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CHAOTIC SCATTERING ON NONCOMPACT SURFACES OF CONSTANT NEGATIVE CURVATURE

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Abstract. The quantization of the chaotic geodesic motion on Riemann surfaces $\Sigma_{g,\kappa}$ of constant negative curvature with genus g and a finite number of points κ infinitely far away (cusps) describing scattering channels is investigated. It is shown that terms in Selberg's trace formula describing scattering states can be expressed in terms of a regularized time delay. Poles in these quantities give rise to resonances reflecting the chaos of the underlying classical dynamics. Illustrative examples for a class of $\Sigma_{g,2}$ surfaces are given.

1. Introduction

Let us consider the two dimensional sphere S^2 with three points removed. This is a two dimensional surface with three holes. A *classical* charged particle confined to move on the inner part of this "box" can enter and leave the box on any one of the holes. Taking these exceptional points infinitely far away with respect to some metric on S^2 we obtain a pants-like leaky surface $\Sigma_{0,3}$. This surface is called by mathematicians a noncompact Riemann-surface with three cusps and genus zero. It can serve as a model of a three channel scattering problem, where the channels are realized topologically. Taking instead of the sphere ($g = 0$), a torus ($g = 1$) or any higher genus multiply connected surface and moving κ points infinitely far away we obtain a wide variety of multichannel scattering systems. These systems describe the *classical* motion of a charged particle inside a noncompact box, modelled by a Riemann surface of type $\Sigma_{g,\kappa}$.

How can we obtain a unified description of such surfaces? According to Riemann uniformization, except for the sphere $\Sigma_{0,0}$, and the torus $\Sigma_{1,0}$ all of our surfaces $\Sigma_{g,\kappa}$ can be obtained by the following method. Take the Poincaré upper half plane $\mathcal{H} \equiv \{z = x + iy \in \mathbb{C};\ y > 0\}$, with the metric $\mathrm{d}s^2 = y^{-2}((\mathrm{d}x)^2 + (\mathrm{d}y)^2)$ of Gaussian curvature $K = -1$, and form the right coset $\Gamma\backslash\mathcal{H}$ where Γ is a Fuchsian group of the first kind acting on $\mathcal{H}$ discontinuously. Uniformization means that we represent our Riemann surface as this right coset viewed as a fundamental domain in $\mathcal{H}$ with its boundary points identified by elements of Γ, i. e. we have $\Sigma_{g,\kappa} \sim \Gamma\backslash\mathcal{H}$. The copies of the fundamental domain give a tessellation of the upper half plane. Γ is a discrete subgroup of $PSL(2,\mathbb{R})$ the group of fractional linear transformations $\gamma = (ab|cd)$, $\gamma z = \dfrac{az+b}{cz+d}$ with the properties $a, b, c, d, \in \mathbb{R}$ and $ad - bc = 1$. Such discrete subgroups have a finite number of generators and the non-euclidean area V of $\Sigma_{g,\kappa}$ is finite. Recall that using the Gauss–Bonnet theorem the area can be expressed as $V = 2\pi(2g - 2 + \kappa)$. The generators of Γ form the letters of an alphabet, and the elements of Γ are the possible words that can be combined from such letters. There are, however some defining relations that can be used to simplify all possible combinations. Moreover, the couple (g, κ) is determined by the structure of the group. Physically this means that our leaky boxes for scattering can be made, by “cutting” some fundamental domain out of $\mathcal{H}$, and gluing the sides of this domain appropriately. The glueing prescription is given by the structure of Γ, wich determines the topology of the box.

From the classical point of view such surfaces provide simple examples of systems with hard chaos [1]. Indeed, it is well-known by now that the *classical* (geodesic) motion on $\Sigma_{g,\kappa}$ is ergodic, and strongly chaotic. Moreover, according to the general philosophy of quantum chaos, in order to learn more about such systems it is instructive to investigate how this irregular behavior manifests itself in the corresponding *quantum* system.

In this contribution, we quantize the geodesic motion on $\Sigma_{g,\kappa}$, and calculate the scattering matrix of the κ channel quantum scattering problem. A physically important quantity is the Wigner–Smith time delay. For usual scattering systems where the interaction is defined by a potential, this time delay is defined to be the difference between the time spent by the scattered particle within the region of the potential, and the time that it would have spent in the same region had it moved without the influence of the potential. Here the scattering problem is purely geometrical, (we have no potential) hence this notion has to be clarified. Our basic tool will be Selberg’s trace formula. This formula as we show relates the *quantum* data (energy eigenvalues, positions of scattering resonances) to the corresponding *classical* data (length spectrum of classical periodic orbits). Illustrating our results we also give examples for a special class of two channel scattering systems. For reasons of space, we shall merely outline the basic methods. Detailed proofs shall be presented elsewhere [2].

2. Chaotic Scattering on Riemann Surfaces

The description of scattering states and the corresponding S matrices for scattering on noncompact surfaces was initiated by Faddeev [3], and developed in the book by Lax and Phillips [4]. The physical interpretation of these results in the context of quantum chaos was given by Gutzwiller [5, 6], the systematic adaptation of these ideas to surfaces with a multitude of cusps (i. e. points infinitely far away) regarding them as multichannel scattering systems is due to Pnueli [7].

The quantum systems arising from the quantization of the geodesic motion on $\Sigma_{g,\kappa}$ are governed by Schrödinger's equation $H\psi = E\psi$, with the Hamiltonian $H = -\triangle$, where $\triangle = y^2(\partial_x^2 + \partial_y^2)$ is the Laplacian on $\mathcal{H}$ corresponding to the Poincaré metric, with $\psi(z)$ subject to the boundary condition $\psi(\gamma z) = \psi(z)$ for all $\gamma \in \Gamma$ and $z \in \mathcal{H}$. We set $\hbar = 2m = 1$ for convenience. The spectrum of H is known to be both discrete and continuous [8]. Scattering solutions corresponding to the continuous spectrum with $E = 1/4 + k^2$, are described by the Eisenstein series which we now briefly review.

The transformations $\gamma \in \Gamma$ are called *parabolic* (*hiperbolic*, and *elliptic*) if $|\operatorname{trace}\gamma| = 2$, ($> 2$ and < 2, respectively). Such transformations can be shown to be conjugate to a translation $z \to z + a, a \in \mathbb{R}$ (dilatation $z \to bz, b > 0$, rotation respectively). Among the generators of Γ there are κ parabolic ones $P_1, P_2, \dots, P_\kappa$. The fixed points of these generators are the cusps. They will be denoted by $z_1, z_2, \dots, z_\kappa$ and taken to be the elements of $\mathbb{R} \cup \infty$ (the boundary of $\mathcal{H}$) since they are infinitely far away with respect to the Poincaré metric. Under the identification of $\Gamma\backslash\mathcal{H}$ and $\Sigma_{g,\kappa}$ the cusps correspond to punctures (leaks) of our surface describing scattering channels. For each $\alpha = 1, 2, \dots \kappa$ the P_α generate an infinite cyclic subgroup Γ_α of Γ, the stability subgroup of cusp α. Since parabolic elements are conjugate to a translation we can choose an element of $\sigma_\alpha \in SL(2, \mathbb{R})$ such that $\sigma_\alpha\infty = z_\alpha$ and $\sigma_\alpha^{-1} P_\alpha \sigma_\alpha = (1 \pm 1|01)$. We denote by Γ_∞ the infinite cyclic group generated by $(11|01)$ with its fixed point beeing ∞ the standard cusp. This is the group consisting of elements of the form $\pm(1b|01)$, $b \in \mathbb{Z}$. The Eisenstein series $\mathcal{E}_\alpha(z, s)$ corresponding to the cusp z_α of Γ is defined for $\operatorname{Re} s > 1$ by the absolutely convergent series

$$\mathcal{E}_\alpha(z, s) = \sum_{\gamma \in \Gamma_\alpha \backslash \Gamma} \operatorname{Im}(\sigma_\alpha^{-1}\gamma z)^s \quad \alpha = 1, 2, \dots, \kappa. \tag{1}$$

The Eisenstein series defined in this way satisfies the Schrödinger equation, i. e. $H\mathcal{E}_\alpha(z, s) = s(1 - s)\mathcal{E}_\alpha$ for each $\alpha = 1, 2, \dots, \kappa$, and the boundary condition $\mathcal{E}_\alpha(\gamma z, s) = \mathcal{E}_\alpha(z, s)$ for all $\gamma \in \Gamma$. Of course we are interested in the choice $s = \dfrac{1}{2} + \mathrm{i}k$ with $k \in \mathbb{R}$. For this purpose we need a meromorphic continuation of $\mathcal{E}_\alpha(z,s)$ over the whole s-plane. This continuation exists and the poles of $\mathcal{E}_\alpha(z,s)$ are all simple and are in the segment $\frac{1}{2} < s \leq 1$. One can derive a Fourier expansion of $\mathcal{E}_\alpha(z,s)$ at the cusp β which is of the form

$$\mathcal{E}_\alpha(\sigma_\beta z,s) = \delta_{\alpha\beta} y^s + \varphi_{\alpha\beta}(s) y^{1-s} + \sum_{n\neq 0} \varphi_{\alpha\beta}(n,s) W_s(nz) \tag{2}$$

where

$$\varphi_{\alpha\beta}(s) = \pi^{1/2} \frac{\Gamma(s-1/2)}{\Gamma(s)} \sum_{c>0} c^{-2s} \mathcal{S}_{\alpha\beta}(0,0;c) \tag{3}$$

$$\varphi_{\alpha\beta}(n,s) = \frac{\pi^s}{\Gamma(s)} |n|^{s-1} \sum_{c>0} c^{-2s} \mathcal{S}_{\alpha\beta}(0,n;c)\,. \tag{4}$$

Here $\mathcal{S}_{\alpha\beta}(m,n;c)$ are the so called Kloosterman–Gauss–Ramanujan sums defined by

$$\mathcal{S}_{\alpha\beta}(m,n;c) = \sum_{\gamma=(a*|cd)\in\Gamma_\infty\backslash\sigma_\alpha^{-1}\Gamma\sigma_\beta/\Gamma_\infty} \mathrm{e}^{2\pi i(ma+nd)/c}\,, \tag{5}$$

and $W_s(z)$ is Whittaker's function on $z \in \mathbb{R}\backslash\mathbb{C}$. Due to the asymptotic behavior $W_s(z) \sim e^{-2\pi y}$, as $y \to \infty$ the nonzero Fourier coefficients in Eq. (2) are dying out exponentially when we approach any of our cusps. Hence only the term $\delta_{\alpha\beta} y^s + \varphi_{\alpha\beta}(s) y^{1-s}$ from Eq. (2) survives near the cusps. Since y^s and y^{1-s} correspond to the incoming and outgoing plane waves in hyperbolic geometry we are left with the correct asymptotic behavior for scattering states. Moreover, from this it is clear that $\varphi_{\alpha\beta}(1/2 + \mathrm{i}k)$ has to be proportional to the scattering matrix $S_{\alpha\beta}^{\Gamma\backslash\mathcal{H}}(k)$ of our surface $\Gamma\backslash\mathcal{H}$. Indeed according to [4]

$$S_{\alpha\beta}^{\Gamma\backslash\mathcal{H}}(q,k) = -q^{-2\mathrm{i}k} \varphi_{\beta\alpha}\left(\frac{1}{2} + \mathrm{i}k\right)\,, \tag{6}$$

where $0 < q \in \mathbb{R}$ is arbitrary. The minus sign in Eq. (6) results in the nice property of $S_{\alpha\beta}^{\Gamma\backslash\mathcal{H}}(k)$ proved in Proposition 8.14 of [4]

$$S_{\alpha\beta}^{\Gamma\backslash\mathcal{H}}(0) = \delta_{\alpha\beta}\,, \tag{7}$$

moreover, $S_{\alpha\beta}^{\Gamma\backslash\mathcal{H}}(k)$ is unitary and symmetric.

In order to know the scattering matrix of a particular surface the quantity $\varphi_{\alpha\beta}$ of Eq. (3) has to be calculated. For this purpose after fixing Γ uniformizing our surface, we have to describe the double cosets appearing in (5), and characterize our cusps which is generally an arduous task. However, in spite of this we are provided with a variety of matrices $\varphi_{\alpha\beta}$ for different groups Γ having been calculated for different purposes by number theorists see in particular Hejhal's book [8].

We have seen that the space uniformizing our Riemann surfaces is the upper half plane $\mathcal{H}$. Hence for later use it will be useful to describe the scattering matrix associated with the Poincaré upper half plane $\mathcal{H}$ too. On $\mathcal{H}$ we have to solve the equation $y^2(\partial_x^2+\partial_y^2)\Psi(x,y)=s(s-1)\Psi(x,y)$. Putting $s=\dfrac{1}{2}+\mathrm{i}k$ gives then the scattering states. We seek solutions in the form $\Psi(x,y)=F(x)G(y)$, this yields after the separation of variables two ordinary differential equations. Solving the equation for $F(x)$ gives $F(x)=\exp(\mathrm{i}\lambda x)$ with λ^2 beeing the separation constant. Moreover, we get for $\lambda=0$, $G(y)=y^s$, for $\lambda\neq 0$, $G(y)=y^{1/2}K_{s-1/2}(|\lambda|y)$. Using the functional equation $K_s(z)=K_{-s}(z)$ for $\mathrm{Re}\,z>0$ and the asymptotics $K_s(z)\sim 2^{s-1}\Gamma(s)z^{-s}$ for $\mathrm{Re}\,s>0$ and $\mathrm{Re}\,z>0$ as $z\to 0$ of K-Bessel functions we obtain the following formula for the asymptotic behavior of $G(y)$

$$G(y)\sim 2^{ik}\Gamma(\mathrm{i}k)|\lambda|^{-\mathrm{i}k}y^{\frac{1}{2}-\mathrm{i}k}+2^{-\mathrm{i}k}\Gamma(-\mathrm{i}k)|\lambda|^{\mathrm{i}k}y^{\frac{1}{2}+\mathrm{i}k}\,,\quad \text{as } y\to 0^+. \tag{8}$$

From this we can read off the S matrix

$$S^{\mathcal{H}}(|\lambda|,k)=\left(\frac{|\lambda|}{2}\right)^{-2\mathrm{i}k}\frac{\Gamma(\mathrm{i}k)}{\Gamma(-\mathrm{i}k)}=-\left(\frac{|\lambda|}{2}\right)^{-2\mathrm{i}k}\frac{\Gamma(1+\mathrm{i}k)}{\Gamma(1-\mathrm{i}k)}\,,\quad \lambda\neq 0\,. \tag{9}$$

3. Selberg's Trace Formula and the Renormalized Time Delay

Let $h(k)$ be a function satisfying the following conditions

$$h(k):\begin{cases}\text{is even}\,,\\ \text{holomorphic in the strip } |\mathrm{Im}\,k|\leq\frac{1}{2}+\varepsilon,\\ \ll(|k|+1)^{-2-\varepsilon}\text{ in this strip}\,.\end{cases} \tag{10}$$

Let moreover $g(u)=\dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{+\infty}\mathrm{e}^{-\mathrm{i}uk}h(k)\,\mathrm{d}k$ be the Fourier transform of $h(k)$.

Then Selberg's trace formula for noncompact surfaces is [8]

$$
\begin{aligned}
\sum_j h(k_j) = \frac{V}{4\pi} \int_{-\infty}^{+\infty} k h(k) \tanh(\pi k)\, \mathrm{d}k + \sum_{\text{ppo}} \sum_{n=1}^{\infty} \frac{l(p)}{2\sinh(nl(p)/2)} g(nl(p)) \\
+ \frac{1}{4} h(0)\, \mathrm{trace}\left[I - \Phi\left(\frac{1}{2}\right)\right] - \kappa g(0) \log 2 \\
- \frac{\kappa}{2\pi} \int_{-\infty}^{\infty} h(k) \psi(1 + \mathrm{i}k)\, \mathrm{d}k \\
+ \frac{1}{4\pi} \int_{-\infty}^{\infty} h(k)\, \mathrm{trace}\left[\Phi'\left(\frac{1}{2} + \mathrm{i}k\right) \Phi^{-1}\left(\frac{1}{2} + \mathrm{i}k\right)\right] \mathrm{d}k\,.
\end{aligned} \tag{11}
$$

Here $V = 2\pi(2g - 2 + \kappa)$ is the area of the surface, Φ is the $\kappa \times \kappa$ matrix with entries $\varphi_{\alpha\beta}$ as given by (3), $\psi(z) \equiv \frac{\Gamma'}{\Gamma}(z)$ is the digamma function, I is the $\kappa \times \kappa$ identity matrix. $E_j \equiv 1/4 + k_j^2$ are the energy values belonging to the discrete part of the spectrum. The first two terms on the rhs. of the formula are well-known from studies concerning the compact case [6]. The first is the so called Weyl term, and the second contains the sum over the primitive periodic orbits (ppo) of primitive length $l(p)$. The repetitions of the orbits are indexed by n. The remaining four terms, our main concern here, correspond to the modification of the trace formula due to the presence of scattering states. In the following we shall refer to these terms as the *parabolic contribution* to the trace formula.

Our aim is now to rewrite the parabolic contribution in a physically more transparent form. Indeed in the mathematical literature during the algebraic manipulations the physical origin and meaning of these terms is by no means clear. As a first step using the fact that $h(k)$ is even we rewrite the second and third term of this contribution in the form

$$
\begin{aligned}
-\frac{\kappa}{4\pi} \int_{-\infty}^{\infty} h(k) \left(2\log 2 + \psi(1 + \mathrm{i}k) + \psi(1 - \mathrm{i}k)\right) \mathrm{d}k = \\
= \mathrm{i} \frac{\kappa}{4\pi} \int_{-\infty}^{\infty} h(k) \partial_k \log\left(2^{2\mathrm{i}k} \frac{\Gamma(1 + ik)}{\Gamma(1 - ik)}\right) \mathrm{d}k\,.
\end{aligned} \tag{12}
$$

Similarly in the fourth term using the identity $\mathrm{trace} \log M = \log\det M$ we get

$$
\frac{1}{4\pi} \int_{-\infty}^{\infty} h(k)\, \mathrm{trace}\left(\Phi'(\frac{1}{2} + \mathrm{i}k) \Phi^{-1}(\frac{1}{2} + \mathrm{i}k)\right) \mathrm{d}k =
$$

$$= -\mathrm{i}\frac{1}{4\pi}\int_{-\infty}^{\infty} h(k)\partial_k \log\det\left(\Phi(\frac{1}{2}+\mathrm{i}k)\right)\,\mathrm{d}k\,. \quad (13)$$

Let us now define the $\kappa \times \kappa$ matrix using Eq. (9)

$$S^{\mathcal{H}}_{\alpha\beta}(|\lambda|,k) \equiv S^{\mathcal{H}}(|\lambda|,k)\delta_{\alpha\beta}\,. \quad (14)$$

Recall moreover Eq. (6). Then we have two matrices $S^{\mathcal{H}}_{\alpha\beta}(|\lambda|,k)$ and $S^{\Gamma\backslash\mathcal{H}}_{\alpha\beta}(q,k)$. They are depending also on the *positive* quantities q and $|\lambda|$. In order to clarify the meaning of q we proceed as follows [6]. We can put a ring (in hyperbolic geometry this is called a horocycle) on each cusp, regularizing the infinite length of the geodesic corresponding to the scattering trajectory coming from and then going to infinity through the leak. By the transformations σ_α the neighbourhood of each cusp (cuspidal zone) can be mapped to the semi strip $F_q \equiv \{z \in \mathcal{H};\ y > q, 0 < x < 1\}$. Hence the value of q defines the horocycle which plays the role of a monitoring station, this is the place where the particle is registered after beeing scattered. This choice tells us where the "free dynamics" starts. Hence, with the help of the arbitrariness of q we can refer our dynamics on $\Gamma\backslash\mathcal{H}$ to the dynamics on $\mathcal{H}$ by giving q and $|\lambda|$ *the same values*. With this convention the quantity

$$-i\frac{1}{4\pi}\int_{-\infty}^{\infty} h(k)\partial_k \log\det S(k)\,\mathrm{d}k, \qquad S(k) \equiv S^{\Gamma\backslash\mathcal{H}}(k)(S^{\mathcal{H}}(k))^{-1} \quad (15)$$

is independent of $q = |\lambda|$ and equals the last three terms of our parabolic contribution. Moreover, by virtue of Eq. (7) the first term from the parabolic contribution is $\frac{\kappa}{2}h(0)$. Since scattering matrices always occur in the combination as shown in (15), in the following we shall refer to them as $S^{\Gamma\backslash\mathcal{H}}(k)$ and $S^{\mathcal{H}}(k)$, i. e as the ones independent of q and $|\lambda|$. Moreover, we call $S(k)$ the renormalized S matrix.

Introducing the Wigner–Smith time delay [9] for the corresponding S matrices by

$$T^{\Gamma\backslash\mathcal{H}}(k) \equiv \frac{\mathrm{i}}{2k}\partial_k \log\det S^{\Gamma\backslash\mathcal{H}}(k)\,, \quad (16)$$

(and similarly for $S^{\mathcal{H}}(k)$) we can write finally the parabolic contribution in the nice form

$$\frac{\kappa}{2}h(0) - \frac{1}{2\pi}\int_0^{\infty} h(E)T(E)dE, \quad (17)$$

where $T(E) \equiv T^{\Gamma\backslash\mathcal{H}}(E) - T^{\mathcal{H}}(E)$.

We shall refer to $T(E)$ as the *renormalized time delay*. This quantity is the time delay associated with the surface in question *minus* the time delay corresponding to the scattering problem on the Poincaré upper half plane uniformizing our surface. This procedure can also be justified by the fact that the role of interaction is not played by a potential. Now interaction is just restriction of the motion from $\mathcal{H}$ to the fundamental domain $\Gamma\backslash\mathcal{H}$ in $\mathcal{H}$ via the *special boundary condition* $\psi(gz) = \psi(z) \quad g \in \Gamma$ we impose on the wave function. Identifying the free and interacting dynamics in this way, beeing a time difference we can alternatively regard the renormalized time delay of Eq. (17) as the *time delay* for a scattering problem on $\Sigma_{g,\kappa}$.

4. Resonances

The renormalized time delay introduced in the previous section is a function of the scattering energy. It is a measure of the time spent by the particle in the leaky box. It can happen that for special values of the energy the particle is captured for much longer time. For such values we are having a *resonance*.
The $T^{\mathcal{H}}(k)$ part of the renormalized time delay is

$$\begin{aligned} T(k)^{\mathcal{H}} &= -\frac{\kappa}{k}(\log 2 + \operatorname{Re}\psi(1+ik)) \\ &= -\frac{\kappa}{k}\left(\log 2 - \gamma + k^2 \sum_{n=1}^{\infty} \frac{1}{n(n^2+k^2)}\right), \end{aligned} \tag{18}$$

where $\gamma = 0.57721\ldots$ is Euler's constant. It is easy to show that $-kT^{\mathcal{H}}(k)$ is a slowly varying function of k increasing monotonically from $\kappa(\log 2 - \gamma) > 0$ with an asymptotic behavior $\kappa \log 2k$. Hence $T^{\mathcal{H}}(k)$ gives merely a slowly varying smooth contribution to $T(k) = T^{\Gamma\backslash\mathcal{H}}(k) - T^{\mathcal{H}}(k)$.
In order to find resonances we have to investigate the pole structure of the $T^{\Gamma\backslash\mathcal{H}}(s)$ part of $T(s)$ as a function of the *complex* variable. From the formula (see Eqs. (6) and (16)) it is clear that $T^{\Gamma\backslash\mathcal{H}}(s) \sim \dfrac{1}{2s-1}\partial_s \log\det\Phi(s)$. It is known [8] that for $\operatorname{Re} s \geq 1/2$, $\det\Phi(s)$ has a *finite* number of poles $s_a = \varrho_a$ $a = 0, 1, 2\ldots M$ all in the interval $1/2 < s_a \leq 1$. These poles give rise to the eigenvalues $0 \leq E_a = s_a(1-s_a) < 1/4$ in the so called *residual spectrum*. The value $E_0 = 0$ with the value $s_0 = 1$ corresponds to the constant normalized solution $\Psi_0 \equiv V^{-1/2}$ of the Hamiltonian $H = -\triangle$. For $\operatorname{Re} s < 1/2$ the poles are denoted by $s_\mu = \varrho_\mu + i\eta_\mu$ $\mu = 1, 2, \ldots$. Then we have the formula [8]

$$-\partial_s \log\det\Phi(s) = \sum_j \left(\frac{1}{s-s_j} - \frac{1}{s-1+s_j^*}\right) + 2\log g_1 \tag{19}$$

where the sum for j is over $a = 1, 2 \dots M$ and $\mu = 1, 2, \dots,$ and g_1 is a constant. Hence on the critical line $s = 1/2 + ik$ for $T^{\Gamma\backslash\mathcal{H}}(k)$ we have

$$
\begin{aligned}
T^{\Gamma\backslash\mathcal{H}}(k) = \frac{1}{2k}\Bigg(&\sum_{\mu} \frac{1-2\varrho_\mu}{(1/2-\varrho_\mu)^2 + (k-\eta_\mu)^2} \\
&+ \sum_{a=0}^{M} \frac{1-2\varrho_a}{(1/2-\varrho_a)^2 + k^2} + 2\log g_1\Bigg).
\end{aligned}
\tag{20}
$$

Notice that the first sum on the right hand side of this formula is *positive* and the second (corresponding to the possible presence of the residual spectrum) is *negative*. It is known that for a large and important class of groups Γ (congruence groups) we have no residual spectrum besides the obvious point $s_0 = 1$ [10]. Hence in this case the second sum gives merely the term $-\dfrac{1}{2k}\dfrac{1}{1/4+k^2}$. Terms coming from the first sum with $\operatorname{Re} s_\mu < 1/2$ give rise to poles corresponding to resonances. The distribution of these poles shows the irregular behavior of the quantum scattering problem, hence reflecting the chaotic nature of the associated classical dynamics.

Since $T^{\mathcal{H}}(k)$ gives merely a slowly varying smooth contribution to $T(k) = T^{\Gamma\backslash\mathcal{H}}(k) - T^{\mathcal{H}}(k)$, the expression for $T(k)$ is dominated by the terms of the form

$$
T(k) \sim \frac{1}{k}\sum_{\mu} \frac{\Gamma_\mu/2}{(k-k_\mu)^2 + (\Gamma_\mu/2)^2}
\tag{21}
$$

which is a collection of Lorentzians centered at $k_\mu \equiv \eta_\mu$ with a half width $\Gamma_\mu/2 = 1/2 - \varrho_\mu$. The quantity $((1/2-\varrho_\mu)\eta_\mu)^{-1}$ can be thought as a lifetime of the resonance. The allowed values for the quantities k_μ and Γ_μ are determined by the number theoretic properties of the Dirichlet series appearing in the determinant of the S matrix. These properties in turn can be traced back to the behavior of the Kloosterman sum in (4).

In order to use the trace formula of Eq. (11) to relate the quantal data to the classical, we chose a special function $h(k)$ and exploit the pole structure of the renormalized time delay. Let us chose h and g as follows

$$
\begin{aligned}
h_{s,\sigma}(k) &= \left[\left(s-\frac{1}{2}\right)^2 + k^2\right]^{-1} - \left[\left(\sigma-\frac{1}{2}\right)^2 + k^2\right]^{-1}, \\
g_{s,\sigma}(u) &= \frac{\mathrm{e}^{-|u|(s-1/2)}}{2s-1} - \frac{\mathrm{e}^{-|u|(\sigma-1/2)}}{2\sigma-1}.
\end{aligned}
\tag{22}
$$

Here $\sigma > \operatorname{Re} s > 1$, and σ is the regulator. This constraint is sufficient for ensuring the (10) conditions for $h(k)$. The final formula in this case can be written in the form

$$\sum_j \left(\frac{1}{(s-1/2)^2 + k_j^2} - \frac{1}{(\sigma - 1/2)^2 + k_j^2} \right) = \mathcal{F}(s) - \mathcal{F}(\sigma), \tag{23}$$

where

$$\begin{aligned} \mathcal{F}(s) = & -\frac{V}{2\pi}\psi(s) + \frac{2}{2s-1} \sum_{\text{ppo}} \sum_{m=0}^{\infty} \frac{l(p)}{\mathrm{e}^{(s+m)l(p)} - 1} \\ & - \frac{2\kappa}{(2s-1)^2} - \frac{1}{2\pi} \int_0^{\infty} h_s(E) T(E) \, \mathrm{d}E, \end{aligned} \tag{24}$$

where $h_s(E) \equiv (s(s-1)+E)^{-1}$, and $T(E)$ is given by Eq. (17). The second term on the right hand side of Eq. (24) represents the sum over classical periodic orbits well-known from studies of quantum chaos on compact surfaces. Introducing Selberg's zeta function

$$Z(s) = \prod_{\text{ppo}} \prod_{m=0}^{\infty} (1 - \mathrm{e}^{-(s+m)l(p)}), \quad \mathrm{Re}(s) > 1 \tag{25}$$

this sum over the periodic orbits can be written with the help of the logarithmic derivative of $Z(s)$ as

$$\frac{1}{2s-1} \frac{Z'}{Z}(s) = \frac{1}{2s-1} \sum_{\text{ppo}} \sum_{m=0}^{\infty} \frac{l(p)}{\mathrm{e}^{(s+m)l(p)} - 1}. \tag{26}$$

We expect that the trace formula will give us the analytic continuation of $\dfrac{Z'}{Z}(s)$ to the whole complex plane. This quantity can be calculated using purely *classical* data, the length spectra of classical periodic orbits. On the other hand we expect its zeros to be related to the exact locations of *quantum* data namely the energy eigenvalues and scattering resonances. It can be proved that it is really the case. Using the results presented in the book by Venkov [11] one can show that the nontrivial zeros of $Z(s)$ are as follows.

a They are on the line $\mathrm{Re}\, s = 1/2$ localized symmetrically with respect to the real axis, or $s_j \in [0,1]$ symmetrically with respect to $s = 1/2$. They correspond to the eigenvalues of H, $E_j = s_j(1-s_j)$ of the form $s_j = 1/2 + \mathrm{i}k_j$ corresponding to the discrete part of the spectrum. The multiplicity of the zeros equals the multiplicity of E_j.

b They are at the points $s_\mu \equiv z_\mu$ which correspond to the poles of $\det \Phi(s)$, i. e. the poles of the determinant of the scattering matrix on $\Gamma \backslash \mathcal{H}$ with the property $\mathrm{Re}\, s < 1/2$. The multiplicity is not larger than κ i. e. the number of scattering channels. These zeros correspond to the scattering resonances our main concern here.

c There are zeros in $Z(s)$ coming from the poles of $-\partial_s \log\det\Phi(s)$, $s_a = 1-z_a$, and $s_a = z_a \quad a = 0, 1, 2 \ldots M$ corresponding to the residual spectrum. If we have no residual spectrum (e. g. for congruence groups) we have merely the obvious point $z_0 = 1$.

5. Examples

In this section we study a class of two channel scattering problems on leaky boxes of the type $\Sigma_{g,2}$. In order to do this we have to find suitable groups Γ uniformizing the surfaces $\Sigma_{g,2}$. For this purpose let $SL(2,\mathbb{Z})$ be the matrix group of 2×2 unimodular matrices with *integer* entries, i. e. $a, b, c, d \in \mathbb{Z}$ and $ad - bc = 1$. Then let Γ be $\Gamma_0(p)$, i. e. the Hecke congruence group with p prime. Note that $\Gamma_0(p)$ is a subgroup of $SL(2,\mathbb{Z})$ defined as

$$\Gamma_0(p) \equiv \left\{\gamma \in SL(2,\mathbb{Z})\colon\ \gamma \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \bmod p\right\} . \tag{27}$$

It can be shown [12] that for p prime, the number of scattering channels is two ($\kappa = 2$). However, we cannot chose the prime number p arbitrarily, since generally the resulting surface $\Gamma\backslash\mathcal{H}$ will *not* be a manifold, only an orbifold. The reason for this is the presence of a finite number of fixed points of some elliptic transformations. Luckily, it can be shown [2] that for certain values of p these points are missing. These primes are the ones that can be found in the arithmetic progression $11 + 12n$, $n = 0, 1, \ldots$. Moreover, in this case for the genus we have the formula $g_n = n + 1$. Hence our surfaces $\Sigma_{g,2}$ uniformized by $\Gamma_0(p)$ can have only genus $g = 1, 2, 4, 5, 6, 7, 9, 11, \ldots$. The genus one case corresponds to the choice of the prime $p = 11$.

Using the results of [8], for such surfaces the scattering matrix $S^{\Gamma\backslash\mathcal{H}}_{\alpha\beta}$ can be explicitly calculated. The result is

$$S^{\Gamma\backslash\mathcal{H}}_{\alpha\beta}(q,p,k) = -\left(\frac{q}{\pi}\right)^{-2\mathrm{i}k} \frac{\Gamma(1/2-\mathrm{i}k)}{\Gamma(1/2+\mathrm{i}k)} \frac{\zeta(1-2\mathrm{i}k)}{\zeta(1+2\mathrm{i}k)} \mathcal{R}_{\alpha\beta}(p,k), \tag{28}$$
$$\alpha, \beta = 1, 2$$

where

$$\mathcal{R}(p,k) = M_p^{-1}(1/2+\mathrm{i}k) M_p(1/2-\mathrm{i}k), \quad M_p(s) = \begin{pmatrix} 1 & p^s \\ p^s & 1 \end{pmatrix}, \tag{29}$$

and $\zeta(s)$ is Riemann's zeta function. For the renormalized time delay straightforward calculation yields the result

$$\begin{aligned}T(k) = \frac{1}{k}\left(\log 2\pi - \operatorname{Re}\frac{\zeta'}{\zeta}(2\mathrm{i}k)\right) \\ + \frac{2}{k}p\log p\left(\frac{p-\cos(2k\log p)}{1-2p\cos(2k\log p)+p^2}\right).\end{aligned} \tag{30}$$

Parametrizing instead of the primes p_n, by the genus g_n by writing $p_n = 11 + 12(g_n - 1)$ in Eq. (30) the last part of the time delay besides the energy, *depends on the genus* of the surface.

Now we use the formula

$$-\frac{\zeta'}{\zeta}(s) = \frac{s}{s-1} - \sum_{\varrho}\frac{s}{\varrho(s-\varrho)} + \sum_{n=1}^{\infty}\frac{s}{2n(s+2n)} - \frac{\zeta'}{\zeta}(0) \tag{31}$$

with $\frac{\zeta'}{\zeta}(0) = \log 2\pi$ to write $T(k)$ in terms of the Riemann zeros $\varrho = 1/2 - 2\mathrm{i}k_\varrho$ of $\zeta(s)$.

$$\begin{aligned}kT(k) = {} & \log 2 - \gamma + \sum_{n=1}^{\infty}\frac{k^2}{n(n^2+k^2)} - \frac{1/2}{(1/2)^2+k^2} \\ & + 2p\log p\left(\frac{p-\cos(2k\log p)}{1-2p\cos(2k\log p)+p^2}\right) \\ & + \sum_{k_\varrho>0}\left(\frac{1/4}{(1/4)^2+(k+k_\varrho)^2} + \frac{1/4}{(1/4)^2+(k-k_\varrho)^2}\right).\end{aligned} \tag{32}$$

The first three terms correspond to the time delay of $-T^{\mathcal{H}}$, only the fourth term is depending on the genus via p, the fifth term corresponds to the obvious point $s_0 = 1, \quad E_0 = 0$ in the residual spectrum. The sixth term produces the resonances we are interested in. We see that the special form of $T(k)$ fits into the general scheme suggested by equations (20, 21). Moreover we see that $\Gamma_\mu = 1/2$, and $k_\mu = r_\mu/2$, where $r_\mu \quad \mu = 1, 2, \ldots$ ranges over the zeroes of Riemann's zeta function. Hence the energy values for which the particle is captured for a long time, are related to the famous Riemann zeroes. The irregular distribution of these zeros reflects the chaos of the associated scattering problem.

6. Conclusions

In this paper we considered the problem of quantizing the geodesic motion on noncompact surfaces of constant negative curvature. This problem can be regarded as a model of multichannel quantum scattering. Knowing that the geodesic motion on such surfaces is chaotic, we examined how the chaos of the underlying classical dynamics manifests itself in the corresponding quantum system. We calculated the scattering matrix, and introduced the associated time delays. With the help of Selberg's trace formula we established a connection between the classical periodic orbits, and the quantum resonances, and energy eigenvalues. Illustrative examples for a class of $\Sigma_{g,2}$ surfaces were given.

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