $\{q^b;\ b = 1, \cdots, n\}$ specifying a point $\vec{x} = (x^A)$ on $\mathbf{M}^n \subset \mathbb{R}^p$; $x^A = x^A(q^b)$, $A = 1, \cdots, p$. At each point $\vec{x}(q^b)$, there is a set of orthogonal unit vectors $\{N_U{}^A(q^b);\ U = n + 1, \cdots, p\}$, which are normal to $\mathbf{M}^n$ at $\vec{x}(q^b)$. Any point $X^A$ in $\Xi(\mathbf{M}^n)$, when taken to be sufficiently "thin", is assumed to be expressed as

$$X^A(q^b, q^U) = x^A(q^b) + \sum_{U=n+1}^{p} q^U\, N_U{}^A(q)\,, \quad A = 1, \ldots, p\,; \tag{1}$$

here new parameters $\{q^U;\ U = n+1, \cdots, p\}$ are called the normal coordinates. The induced metric on $\mathbf{M}^n$ is written as $g_{ab}(q) = B_a{}^A(q)\eta_{AB}B_b{}^B(q)$, where $B_a{}^A(q) := \partial x^A(q)/\partial q^a$, tangent to $\mathbf{M}^n$ at $\vec{x}(q)$. The fundamental equations for $\mathbf{M}^n$, i.e. the generalized Frenet–Serret equations, are expressed as [4, 6]

$$\partial_a B_b{}^A = \{_a{}^d{}_b\}B_d{}^A + H_{Wab}\eta^{WU}N_U{}^A\,, \tag{2}$$

$$\partial_a N_V{}^A = -H_{Va}{}^d B_d{}^A - T_{VW,a}\eta^{WU}N_U{}^A\,, \tag{3}$$

where $\{_a{}^d{}_b\}$ is Christoffel symbol written in terms of $g_{ab}$; $H_{Va}{}^d = H_{Vab}g^{bd}$; $T_{VW,a}\ (= -T_{WV,a})$ is expressed as

$$T_{VW,a} = \vec{N}_V \cdot \partial_a \vec{N}_W\,, \tag{4}$$

called the N-connection. This quantity is seen to appear in the kinetic term of the effective Hamiltonian on $\mathbf{M}^n$, obtained from $\dot{X}^A\eta_{AB}\dot{X}^B/2$, only through the combination

$$\pi_a = p_a + \frac{1}{2}T_{VW,a}L^{VW}\,; \tag{5}$$

here, $L^{VW} := q^V\eta^{WU}p_U - q^U\eta^{WV}p_V$; $p_a$ is the momentum operator conjugate to $q^a$, and satisfies $[q^a, p_b] = i\hbar\delta^a{}_b$, $[p_a, p_b] = 0$. The thin layer approximation, under which the effective Hamiltonian is obtained, means that geometric

properties along $\mathbf{M}^n$ changes slowly in comparison with those along directions normal to $\mathbf{M}^n$.

Characteristic features and consequences of CPA are summarized as follows.

1) Under an orthogonal transformation of a set $\{N_U{}^A, \ U = n + 1, \cdots, p\}$ to another set $\{N'_U{}^A\}$, i. e.

$$N'_U{}^A(q) = N_W{}^A(q)\Lambda^W{}_V(q), \qquad \Lambda(q) \in O(p) \qquad (6)$$

the $N$-connection is transformed as [4, 6]

$$T'_{VW,b}(q) = \left[\Lambda^{-1}(q)\, T_{,b}(q)\Lambda(q)\right]_{VW} + \left[\Lambda^{-1}(q)\, \partial_b\Lambda(q)\right]_{VW}. \qquad (7)$$

2) Employing $R_{ab,VW}$, defined by

$$R_{db,VW} := -\partial_d T_{VW,b} + \partial_b T_{VW,d} + T_{XV,d}\eta^{XY}T_{YW,b} - T_{XV,b}\eta^{XY}T_{YW,d}, \quad (8)$$

we obtain

$$[\pi_a, \pi_b] = \frac{i\hbar}{2}\, R_{ab,VW}L^{VW}. \qquad (9)$$

3) In the case of $\mathbf{M}^1 \subset \mathbb{R}^p$ (with $p \geq 3$), when the tangent vector $\vec{B}_1$ of $\mathbf{M}^1$ is mapped to a point on $\mathbb{S}^{p-1}$ in the parameter space $\mathbb{R}^p(\vec{B}_1)$ through Gauss mapping, the gauge potential on $\mathbb{S}^{p-1}$ derived by Ohnuki and Kitakado [1] is shown [6] to have the same structure as the mapped form of the $N$-connection. Further, it is shown [6] that the gauge field found in Ref. [1] coincides with the spin-connection on $\mathbb{S}^{p-1} \subset \mathbb{R}^p(\vec{B}_1)$.

4) In the special case with $p = 3$, an integral of the geodesic curvature along a closed path on $\mathbb{S}^2 \subset \mathbb{R}^3(\vec{B}_1)$ is related to the area (or the solid angle) surrounded by that closed path (due to Gauss–Bonnet theorem) and at the same time is equal to some integral of the $N$-connection (i. e. torsion) along $\mathbf{M}^1(\subset \mathbb{R}^3)$; the area on $\mathbb{S}^2$ is equal to the magnetic flux through that surface when a magnetic monopole with unit strength exists at the origin of $\mathbb{R}^3(\vec{B}_1)$ [4, 6].

5) The case examined in [1] corresponds to the case of $\mathbb{S}^{p-1} \simeq SO(p)/SO(p-1)$; $SO(p-1)$ is Wigner's little group, which leaves one frame in $\mathbb{R}^3(\vec{B}_1)$ invariant. The procedure employed in 3) to establish the relation of the $N$-connection to the induced gauge field on $\mathbb{S}^{p-1}$ can be easily extended to the case of

$$SO(p)/SO(p-n) \simeq V_{p,n}(\mathbb{R}), \quad p - n \geq 2; \qquad (10)$$

here, $V_{p,n}(\mathbb{R})$ is the real Stiefel manifold constructed by $n$-frames; the subgroup $SO(p-n)$ leaves a set of $n$-frames invariant [7].

6) By extending formally the formalism of CPA to an infinite dimensional case, the path-integral formula for the real scalar theory which allows classical solutions has been given [6]. In this formula, there appears the induced gauge term, which is dropped in the formula given by Gervais and others [9]. It is required for us to explore the physical role of such an induced gauge term.

## 3. Properties of the Gauge Field Emerging in $SU(3)$ Skyrme Model

Following the quantum-mechanical treatment [8] of the $SU(3)$ Skyrme–Witten model in the collective-coordinate approximation, we examine the properties of the gauge field coming from the Wess–Zumino–Witten term in the action induced in the form

$$N_c \Gamma^{WZW} = \frac{N_c}{240\pi^2} \int_{M^5} \mathrm{d}^5 x \, \epsilon^{ijklm} \, \mathrm{Tr}[U_i U_j U_k U_l U_m] \, ; \tag{11}$$

here the sum with respect to the dummy indices is carried out from 0 to 4; $U_j(x) := -iU(x)^\dagger \partial_j U(x)$ for a $3 \times 3$ unitary matrix-valued field $U(x)$; $\epsilon^{12304} = \epsilon^{1230} = 1$; $\partial M^5 =$ Minkowski space $M^4$ (with the metric $(\eta_{jk}) = \mathrm{diag}(-1, 1, 1, 1)$). We adopt Ansatz for the stationary $SU(3)$ soliton expressed as

$$\sigma(\vec{x}) = \begin{pmatrix} \sigma_0(\vec{x}) & 0 \\ 0 & 1 \end{pmatrix}, \tag{12}$$

where $\sigma_0(\vec{x})$ is the $SU(2)$ soliton solution of hedgehog type

$$\sigma_0(\vec{x}) = \exp[iF(r)\vec{x} \cdot \vec{\tau}/r], \quad r = |\vec{x}| \, , \tag{13}$$

satisfying $F(0) = n\pi$, $n \in \mathbb{Z}$, $F(\infty) = 0$ due to the uniqueness of $\sigma_0(\vec{x})$ at the origin and the condition of finite energy of the soliton. In the framework of the present approximation, $U(\vec{x}, t)$ is set equal to

$$U(\vec{x}, t) = A(t)\sigma(\vec{x})A(t)^\dagger, \quad A(t) \in SU(3) \, . \tag{14}$$

Then, $A(t)$ is specified by a set of time-dependent curvilinear coordinates $\{q^b = q^b(t); b = 1, \ldots, 7\}$ because of the invariance of $U(x)$ under

$$A(q(t)) \to A'(q(t)) = A(q(t)) \exp(i\lambda_8 \beta(q)/2) \tag{15}$$

with arbitrary real function $\beta(q)$. Thus we treat the 7-dimensional coset space $SU(3)/U(1)_Y$.

We decompose $A^\dagger \partial_b A$ [8] by employing Gell-Mann's matrices $\lambda_\beta$'s as

$$A^\dagger(q)\partial_b A(q) = \frac{1}{2}\sum_{m=1}^{7} \lambda_m C(q)_b{}^m + \frac{i}{2}\lambda_8 h_b(q)\,. \qquad (16)$$

Following the quantum-mechanical treatment in Ref. [8], we obtain the Lagrangian expressed in terms of $q^b$'s and $\dot{q}^b$'s in the form

$$L(q,\dot{q}) = \frac{i}{2}\dot{q}^b g_{bd}(q)\dot{q}^d + \frac{\lambda}{2}\left(\dot{q}^b h_b + h_b \dot{q}^b\right) + v(q)\,; \qquad (17)$$

here, $\lambda = N_c B/(2\sqrt{3})$, $B =$ the baryon number; $g_{bd} = C(q)_b{}^m \gamma_{mn} C(q)_d{}^n$, $(\gamma_{mn}) = \text{diag}(a,a,a,b,b,b,b)$ [8]. The $\lambda$-term in (17) comes from $\Gamma^{WZW}$ term in the action, and leads to the gauge term in the momentum operators;

$$p_b := \frac{\partial L(q,\dot{q})}{\partial \dot{q}^b} = \langle \dot{q}^d, g_{db}\rangle + \lambda h_b\,. \qquad (18)$$

$q^b$'s and $p^b$'s are required to satisfy the ordinary canonical commutation relations. Under (16), $h_b$ is transformed as

$$h_b(q)' = h_b + \partial_b \beta(q)\,. \qquad (19)$$

We can express the gauge field $h_b$ in terms of the spin connection defined for the siebenbein $C_b{}^m$.

The commutator $[\pi_a, \pi_b]$ for $\pi_b := p_b - \lambda h_b$ is

$$[\pi_a, \pi_b] = i\hbar^2 \lambda(\partial_b h_d - \partial_d h_b) = i\hbar^2 C_b{}^m C_d{}^n f_{mn}{}^8\,, \qquad (20)$$

which is shown to be expressed is terms of the curvature tensor defined by $g_{bd}$. We can define the $SU(3)$ generators $\{L_\rho;\ \rho = 1,\ldots,8\}$ as

$$L_\rho = -\langle D(A)_\rho{}^m C^b{}_m,\ p_b - \hbar\lambda h_b\rangle - \hbar\lambda D(A)_\rho{}^8\,; \qquad (21)$$

here, $C^b{}_m$ is the dual of $C_b{}^m$; $D(A)_\rho{}^\sigma$ is defined by $A^\dagger \lambda_\rho A = \sum_{\sigma=1}^{8} D(A)_\rho{}^\sigma \lambda_\sigma$. We see (21) has the form fully analogous to that of the total angular momentum for the system composed of a moving charged particle and a magnetic monopole at the origin of $\mathbb{R}^3$. $L_\rho$ is shown to coincide with the generator constructed in accordance with the prescription proposed by Biedenharn and Dothan [10].

## 4. Conclusion

In the classical mechanics, we have various definitions of the constrained system with the holonomic constraint [11]; e. g. the definition of the system subject to d'Alembert principle of virtual action is equivalent to the definition on the basis of CPA, where the potential wall is so high that the motion is realized along the bottom consistent with the constraint. By treating quantum-mechanically, such an equivalence has been shown not to hold due to appearance of the gauge structure.

In Sect. 3 we have examined the properties of the gauge field $h_b$ emerging from W-Z-W term in $SU(3)$ Skyrme–Witten model. Miyazaki and Tsutsui [12] have pointed out, by examining the $1+1$ dimensional $SU(2)$ principal chiral model, the possibility that W-Z term may be induced quantum-mechanically. It seems necessary for us to consider such possibility also from the viewpoint of CPA.

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