VIII. Integration, 523-554

DOI: 10.3792/euclid/9798989504206-9

from

Lie Groups
Beyond an Introduction
Digital Second Edition, 2023

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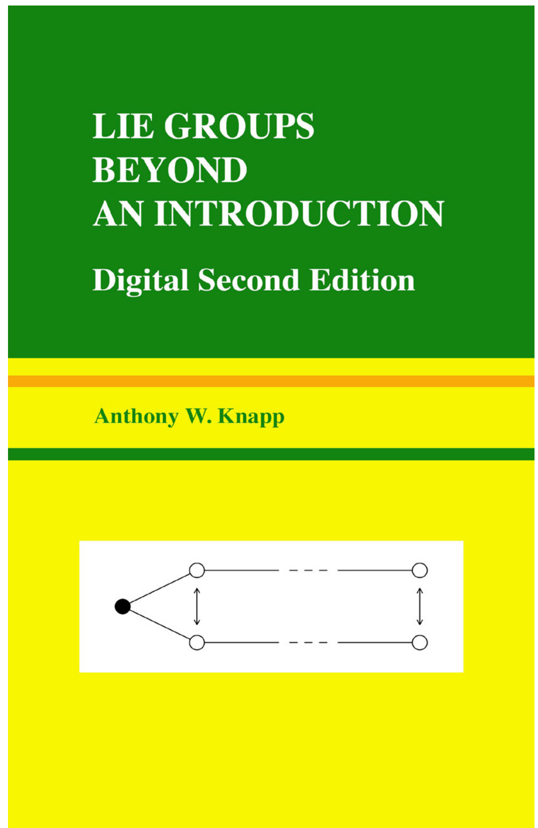

Full Book DOI: 10.3792/euclid/9798989504206
ISBN: 979-8-9895042-0-6

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Lie Groups Beyond an Introduction, Digital Second Edition

Pages vii–xviii and 1–812 are the same in the digital and printed second editions. A list of corrections as of June 2023 has been included as pages 813–820 of the digital second edition. The corrections have not been implemented in the text.

Cover: Vogan diagram of $\mathfrak{sl}(2n, \mathbb{R})$. See page 399.

AMS Subject Classifications: 17-01, 22-01

CHAPTER VIII

Integration

Abstract. An m-dimensional manifold M that is oriented admits a notion of integration $f \mapsto \int_M f\omega$ for any smooth m form ω. Here f can be any continuous real-valued function of compact support. This notion of integration behaves in a predictable way under diffeomorphism. When ω satisfies a positivity condition relative to the orientation, the integration defines a measure on M. A smooth map $M \to N$ with $\dim M < \dim N$ carries M to a set of measure zero.

For a Lie group G, a left Haar measure is a nonzero Borel measure invariant under left translations. Such a measure results from integration of ω if $M = G$ and if the form ω is positive and left invariant. A left Haar measure is unique up to a multiplicative constant. Left and right Haar measures are related by the modular function, which is given in terms of the adjoint representation of G on its Lie algebra. A group is unimodular if its Haar measure is two-sided invariant. Unimodular Lie groups include those that are abelian or compact or semisimple or reductive or nilpotent.

When a Lie group G has the property that almost every element is a product of elements of two closed subgroups S and T with compact intersection, then the left Haar measures on G, S, and T are related. As a consequence, Haar measure on a reductive Lie group has a decomposition that mirrors the Iwasawa decomposition, and also Haar measure satisfies various relationships with the Haar measures of parabolic subgroups. These integration formulas lead to a theorem of Helgason that characterizes and parametrizes irreducible finite-dimensional representations of G with a nonzero K fixed vector.

The Weyl Integration Formula tells how to integrate over a compact connected Lie group by first integrating over conjugacy classes. It is a starting point for an analytic treatment of parts of representation theory for such groups. Harish-Chandra generalized the Weyl Integration Formula to reductive Lie groups that are not necessarily compact. The formula relies on properties of Cartan subgroups proved in Chapter VII.

1. Differential Forms and Measure Zero

Let M be an m-dimensional manifold, understood to be smooth and to have a countable base for its topology; M need not be connected. We say that M is **oriented** if an atlas of compatible charts $(U_\alpha, \varphi_\alpha)$ is given with the property that the m-by-m derivative matrices of all coordinate changes

$$\varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \to \varphi_\beta(U_\alpha \cap U_\beta) \tag{8.1}$$

have everywhere positive determinant. When M is oriented, a compatible chart (U, φ) is said to be **positive** relative to $(U_\alpha, \varphi_\alpha)$ if the derivative matrix of $\varphi \circ \varphi_\alpha^{-1}$ has everywhere positive determinant for all α. We always have the option of adjoining to the given atlas of charts for an oriented M any or all other compatible charts (U, φ) that are positive relative to all $(U_\alpha, \varphi_\alpha)$, and M will still be oriented.

On an oriented M as above, there is a well defined notion of integration involving smooth m forms, which is discussed in Chapter V of Chevalley [1946], Chapter X of Helgason [1962], and elsewhere. In this section we shall review the definition and properties, and then we shall apply the theory in later sections in the context of Lie groups.

We shall make extensive use of **pullbacks** of differential forms. If $\Phi : M \to N$ is smooth and if ω is a smooth k form on N, then $\Phi^*\omega$ is the smooth k form on M given by

$$(8.2) \qquad (\Phi^*\omega)_p(\xi_1, \dots, \xi_k) = \omega_{\Phi(p)}(d\Phi_p(\xi_1), \dots, d\Phi_p(\xi_k))$$

for p in M and $\xi_1, \dots, \xi_k$ in the tangent space $T_p(M)$; here $d\Phi_p$ is the differential of Φ at p. In case M and N are open subsets of $\mathbb{R}^m$ and ω is the smooth m form $F(y_1, \dots, y_m)\, dy_1 \wedge \cdots \wedge dy_m$ on N, the formula for $\Phi^*\omega$ on M is

$$(8.3) \quad \Phi^*\omega = (F \circ \Phi)(x_1, \dots, x_m) \det(\Phi'(x_1, \dots, x_m))\, dx_1 \wedge \cdots \wedge dx_m,$$

where Φ has m entries $y_1(x_1, \dots, x_m), \dots, y_m(x_1, \dots, x_m)$ and where Φ' denotes the derivative matrix $\left(\dfrac{\partial y_i}{\partial x_j}\right)$.

Let ω be a smooth m form on M. The theory of integration provides a definition of $\int_M f\omega$ for all f in the space $C_{\text{com}}(M)$ of continuous functions of compact support on M. Namely we first assume that f is compactly supported in a coordinate neighborhood U_α. The local expression for ω in $\varphi_\alpha(U_\alpha)$ is

$$(8.4) \qquad (\varphi_\alpha^{-1})^*\omega = F_\alpha(x_1, \dots, x_m)\, dx_1 \wedge \cdots \wedge dx_m$$

with $F_\alpha : \varphi_\alpha(U_\alpha) \to \mathbb{R}$ smooth. Since $f \circ \varphi_\alpha^{-1}$ is compactly supported in $\varphi_\alpha(U_\alpha)$, it makes sense to define

$$(8.5a) \quad \int_M f\omega = \int_{\varphi_\alpha(U_\alpha)} (f \circ \varphi_\alpha^{-1})(x_1, \dots, x_m) F_\alpha(x_1, \dots, x_m)\, dx_1 \cdots dx_m.$$

If f is compactly supported in an intersection $U_\alpha \cap U_\beta$, then the integral is given also by

$$\text{(8.5b)}\quad \int_M f\omega = \int_{\varphi_\beta(U_\beta)} (f \circ \varphi_\beta^{-1})(y_1, \dots, y_m) F_\beta(y_1, \dots, y_m)\, dy_1 \cdots dy_m.$$

To see that the right sides of (8.5) are equal, we use the change of variables formula for multiple integrals. The change of variables $y = \varphi_\beta \circ \varphi_\alpha^{-1}(x)$ in (8.1) expresses $y_1, \dots, y_m$ as functions of $x_1, \dots, x_m$, and (8.5b) therefore is

$$\begin{aligned} &= \int_{\varphi_\beta(U_\alpha \cap U_\beta)} (f \circ \varphi_\beta^{-1})(y_1, \dots, y_m) F_\beta(y_1, \dots, y_m)\, dy_1 \cdots dy_m \\ &= \int_{\varphi_\alpha(U_\alpha \cap U_\beta)} f \circ \varphi_\beta^{-1} \circ \varphi_\beta \circ \varphi_\alpha^{-1}(x_1, \dots, x_m) \\ &\qquad \times F_\beta \circ \varphi_\beta \circ \varphi_\alpha^{-1}(x_1, \dots, x_m) |\det(\varphi_\beta \circ \varphi_\alpha^{-1})'|\, dx_1 \cdots dx_m. \end{aligned}$$

The right side here will be equal to the right side of (8.5a) if it is shown that

$$\text{(8.6)}\qquad F_\alpha \stackrel{?}{=} (F_\beta \circ \varphi_\beta \circ \varphi_\alpha^{-1}) |\det(\varphi_\beta \circ \varphi_\alpha^{-1})'|.$$

Now

$$\begin{aligned} F_\alpha\, dx_1 \wedge \cdots \wedge dx_m &= (\varphi_\alpha^{-1})^* \omega && \text{from (8.4)} \\ &= (\varphi_\beta \circ \varphi_\alpha^{-1})^* (\varphi_\beta^{-1})^* \omega \\ &= (\varphi_\beta \circ \varphi_\alpha^{-1})^* (F_\beta\, dy_1 \wedge \cdots dy_m) && \text{from (8.4)} \\ &= (F_\beta \circ \varphi_\beta \circ \varphi_\alpha^{-1}) \det(\varphi_\beta \circ \varphi_\alpha^{-1})'\, dx_1 \wedge \cdots \wedge dx_m && \text{by (8.3).} \end{aligned}$$

Thus

$$\text{(8.7a)}\qquad F_\alpha = (F_\beta \circ \varphi_\beta \circ \varphi_\alpha^{-1}) \det(\varphi_\beta \circ \varphi_\alpha^{-1})'.$$

Since $\det(\varphi_\beta \circ \varphi_\alpha^{-1})'$ is everywhere positive, (8.6) follows from (8.7a). Therefore $\int_M f\omega$ is well defined if f is compactly supported in $U_\alpha \cap U_\beta$.

For future reference we rewrite (8.7a) in terms of coordinates as

$$\text{(8.7b)}\qquad F_\beta(y_1, \dots, y_m) = F_\alpha(x_1, \dots, x_m) \det \left(\frac{\partial y_i}{\partial x_j} \right)^{-1}.$$

To define $\int_M f\omega$ for general f in $C_{\text{com}}(M)$, we make use of a smooth partition of unity $\{\psi_\alpha\}$ such that ψ_α is compactly supported in U_α and only finitely many ψ_α are nonvanishing on each compact set. Then $f = \sum \psi_\alpha f$ is actually a finite sum, and we can define

$$\int_M f\omega = \sum \int_M (\psi_\alpha f)\omega. \tag{8.8}$$

Using the consistency result proved above by means of (8.6), one shows that this definition is unchanged if the partition of unity is changed, and then $\int_M f\omega$ is well defined. (For a proof one may consult either of the above references.)

When ω is fixed, it is apparent from (8.5a) and (8.8) that the map $f \mapsto \int_M f\omega$ is a linear functional on $C_{\text{com}}(M)$. We say that ω is **positive** relative to the given atlas if each local expression (8.4) has $F_\alpha(x_1, \ldots, x_m)$ everywhere positive on $\varphi_\alpha(U_\alpha)$. In this case the linear functional $f \mapsto \int_M f\omega$ is positive in the sense that $f \geq 0$ implies $\int_M f\omega \geq 0$. By the Riesz Representation Theorem there exists a Borel measure $d\mu_\omega$ on M such that $\int_M f\omega = \int_M f(x)\,du_\omega(x)$ for all $f \in C_{\text{com}}(M)$. The first two propositions tell how to create and recognize positive ω's.

Proposition 8.9. If an m-dimensional manifold M admits a nowhere-vanishing m form ω, then M can be oriented so that ω is positive.

PROOF. Let $\{(U_\alpha, \varphi_\alpha)\}$ be an atlas for M. The components of each U_α are open and cover U_α. Thus there is no loss of generality in assuming that each coordinate neighborhood U_α is connected. For each U_α, let F_α be the function in (8.4) in the local expression for ω in $\varphi_\alpha(U_\alpha)$. Since ω is nowhere vanishing and U_α is connected, F_α has constant sign. If the sign is negative, we redefine φ_α by following it with the map $(x_1, x_2, \ldots, x_m) \mapsto (-x_1, x_2, \ldots, x_m)$, and then F_α is positive. In this way we can arrange that all F_α are positive on their domains. Referring to (8.7b), we see that each function $\det\left(\dfrac{\partial y_i}{\partial x_j}\right)$ is positive on its domain. Hence M is oriented. Since the F_α are all positive, ω is positive relative to this orientation.

Proposition 8.10. If a connected manifold M is oriented and if ω is a nowhere-vanishing smooth m form on M, then either ω is positive or $-\omega$ is positive.

PROOF. At each point p of M, all the functions F_α representing ω locally as in (8.4) have $F_\alpha(\varphi_\alpha(p))$ nonzero of the same sign because of (8.7b), the nowhere-vanishing of ω, and the fact that M is oriented. Let S be the set where this common sign is positive. Possibly replacing ω by $-\omega$, we may assume that S is nonempty. We show that S is open and closed. Let p be in S and let U_α be a coordinate neighborhood containing p. Then $F_\alpha(\varphi_\alpha(p)) > 0$ since p is in S, and hence $F_\alpha \circ \varphi_\alpha$ is positive in a neighborhood of p. Hence S is open. Let $\{p_n\}$ be a sequence in S converging to p in M, and let U_α be a coordinate neighborhood containing p. Then $F_\alpha(\varphi_\alpha(p_n)) > 0$ and $F_\alpha(\varphi_\alpha(p)) \neq 0$. Since $\lim F_\alpha(\varphi_\alpha(p_n)) = F_\alpha(\varphi_\alpha(p))$, $F_\alpha(\varphi_\alpha(p))$ is > 0. Therefore p is in S, and S is closed. Since M is connected and S is nonempty open closed, $S = M$.

The above theory allows us to use nowhere-vanishing smooth m forms to define measures on manifolds. But we can define sets of measure zero without m forms and orientations. Let $\{(U_\alpha, \varphi_\alpha)\}$ be an atlas for the m-dimensional manifold M. We say that a subset S of M has **measure zero** if $\varphi_\alpha(S \cap U_\alpha)$ has m-dimensional Lebesgue measure 0 for all α.

Suppose that M is oriented and ω is a positive m form. If $d\mu_\omega$ is the associated measure and if ω has local expressions as in (8.4), then (8.5a) shows that

$$d\mu_\omega(S \cap U_\alpha) = \int_{\varphi_\alpha(S\cap U_\alpha)} F_\alpha(x_1, \dots, x_m)\, dx_1 \cdots dx_m. \tag{8.11}$$

If S has measure zero in the sense of the previous paragraph, then the right side is 0 and hence $d\mu_\omega(S \cap U_\alpha) = 0$. Since a countable collection of U_α's suffices to cover M, $d\mu_\omega(S) = 0$. Thus a set a measure zero as in the previous paragraph has $d\mu_\omega(S) = 0$.

Conversely if ω is a nowhere-vanishing positive m form, $d\mu_\omega(S) = 0$ implies that S has measure zero as above. In fact, the left side of (8.11) is 0, and the integrand on the right side is > 0 everywhere. Therefore $\varphi_\alpha(S \cap U_\alpha)$ has Lebesgue measure 0.

Let $\Phi : M \to N$ be a smooth map between m-dimensional manifolds. A **critical point** p of Φ is a point where $d\Phi_p$ has rank $< m$. In this case, $\Phi(p)$ is called a **critical value**.

Theorem 8.12 (Sard's Theorem). If $\Phi : M \to N$ is a smooth map between m-dimensional manifolds, then the set of critical values of Φ has measure zero in N.

PROOF. About each point of M, we can choose a compatible chart (U, φ) so that $\Phi(U)$ is contained in a coordinate neighborhood of N. Countably many of these charts in M cover M, and it is enough to consider one of them. We may then compose with the coordinate mappings to see that it is enough to treat the following situation: Φ is a smooth map defined on a neighborhood of $C = \{x \in \mathbb{R}^m \mid 0 \le x_i \le 1 \text{ for } 1 \le i \le m\}$ with values in $\mathbb{R}^m$, and we are to prove that Φ of the critical points in C has Lebesgue measure 0 in $\mathbb{R}^m$.

For points $x = (x_1, \dots, x_m)$ and $x' = (x'_1, \dots, x'_m)$ in $\mathbb{R}^m$, the Mean Value Theorem gives

$$\Phi_i(x') - \Phi_i(x) = \sum_{j=1}^{m} \frac{\partial \Phi_i}{\partial x_j}(z_i)(x'_j - x_j), \tag{8.13}$$

where z_i is a point on the line segment from x to x'. Since the $\dfrac{\partial \Phi_i}{\partial x_j}$ are bounded on C, we see as a consequence that

$$\|\Phi(x') - \Phi(x)\| \le a\|x' - x\| \tag{8.14}$$

with a independent of x and x'. Let $L_x(x') = (L_{x,1}(x'), \dots, L_{x,m}(x'))$ be the best first-order approximation to Φ about x, namely

$$L_{x,i}(x') = \Phi_i(x) + \sum_{j=1}^{m} \frac{\partial \Phi_i}{\partial x_j}(x)(x'_j - x_j). \tag{8.15}$$

Subtracting (8.15) from (8.13), we obtain

$$\Phi_i(x') - L_{x,i}(x') = \sum_{j=1}^{m} \left(\frac{\partial \Phi_i}{\partial x_j}(z_i) - \frac{\partial \Phi_i}{\partial x_j}(x) \right) (x'_j - x_j).$$

Since $\dfrac{\partial \Phi_i}{\partial x_j}$ is smooth and $\|z_i - x\| \le \|x' - x\|$, we deduce that

$$\|\Phi(x') - L_x(x')\| \le b\|x' - x\|^2 \tag{8.16}$$

with b independent of x and x'.

If x is a critical point, let us bound the image of the set of x' with $\|x' - x\| \le c$. The determinant of the linear part of L_x is 0, and hence L_x has image in a hyperplane. By (8.16), $\Phi(x')$ has distance $\le bc^2$ from this hyperplane. In each of the $m - 1$ perpendicular directions, (8.14) shows that $\Phi(x')$ and $\Phi(x)$ are at distance $\le ac$ from each other. Thus $\Phi(x')$ is contained in a rectangular solid about $\Phi(x)$ of volume $2^m(ac)^{m-1}(bc^2) = 2^m a^{m-1} b c^{m+1}$.

We subdivide C into N^m smaller cubes of side $1/N$. If one of these smaller cubes contains a critical point x, then any point x' in the smaller cube has $\|x' - x\| \leq \sqrt{m}/N$. By the result of the previous paragraph, Φ of the cube is contained in a solid of volume $2^m a^{m-1} b(\sqrt{m}/N)^{m+1}$. The union of these solids, taken over all small cubes containing a critical point, contains the critical values. Since there are at most N^m cubes, the outer measure of the set of critical values is $\leq 2^m a^{m-1} b m^{\frac{1}{2}(m+1)} N^{-1}$. This estimate is valid for all N, and hence the set of critical values has Lebesgue measure 0.

Corollary 8.17. If $\Phi : M \to N$ is a smooth map between manifolds with $\dim M < \dim N$, then the image of Φ has measure zero in N.

PROOF. Let $\dim M = k < m = \dim N$. Without loss of generality we may assume that $M \subseteq \mathbb{R}^k$. Sard's Theorem (Theorem 8.12) applies to the composition of the projection $\mathbb{R}^m \to \mathbb{R}^k$ followed by Φ. Every point of the domain is a critical point, and hence every point of the image is a critical value. The result follows.

We define a **lower-dimensional set** in N to be any set contained in the countable union of smooth images of manifolds M with $\dim M < \dim N$. It follows from Corollary 8.17 that

(8.18) any lower-dimensional set in N has measure zero.

Let M and N be oriented m-dimensional manifolds, and let $\Phi : M \to N$ be a diffeomorphism. We say that Φ is **orientation preserving** if, for every chart $(U_\alpha, \varphi_\alpha)$ in the atlas for M, the chart $(\Phi(U_\alpha), \varphi_\alpha \circ \Phi^{-1})$ is positive relative to the atlas for N. In this case the atlas of charts for N can be taken to be $\{(\Phi(U_\alpha), \varphi_\alpha \circ \Phi^{-1})\}$. Then the change of variables formula for multiple integrals may be expressed using pullbacks as in the following proposition.

Proposition 8.19. Let M and N be oriented m-dimensional manifolds, and let $\Phi : M \to N$ be an orientation-preserving diffeomorphism. If ω is a smooth m form on N, then

$$\int_N f\omega = \int_M (f \circ \Phi)\Phi^*\omega$$

for every $f \in C_{\text{com}}(N)$.

PROOF. Let the atlas for M be $\{(U_\alpha, \varphi_\alpha)\}$, and take the atlas for N to be $\{(\Phi(U_\alpha), \varphi_\alpha \circ \Phi^{-1})\}$. It is enough to prove the result for f compactly supported in a particular $\Phi(U_\alpha)$. For such f, (8.5) gives

(8.20a)
$$\int_N f\omega = \int_{\varphi_\alpha\circ\Phi^{-1}(\Phi(U_\alpha))} f\circ\Phi\circ\varphi_\alpha^{-1}(x_1,\dots,x_m)F_\alpha(x_1,\dots,x_m)\,dx_1\cdots dx_m,$$

where F_α is the function with

$$\text{(8.20b)} \qquad ((\varphi_\alpha \circ \Phi^{-1})^{-1})^*\omega = F_\alpha(x_1,\dots,x_m)\,dx_1\wedge\cdots\wedge dx_m.$$

The function $f\circ\Phi$ is compactly supported in U_α, and (8.5) gives also

(8.20c)
$$\int_M (f\circ\Phi)\Phi^*\omega = \int_{\varphi_\alpha(U_\alpha)} f\circ\Phi\circ\varphi_\alpha^{-1}(x_1,\dots,x_m)F_\alpha(x_1,\dots,x_m)\,dx_1\cdots dx_m$$

since

$$(\varphi_\alpha^{-1})^*\Phi^*\omega = ((\varphi_\alpha\circ\Phi^{-1})^{-1})^*\omega = F_\alpha(x_1,\dots,x_m)\,dx_1\wedge\cdots\wedge dx_m$$

by (8.20b). The right sides of (8.20a) and (8.20c) are equal, and hence so are the left sides.

2. Haar Measure for Lie Groups

Let G be a Lie group, and let $\mathfrak{g}$ be its Lie algebra. For $g \in G$, let $L_g : G \to G$ and $R_g : G \to G$ be the left and right translations $L_g(x) = gx$ and $R_g(x) = xg$. A smooth k form ω on G is **left invariant** if $L_g^*\omega = \omega$ for all $g \in G$, **right invariant** if $R_g^*\omega = \omega$ for all $g \in G$.

Regarding $\mathfrak{g}$ as the tangent space at 1 of G, let $X_1, \dots, X_m$ be a basis of $\mathfrak{g}$, and let $\widetilde{X}_1, \dots, \widetilde{X}_m$ be the corresponding left-invariant vector fields on G. We can define smooth 1 forms $\omega_1, \dots, \omega_m$ on G by the condition that $(\omega_i)_p((\widetilde{X}_j)_p) = \delta_{ij}$ for all p. Then $\omega_1, \dots, \omega_m$ are left invariant, and at each point of G they form a basis of the dual of the tangent space at that point. The differential form $\omega = \omega_1 \wedge \cdots \wedge \omega_m$ is therefore a smooth m form that is nowhere vanishing on G. Since pullback commutes with $\wedge$, ω is left invariant. Using Proposition 8.9, we can orient G so that ω is positive. This proves part of the following theorem.

Theorem 8.21. If G is a Lie group of dimension m, then G admits a nowhere-vanishing left-invariant smooth m form ω. Then G can be oriented so that ω is positive, and ω defines a nonzero Borel measure $d\mu_l$ on G that is left invariant in the sense that $d\mu_l(L_gE) = d\mu_l(E)$ for all $g \in G$ and every Borel set E in G.

PROOF. We have seen that ω exists and that G may be oriented so that ω is positive. Let $d\mu_l$ be the associated measure, so that $\int_G f\omega = \int_G f(x)\,d\mu_l(x)$ for all $f \in C_{\text{com}}(G)$. From Proposition 8.19 and the equality $L_g^*\omega = \omega$, we have

$$\int_G f(gx)\,d\mu_l(x) = \int_G f(x)\,d\mu_l(x) \tag{8.22}$$

for all $f \in C_{\text{com}}(G)$. If K is a compact set in G, we can apply (8.22) to all f that are $\geq$ the characteristic function of K. Taking the infimum shows that $d\mu_l(L_{g^{-1}}K) = d\mu_l(K)$. Since G has a countable base, the measure $d\mu_l$ is automatically regular, and hence $d\mu_l(L_{g^{-1}}E) = d\mu_l(E)$ for all Borel sets E.

A nonzero Borel measure on G invariant under left translation is called a **left Haar measure** on G. Theorem 8.21 thus says that a left Haar measure exists.

In the construction of the left-invariant m form ω before Theorem 8.21, a different basis of G would have produced a multiple of ω, hence a multiple of the left Haar measure in Theorem 8.21. If the second basis is $Y_1, \ldots, Y_m$ and if $Y_j = \sum_{i=1}^m a_{ij}X_i$, then the multiple is $\det(a_{ij})^{-1}$. When the determinant is positive, we are led to orient G in the same way, otherwise oppositely. The new left Haar measure is $|\det(a_{ij})|^{-1}$ times the old. The next result strengthens this assertion of uniqueness of Haar measure.

Theorem 8.23. If G is a Lie group, then any two left Haar measures on G are proportional.

PROOF. Let $d\mu_1$ and $d\mu_2$ be left Haar measures. Then the sum $d\mu = d\mu_1 + d\mu_2$ is a left Haar measure, and $d\mu(E) = 0$ implies $d\mu_1(E) = 0$. By the Radon–Nikodym Theorem there exists a Borel function $h_1 \geq 0$ such that $d\mu_1 = h_1\,d\mu$. Fix g in G. By the left invariance of $d\mu_1$ and $d\mu$, we have

$$\begin{aligned}\int_G f(x)h_1(g^{-1}x)\,d\mu(x) &= \int_G f(gx)h_1(x)\,d\mu(x) = \int_G f(gx)\,d\mu_1(x)\\ &= \int_G f(x)\,d\mu_1(x) = \int_G f(x)h_1(x)\,d\mu(x)\end{aligned}$$

for every Borel function $f \geq 0$. Therefore the measures $h_1(g^{-1}x)\,d\mu(x)$ and $h_1(x)\,d\mu(x)$ are equal, and $h_1(g^{-1}x) = h_1(x)$ for almost every $x \in G$ (with respect to $d\mu$). We can regard $h_1(g^{-1}x)$ and $h_1(x)$ as functions of $(g, x) \in G \times G$, and these are Borel functions since the group operations are continuous. For each g, they are equal for almost every x. By Fubini's Theorem they are equal for almost every pair (g, x) (with respect to the product measure), and then for almost every x they are equal for almost every g. Pick such an x, say x_0. Then it follows that $h_1(x) = h_1(x_0)$ for almost every x. Thus $d\mu_1 = h_1(x_0)\,d\mu$. So $d\mu_1$ is a multiple of $d\mu$, and so is $d\mu_2$.

A **right Haar measure** on G is a nonzero Borel measure invariant under right translations. Such a measure may be constructed similarly by starting from right-invariant 1 forms and creating a nonzero right-invariant m form. As is true for left Haar measures, any two right Haar measures are proportional. To simplify the notation, we shall denote particular left and right Haar measures on G by $d_l x$ and $d_r x$, respectively.

An important property of left and right Haar measures is that

(8.24) any nonempty open set has nonzero Haar measure.

In fact, in the case of a left Haar measure if any compact set is given, finitely many left translates of the given open set together cover the compact set. If the open set had 0 measure, so would its left translates and so would every compact set. Then the measure would be identically 0 by regularity.

Another important property is that

(8.25) any lower-dimensional set in G has 0 Haar measure.

In fact, Theorems 8.21 and 8.23 show that left and right Haar measures are given by nowhere-vanishing differential forms. The sets of measure 0 relative to Haar measure are therefore the same as the sets of measure zero in the sense of Sard's Theorem, and (8.25) is a special case of (8.18).

Since left translations on G commute with right translations, $d_l(\,\cdot\, t)$ is a left Haar measure for any $t \in G$. Left Haar measures are proportional, and we therefore define the **modular function** $\Delta : G \to \mathbb{R}^+$ of G by

$$d_l(\,\cdot\, t) = \Delta(t)^{-1} d_l(\,\cdot\,). \tag{8.26}$$

Proposition 8.27. If G is a Lie group, then the modular function for G is given by $\Delta(t) = |\det \mathrm{Ad}(t)|$.

PROOF. If X is in $\mathfrak{g}$ and $\widetilde{X}$ is the corresponding left-invariant vector field, then we can use Proposition 1.86 to make the computation

$$\begin{aligned}(dR_{t^{-1}})_p(\widetilde{X}_p)h &= \widetilde{X}_p(h \circ R_{t^{-1}}) = \frac{d}{dr}\, h(p(\exp rX)t^{-1})|_{r=0} \\ &= \frac{d}{dr}\, h(pt^{-1}\exp r\operatorname{Ad}(t)X)|_{r=0} = (\operatorname{Ad}(t)X)^{\sim} h(pt^{-1}),\end{aligned}$$

and the conclusion is that

$$(dR_{t^{-1}})_p(\widetilde{X}_p) = (\operatorname{Ad}(t)X)^{\sim}_{pt^{-1}}. \tag{8.28}$$

Therefore the left-invariant m form ω has

$$\begin{aligned}&(R^*_{t^{-1}}\omega)_p((\widetilde{X}_1)_p, \dots, (\widetilde{X}_m)_p) \\ &\qquad = \omega_{pt^{-1}}((dR_{t^{-1}})_p(\widetilde{X}_1)_p, \dots, (dR_{t^{-1}})_p(\widetilde{X}_m)_p) \\ &\qquad = \omega_{pt^{-1}}((\operatorname{Ad}(t)X_1)^{\sim}_{pt^{-1}}, \dots, (\operatorname{Ad}(t)X_m)^{\sim}_{pt^{-1}}) \qquad \text{by (8.28)} \\ &\qquad = (\det\operatorname{Ad}(t))\omega_{pt^{-1}}((\widetilde{X}_1)_{pt^{-1}}, \dots, (\widetilde{X}_m)_{pt^{-1}}),\end{aligned}$$

and we obtain

$$R^*_{t^{-1}}\omega = (\det\operatorname{Ad}(t))\omega. \tag{8.29}$$

The assumption is that ω is positive, and therefore $R^*_{t^{-1}}\omega$ or $-R^*_{t^{-1}}\omega$ is positive according as the sign of $\det\operatorname{Ad}(t)$. When $\det\operatorname{Ad}(t)$ is positive, Proposition 8.19 and (8.29) give

$$\begin{aligned}(\det\operatorname{Ad}(t))\int_G f(x)\,d_lx &= (\det\operatorname{Ad}(t))\int_G f\omega = \int_G f\, R^*_{t^{-1}}\omega \\ &= \int_G (f\circ R_t)\omega = \int_G f(xt)\,d_lx \\ &= \int_G f(x)\,d_l(xt^{-1}) = \Delta(t)\int_G f(x)\,d_lx,\end{aligned}$$

and thus $\det\operatorname{Ad}(t) = \Delta(t)$. When $\det\operatorname{Ad}(t)$ is negative, every step of this computation is valid except for the first equality of the second line. Since $-R^*_{t^{-1}}\omega$ is positive, Proposition 8.19 requires a minus sign in its formula in order to apply to $\Phi = R_{t^{-1}}$. Thus $-\det\operatorname{Ad}(t) = \Delta(t)$. For all t, we therefore have $\Delta(t) = |\det\operatorname{Ad}(t)|$.

Corollary 8.30. The modular function Δ for G has the properties that

(a) $\Delta : G \to \mathbb{R}^+$ is a smooth homomorphism,
(b) $\Delta(t) = 1$ for t in any compact subgroup of G and in any semisimple analytic subgroup of G,
(c) $d_l(x^{-1})$ and $\Delta(x)\,d_l x$ are right Haar measures and are equal,
(d) $d_r(x^{-1})$ and $\Delta(x)^{-1}\,d_r x$ are left Haar measures and are equal,
(e) $d_r(t\,\cdot\,) = \Delta(t)\,d_r(\,\cdot\,)$ for any right Haar measure on G.

PROOF. Conclusion (a) is immediate from Proposition 8.27. The image under Δ of any compact subgroup of G is a compact subgroup of $\mathbb{R}^+$ and hence is $\{1\}$. This proves the first half of (b), and the second half follows from Lemma 4.28.

In (c) put $d\mu(x) = \Delta(x)\,d_l x$. This is a Borel measure since Δ is continuous (by (a)). Since Δ is a homomorphism, (8.26) gives

$$\begin{aligned}\int_G f(xt)\,d\mu(x) &= \int_G f(xt)\Delta(x)\,d_l x = \int_G f(x)\Delta(xt^{-1})\,d_l(xt^{-1})\\ &= \int_G f(x)\Delta(x)\Delta(t^{-1})\Delta(t)\,d_l x\\ &= \int_G f(x)\Delta(x)\,d_l x = \int_G f(x)\,d\mu(x).\end{aligned}$$

Hence $d\mu(x)$ is a right Haar measure. It is clear that $d_l(x^{-1})$ is a right Haar measure, and thus Theorem 8.23 for right Haar measures implies that $d_l(x^{-1}) = c\Delta(x)\,d_l x$ for some constant $c > 0$. Changing x to x^{-1} in this formula, we obtain

$$d_l x = c\Delta(x^{-1})\,d_l(x^{-1}) = c^2\Delta(x^{-1})\Delta(x)\,d_l x = c^2\,d_l x.$$

Hence $c = 1$, and (c) is proved.

For (d) and (e) there is no loss of generality in assuming that $d_r x = d_l(x^{-1}) = \Delta(x)\,d_l x$, in view of (c). Conclusion (d) is immediate from this identity if we replace x by x^{-1}. For (e) we have

$$\begin{aligned}\int_G f(x)\,d_r(tx) &= \int_G f(t^{-1}x)\,d_r x = \int_G f(t^{-1}x)\Delta(x)\,d_l x\\ &= \int_G f(x)\Delta(tx)\,d_l x\\ &= \Delta(t)\int_G f(x)\Delta(x)\,d_l x = \Delta(t)\int_G f(x)\,d_r x,\end{aligned}$$

and we conclude that $d_r(t\,\cdot\,) = \Delta(t)\,d_r(\,\cdot\,)$.

The Lie group G is said to be **unimodular** if every left Haar measure is a right Haar measure (and vice versa). In this case we can speak of **Haar measure** on G. In view of (8.26), G is unimodular if and only if $\Delta(t) = 1$ for all $t \in G$.

Corollary 8.31. The following kinds of Lie groups are always unimodular:

(a) abelian Lie groups,
(b) compact Lie groups,
(c) semisimple Lie groups,
(d) reductive Lie groups,
(e) nilpotent Lie groups.

PROOF. Conclusion (a) is trivial, and (b) and (c) follow from Corollary 8.30b. For (d) let (G, K, θ, B) be reductive. By Proposition 7.27, $G \cong {}^0G \times Z_{vec}$. A left Haar measure for G may be obtained as the product of the left Haar measures of the factors, and (a) shows that Z_{vec} is unimodular. Hence it is enough to consider ${}^0G$, which is reductive by Proposition 7.27c. The modular function for ${}^0G$ must be 1 on K by Corollary 8.30b, and K meets every component of ${}^0G$. Thus it is enough to prove that ${}^0G_0$ is unimodular. This group is generated by its center and its semisimple part. The center is compact by Proposition 7.27, and the modular function must be 1 there, by Corollary 8.30b. Again by Corollary 8.30b, the modular function must be 1 on the semisimple part. Then (d) follows.

For (e) we appeal to Proposition 8.27. It is enough to prove that $\det \operatorname{Ad}(x) = 1$ for all x in G. By Theorem 1.127 the exponential map carries the Lie algebra $\mathfrak{g}$ onto G. If $x = \exp X$, then $\det \operatorname{Ad}(x) = \det e^{\operatorname{ad} X} = e^{\operatorname{Tr} \operatorname{ad} X}$. Since $\mathfrak{g}$ is nilpotent, (1.31) shows that $\operatorname{ad} X$ is a nilpotent linear transformation. Therefore 0 is the only generalized eigenvalue of $\operatorname{ad} X$, and $\operatorname{Tr} \operatorname{ad} X = 0$. This proves (e).

3. Decompositions of Haar Measure

In this section we let G be a Lie group, and we let $d_l x$ and $d_r x$ be left and right Haar measures for it.

Theorem 8.32. Let G be a Lie group, and let S and T be closed subgroups such that $S \cap T$ is compact, multiplication $S \times T \to G$ is an open map, and the set of products ST exhausts G except possibly for a set of Haar measure 0. Let Δ_T and Δ_G denote the modular functions of T and G. Then the left Haar measures on G, S, and T can be normalized so that

$$\int_G f(x)\,d_l x = \int_{S\times T} f(st)\frac{\Delta_T(t)}{\Delta_G(t)}\,d_l s\,d_l t$$

for all Borel functions $f \geq 0$ on G.

PROOF. Let $\Omega \subseteq G$ be the set of products ST, and let $K = S \cap T$. The group $S \times T$ acts continuously on Ω by $(s,t)\omega = s\omega t^{-1}$, and the isotropy subgroup at 1 is $\operatorname{diag} K$. Thus the map $(s,t) \mapsto st^{-1}$ descends to a map $(S\times T)/\operatorname{diag} K \to \Omega$. This map is a homeomorphism since multiplication $S \times T \to G$ is an open map.

Hence any Borel measure on Ω can be reinterpreted as a Borel measure on $(S \times T)/\operatorname{diag} K$. We apply this observation to the restriction of a left Haar measure $d_l x$ for G from G to Ω, obtaining a Borel measure $d\mu$ on $(S \times T)/\operatorname{diag} K$. On Ω, we have

$$d_l(L_{s_0} R_{t_0^{-1}} x) = \Delta_G(t_0)\,d_l x$$

by (8.26), and the action unwinds to

$$d\mu(L_{(s_0,t_0)} x) = \Delta_G(t_0)\,d\mu(x) \tag{8.33}$$

on $(S \times T)/\operatorname{diag} K$. Define a measure $d\widetilde{\mu}(s,t)$ on $S \times T$ by

$$\int_{S\times T} f(s,t)\,d\widetilde{\mu}(s,t) = \int_{(S\times T)/\operatorname{diag} K} \Big[\int_K f(sk,tk)\,dk\Big]\,d\mu((s,t)K),$$

where dk is a Haar measure on K normalized to have total mass 1. From the formula (8.33) it follows that

$$d\widetilde{\mu}(s_0 s, t_0 t) = \Delta_G(t_0)\,d\widetilde{\mu}(s,t).$$

The same proof as for Theorem 8.23 shows that any two Borel measures on $S \times T$ with this property are proportional, and $\Delta_G(t)\,d_l s\,d_l t$ is such a measure. Therefore

$$d\widetilde{\mu}(s,t) = \Delta_G(t)\,d_l s\,d_l t$$

for a suitable normalization of $d_l s\,d_l t$.

The resulting formula is

$$\int_\Omega f(x)\,d_l x = \int_{S\times T} f(st^{-1})\Delta_G(t)\,d_l s\,d_l t$$

for all Borel functions $f \geq 0$ on Ω. On the right side the change of variables $t \mapsto t^{-1}$ makes the right side become

$$\int_{S\times T} f(st)\Delta_G(t)^{-1}\, d_l s\ \Delta_T(t)\, d_l t,$$

according to Corollary 8.30c, and we can replace Ω by G on the left side since the complement of Ω in G has measure 0. This completes the proof.

If H is a closed subgroup of G, then we can ask whether G/H has a nonzero G invariant Borel measure. Theorem 8.36 below will give a necessary and sufficient condition for this existence, but we need some preparation. Fix a left Haar measure $d_l h$ for H. If f is in $C_{\text{com}}(G)$, define

(8.34a) $$f^{\#}(g) = \int_H f(gh)\, d_l h.$$

This function is invariant under right translation by H, and we can define

(8.34b) $$f^{\#\#}(gH) = f^{\#}(g).$$

The function $f^{\#\#}$ has compact support on G/H.

Lemma 8.35. The map $f \mapsto f^{\#\#}$ carries $C_{\text{com}}(G)$ onto $C_{\text{com}}(G/H)$, and a nonnegative member of $C_{\text{com}}(G/H)$ has a nonnegative preimage in $C_{\text{com}}(G)$.

PROOF. Let $\pi : G \to G/H$ be the quotient map. Let $F \in C_{\text{com}}(G/H)$ be given, and let K be a compact set in G/H with $F = 0$ off K. We first produce a compact set $\widetilde{K}$ in G with $\pi(\widetilde{K}) = K$. For each coset in K, select an inverse image x and let N_x be a compact neighborhood of x in G. Since π is open, π of the interior of N_x is open. These open sets cover K, and a finite number of them suffices. Then we can take $\widetilde{K}$ to be the intersection of $\pi^{-1}(K)$ with the union of the finitely many N_x's.

Next let K_H be a compact neighborhood of 1 in H. By (8.24) the left Haar measure on H is positive on K_H. Let $\widetilde{K}'$ be the compact set $\widetilde{K}' = \widetilde{K}K_H$, so that $\pi(\widetilde{K}') = \pi(\widetilde{K}) = K$. Choose $f_1 \in C_{\text{com}}(G)$ with $f_1 \geq 0$ everywhere and with $f_1 = 1$ on $\widetilde{K}'$. If g is in $\widetilde{K}'$, then $\int_H f_1(gh)\, d_l h$ is $\geq$ the H measure of K_H, and hence $f_1^{\#\#}$ is > 0 on K. Define

$$f(g) = \begin{cases} f_1(g)\,\dfrac{F(\pi(g))}{f_1^{\#\#}(\pi(g))} & \text{if } \pi(g) \in K \\ 0 & \text{otherwise.} \end{cases}$$

Then $f^{\#\#}$ is F on K and is 0 off K, so that $f^{\#\#} = F$ everywhere.

Certainly f has compact support. To see that f is continuous, it suffices to check that the two formulas for $f(g)$ fit together continuously at points g of $\pi^{-1}(K)$. It is enough to check points where $f(g) \neq 0$. Say $g_n \to g$. We must have $F(\pi(g)) \neq 0$. Since F is continuous, $F(\pi(g_n)) \neq 0$ eventually. Thus for all n sufficiently large, $f(g_n)$ is given by the first of the two formulas. Thus f is continuous.

Theorem 8.36. Let G be a Lie group, let H be a closed subgroup, and let Δ_G and Δ_H be the respective modular functions. Then a necessary and sufficient condition for G/H to have a nonzero G invariant Borel measure is that the restriction to H of Δ_G equal Δ_H. In this case such a measure $d\mu(gH)$ is unique up to a scalar, and it can be normalized so that

$$\int_G f(g)\, d_l g = \int_{G/H} \Big[\int_H f(gh)\, d_l h \Big]\, d\mu(gH) \tag{8.37}$$

for all $f \in C_{\text{com}}(G)$.

PROOF. Let $d\mu(gH)$ be such a measure. In the notation of (8.34), we can define a measure $d\widetilde{\mu}(g)$ on G by

$$\int_G f(g)\, d\widetilde{\mu}(g) = \int_{G/H} f^{\#\#}(gH)\, d\mu(gH).$$

Since $f \mapsto f^{\#\#}$ commutes with left translation by G, $d\widetilde{\mu}$ is a left Haar measure on G. By Theorem 8.23, $d\widetilde{\mu}$ is unique up to a scalar; hence $d\mu(gH)$ is unique up to a scalar.

Under the assumption that G/H has a nonzero invariant Borel measure, we have just seen in essence that we can normalize the measure so that (8.37) holds. If we replace f in (8.37) by $f(\,\cdot\, h_0)$, then the left side is multiplied by $\Delta_G(h_0)$, and the right side is multiplied by $\Delta_H(h_0)$. Hence $\Delta_G|_H = \Delta_H$ is necessary for existence.

Let us prove that this condition is sufficient for existence. Given h in $C_{\text{com}}(G/H)$, we can choose f in $C_{\text{com}}(G)$ by Lemma 8.35 so that $f^{\#\#} = h$. Then we define $L(h) = \int_G f(g)\, d_l g$. If L is well defined, then it is linear, Lemma 8.35 shows that it is positive, and L certainly is the same on a function as on its G translates. Therefore L defines a G invariant Borel measure $d\mu(gH)$ on G/H such that (8.37) holds.

Thus all we need to do is see that L is well defined if $\Delta_G|_H = \Delta_H$. We are thus to prove that if $f \in C_{\text{com}}(G)$ has $f^{\#} = 0$, then $\int_G f(g)\, d_l g = 0$.

Let ψ be in $C_{\text{com}}(G)$. Then we have

$$\begin{aligned}
0 &= \int_G \psi(g)\, f^{\#}(g)\, d_l g \\
&= \int_G \Big[\int_H \psi(g)\, f(gh)\, d_l h \Big]\, d_l g \\
&= \int_H \Big[\int_G \psi(g)\, f(gh)\, d_l g \Big]\, d_l h \\
&= \int_H \Big[\int_G \psi(gh^{-1})\, f(g)\, d_l g \Big]\, \Delta_G(h)\, d_l h && \text{by (8.26)} \\
&= \int_G f(g) \Big[\int_H \psi(gh^{-1}) \Delta_G(h)\, d_l h \Big]\, d_l g \\
&= \int_G f(g) \Big[\int_H \psi(gh) \Delta_G(h)^{-1} \Delta_H(h)\, d_l h \Big]\, d_l g && \text{by Corollary 8.30c} \\
&= \int_G f(g) \psi^{\#}(g)\, d_l g && \text{since } \Delta_G|_H = \Delta_H.
\end{aligned}$$

By Lemma 8.35 we can choose $\psi \in C_{\text{com}}(G)$ so that $\psi^{\#\#} = 1$ on the projection to G/H of the support of f. Then the right side is $\int_G f(g)\, d_l g$, and the conclusion is that this is 0. Thus L is well defined, and existence is proved.

4. Application to Reductive Lie Groups

Let (G, K, θ, B) be a reductive Lie group. We shall use the notation of Chapter VII, but we drop the subscripts 0 from real Lie algebras since we shall have relatively few occurrences of their complexifications. Thus, for example, the Cartan decomposition of the Lie algebra of G will be written $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$.

In this section we use Theorem 8.32 and Proposition 8.27 to give decompositions of Haar measures that mirror group decompositions in Chapter VII. The group G itself is unimodular by Corollary 8.31d, and we write dx for a two-sided Haar measure. We shall be interested in parabolic subgroups MAN, and we need to compute the corresponding modular function that is given by Proposition 8.27 as

$$\Delta_{MAN}(man) = |\det \mathrm{Ad}_{\mathfrak{m}+\mathfrak{a}+\mathfrak{n}}(man)|.$$

For the element m, $|\det \mathrm{Ad}_{\mathfrak{m}+\mathfrak{a}+\mathfrak{n}}(m)| = 1$ by Corollary 8.30b. The element a acts as 1 on $\mathfrak{m}$ and $\mathfrak{a}$, and hence $\det \mathrm{Ad}_{\mathfrak{m}+\mathfrak{a}+\mathfrak{n}}(a) = \det \mathrm{Ad}_{\mathfrak{n}}(a)$. On an $\mathfrak{a}$ root space $\mathfrak{g}_\lambda$, a acts by $e^{\lambda \log a}$, and thus $\det \mathrm{Ad}_{\mathfrak{n}}(a) = e^{2\rho_A \log a}$, where $2\rho_A$ is the sum of all the positive $\mathfrak{a}$ roots with multiplicities counted. Finally $\det \mathrm{Ad}_{\mathfrak{m}+\mathfrak{a}+\mathfrak{n}}(n) = 1$ for the same reasons as in the proof of Corollary 8.31e. Therefore

$$\Delta_{MAN}(man) = |\det \mathrm{Ad}_{\mathfrak{m}+\mathfrak{a}+\mathfrak{n}}(man)| = e^{2\rho_A \log a}. \tag{8.38}$$

We can then apply Theorem 8.32 and Corollary 8.31 to obtain

$$d_l(man) = \frac{\Delta_N(n)}{\Delta_{MAN}(n)}\, d_l(ma)\, d_l n = dm\, da\, dn. \tag{8.39a}$$

By (8.38) and Corollary 8.30c,

$$d_r(man) = e^{2\rho_A \log a}\, dm\, da\, dn. \tag{8.39b}$$

Similarly for the subgroup AN of MAN, we have

$$\Delta_{AN}(an) = e^{2\rho_A \log a} \tag{8.40}$$

and

$$\begin{aligned} d_l(an) &= da\, dn \\ d_r(an) &= e^{2\rho_A \log a}\, da\, dn. \end{aligned} \tag{8.41}$$

Now we shall apply Theorem 8.32 to G itself. Combining Corollary 8.30c with the fact that G is unimodular, we can write

$$dx = d_l s\, d_r t \tag{8.42}$$

whenever the hypotheses in the theorem for S and T are satisfied.

Proposition 8.43. If $G = KA_\mathfrak{p}N_\mathfrak{p}$ is an Iwasawa decomposition of the reductive Lie group G, then the Haar measures of G, $A_\mathfrak{p}N_\mathfrak{p}$, $A_\mathfrak{p}$, and $N_\mathfrak{p}$ can be normalized so that

$$dx = dk\, d_r(an) = e^{2\rho_{A_\mathfrak{p}} \log a}\, dk\, da\, dn.$$

If the Iwasawa decomposition is written instead as $G = A_\mathfrak{p}N_\mathfrak{p}K$, then the decomposition of measures is

$$dx = d_l(an)\, dk = da\, dn\, dk.$$

PROOF. If G is written as $G = KA_\mathfrak{p}N_\mathfrak{p}$, then we use $S = K$ and $T = A_\mathfrak{p}N_\mathfrak{p}$ in Theorem 8.32. The hypotheses are satisfied since Proposition 7.31 shows that $S \times T \to G$ is a diffeomorphism. The second equality follows from (8.41). The argument when $G = A_\mathfrak{p}N_\mathfrak{p}K$ is similar.

Proposition 8.44. If G is a reductive Lie group and MAN is a parabolic subgroup, so that $G = KMAN$, then the Haar measures of G, MAN, M, A, and N can be normalized so that

$$dx = dk\, d_r(man) = e^{2\rho_A \log a} dk\, dm\, da\, dn.$$

PROOF. We use $S = K$ and $T = MAN$ in Theorem 8.32. Here $S \cap T = K \cap M$ is compact, and we know that $G = KMAN$. Since $A_\mathfrak{p} N_\mathfrak{p} \subseteq MAN$ and $K \times A_\mathfrak{p} N_\mathfrak{p} \to G$ is open, $K \times MAN \to G$ is open. Then Theorem 8.32 gives the first equality, and the second equality follows from (8.39b).

Proposition 8.45. If MAN is a parabolic subgroup of the reductive Lie group G, then $N^- MAN$ is open in G and its complement is a lower-dimensional set, hence a set of measure 0. The Haar measures of G, MAN, N^-, M, A, and N can be normalized so that

$$dx = d\bar{n}\, d_r(man) = e^{2\rho_A \log a} d\bar{n}\, dm\, da\, dn \qquad (\bar{n} \in N^-).$$

PROOF. We use $S = N^-$ and $T = MAN$ in Theorem 8.32. Here $S \cap T = \{1\}$ by Lemma 7.64, and $S \times T \to G$ is everywhere regular (hence open) by Lemma 6.44. We need to see that the complement of $N^- MAN$ is lower dimensional and has measure 0. Let $M_\mathfrak{p} A_\mathfrak{p} N_\mathfrak{p} \subseteq MAN$ be a minimal parabolic subgroup. In the Bruhat decomposition of G as in Theorem 7.40, a double coset of $M_\mathfrak{p} A_\mathfrak{p} N_\mathfrak{p}$ is of the form

$$M_\mathfrak{p} A_\mathfrak{p} N_\mathfrak{p} w M_\mathfrak{p} A_\mathfrak{p} N_\mathfrak{p} = N_\mathfrak{p} w M_\mathfrak{p} A_\mathfrak{p} N_\mathfrak{p} = w(w^{-1} N_\mathfrak{p} w) M_\mathfrak{p} A_\mathfrak{p} N_\mathfrak{p},$$

where w is a representative in $N_K(\mathfrak{a}_\mathfrak{p})$ of a member of $N_K(\mathfrak{a}_\mathfrak{p})/M_\mathfrak{p}$. The double coset is thus a translate of $(w^{-1} N_\mathfrak{p} w) M_\mathfrak{p} A_\mathfrak{p} N_\mathfrak{p}$. To compute the dimension of this set, we observe that

$$\dim \operatorname{Ad}(w)^{-1} \mathfrak{n}_\mathfrak{p} + \dim(\mathfrak{m}_\mathfrak{p} \oplus \mathfrak{a}_\mathfrak{p} \oplus \mathfrak{n}_\mathfrak{p}) = \dim \mathfrak{g}.$$

Now $\operatorname{Ad}(w)^{-1}\mathfrak{n}_\mathfrak{p}$ has 0 intersection with $\mathfrak{m}_\mathfrak{p} \oplus \mathfrak{a}_\mathfrak{p} \oplus \mathfrak{n}_\mathfrak{p}$ if and only if $\operatorname{Ad}(w)^{-1}\mathfrak{n}_\mathfrak{p} = \theta \mathfrak{n}_\mathfrak{p}$, which happens for exactly one coset $wM_\mathfrak{p}$ by Proposition 7.32 and Theorem 2.63. This case corresponds to the open set $N_\mathfrak{p}^- M_\mathfrak{p} A_\mathfrak{p} N_\mathfrak{p}$. In the other cases, there is a closed positive-dimensional subgroup R_w of $w^{-1} N_\mathfrak{p} w$ such that the smooth map

$$w^{-1} N_\mathfrak{p} w \times M_\mathfrak{p} A_\mathfrak{p} N_\mathfrak{p} \to (w^{-1} N_\mathfrak{p} w) M_\mathfrak{p} A_\mathfrak{p} N_\mathfrak{p}$$

given by $(x, y) \mapsto xy^{-1}$ factors to a smooth map

$$(w^{-1}N_{\mathfrak{p}}w \times M_{\mathfrak{p}}A_{\mathfrak{p}}N_{\mathfrak{p}})/\operatorname{diag} R_w \to (w^{-1}N_{\mathfrak{p}}w)M_{\mathfrak{p}}A_{\mathfrak{p}}N_{\mathfrak{p}}.$$

Hence in these cases $(w^{-1}N_{\mathfrak{p}}w)M_{\mathfrak{p}}A_{\mathfrak{p}}N_{\mathfrak{p}}$ is the smooth image of a manifold of dimension $< \dim G$ and is lower dimensional in G.

This proves for $M_{\mathfrak{p}}A_{\mathfrak{p}}N_{\mathfrak{p}}$ that $N_{\mathfrak{p}}^-M_{\mathfrak{p}}A_{\mathfrak{p}}N_{\mathfrak{p}}$ is open with complement of lower dimension. By (8.25) the complement is of Haar measure 0. Now let us consider N^-MAN. Since $M_{\mathfrak{p}}A_{\mathfrak{p}}N_{\mathfrak{p}} \subseteq MAN$, we have

$$\begin{aligned} N_{\mathfrak{p}}^-M_{\mathfrak{p}}A_{\mathfrak{p}}N_{\mathfrak{p}} &= (M_{\mathfrak{p}}A_{\mathfrak{p}}N_{\mathfrak{p}}^-)M_{\mathfrak{p}}A_{\mathfrak{p}}N_{\mathfrak{p}} \\ &\subseteq (MAN^-)MAN = N^-MAN. \end{aligned}$$

Thus the open set N^-MAN has complement of lower dimension and hence of Haar measure 0.

Theorem 8.32 is therefore applicable, and we obtain $dx = d\bar{n}\, d_r(man)$. The equality $d\bar{n}\, d_r(man) = e^{2\rho_A \log a}\, d\bar{n}\, dm\, da\, dn$ follows from (8.39b).

Proposition 8.46. Let MAN be a parabolic subgroup of the reductive Lie group G, and let ρ_A be as in (8.38). For g in G, decompose g according to $G = KMAN$ as

$$g = \kappa(g)\mu(g) \exp H(g)\, n.$$

Then Haar measures, when suitably normalized, satisfy

$$\int_K f(k)\, dk = \int_{N^-} f(\kappa(\bar{n}))e^{-2\rho_A H(\bar{n})}\, d\bar{n}$$

for all continuous functions on K that are right invariant under $K \cap M$.

REMARK. The expressions $\kappa(g)$ and $\mu(g)$ are not uniquely defined, but $H(g)$ is uniquely defined, as a consequence of the Iwasawa decomposition, and $f(\kappa(\bar{n}))$ will be seen to be well defined because of the assumed right invariance under $K \cap M$.

PROOF. Given f continuous on K and right invariant under $K \cap M$, extend f to a function F on G by

$$F(kman) = e^{-2\rho_A \log a} f(k). \tag{8.47}$$

The right invariance of f under $K \cap M$ makes F well defined since $K \cap MAN = K \cap M$. Fix $\varphi \geq 0$ in $C_{\text{com}}(MAN)$ with

$$\int_{MAN} \varphi(man)\, d_l(man) = 1;$$

by averaging over $K \cap M$, we may assume that φ is left invariant under $K \cap M$. Extend φ to G by the definition $\varphi(kman) = \varphi(man)$; the left invariance of φ under $K \cap M$ makes φ well defined. Then

$$\int_{MAN} \varphi(xman)\, d_l(man) = 1 \qquad \text{for all } x \in G.$$

The left side of the formula in the conclusion is

$$\begin{aligned}
&\int_K f(k)\, dk \\
&\quad = \int_K f(k) \Big[\int_{MAN} \varphi(kman)\, d_l(man) \Big] dk \\
&\quad = \int_{K \times MAN} f(k)\varphi(kman) e^{-2\rho_A \log a}\, dk\, d_r(man) && \text{by (8.39)} \\
&\quad = \int_{K \times MAN} F(kman)\varphi(kman)\, dk\, d_r(man) && \text{by (8.47)} \\
&\quad = \int_G F(x)\varphi(x)\, dx && \text{by Proposition 8.44,}
\end{aligned}$$

while the right side of the formula is

$$\begin{aligned}
&\int_{N^-} f(\kappa(\bar n)) e^{-2\rho_A H(\bar n)}\, d\bar n \\
&\quad = \int_{N^-} F(\bar n) \Big[\int_{MAN} \varphi(\bar n man)\, d_l(man) \Big] d\bar n && \text{by (8.47)} \\
&\quad = \int_{N^- \times MAN} F(\bar n) e^{-2\rho_A \log a} \varphi(\bar n man)\, d\bar n\, d_r(man) && \text{by (8.39)} \\
&\quad = \int_{N^- \times MAN} F(\bar n man)\varphi(\bar n man)\, d\bar n\, d_r(man) && \text{by (8.47)} \\
&\quad = \int_G F(x)\varphi(x)\, dx && \text{by Proposition 8.45.}
\end{aligned}$$

The proposition follows.

For an illustration of the use of Proposition 8.46, we shall prove a theorem of Helgason that has important applications in the harmonic analysis of G/K. We suppose that the reductive group G is semisimple and has a complexification $G^{\mathbb{C}}$. We fix an Iwasawa decomposition $G = KA_{\mathfrak{p}}N_{\mathfrak{p}}$.

Let $\mathfrak{t}_\mathfrak{p}$ be a maximal abelian subspace of $\mathfrak{m}_\mathfrak{p}$, so that $\mathfrak{t}_\mathfrak{p} \oplus \mathfrak{a}_\mathfrak{p}$ is a maximally noncompact θ stable Cartan subalgebra of $\mathfrak{g}$. Representations of G yield representations of $\mathfrak{g}$, hence complex-linear representations of $\mathfrak{g}^\mathbb{C}$. Then the theory of Chapter V is applicable, and we use the complexification of $\mathfrak{t}_\mathfrak{p} \oplus \mathfrak{a}_\mathfrak{p}$ as Cartan subalgebra for that purpose. Let Δ and Σ be the sets of roots and restricted roots, respectively, and let Σ^+ be the set of positive restricted roots relative to $\mathfrak{n}_\mathfrak{p}$.

Roots and weights are real on $i\mathfrak{t}_\mathfrak{p} \oplus \mathfrak{a}_\mathfrak{p}$, and we introduce an ordering such that the nonzero restriction to $\mathfrak{a}_\mathfrak{p}$ of a member of Δ^+ is a member of Σ^+. By a **restricted weight** of a finite-dimensional representation, we mean the restriction to $\mathfrak{a}_\mathfrak{p}$ of a weight. We introduce in an obvious fashion the notions of **restricted-weight spaces** and **restricted-weight vectors**. Because of our choice of ordering, the restriction to $\mathfrak{a}_\mathfrak{p}$ of the highest weight of a finite-dimensional representation is the highest restricted weight.

Lemma 8.48. Let the reductive Lie group G be semisimple. If π is an irreducible complex-linear representation of $\mathfrak{g}^\mathbb{C}$, then $\mathfrak{m}_\mathfrak{p}$ acts in each restricted weight space of π, and the action by $\mathfrak{m}_\mathfrak{p}$ is irreducible in the highest restricted-weight space.

PROOF. The first conclusion follows at once since $\mathfrak{m}_\mathfrak{p}$ commutes with $\mathfrak{a}_\mathfrak{p}$. Let $v \neq 0$ be a highest restricted-weight vector, say with weight ν. Let V be the space for π, and let V_ν be the restricted-weight space corresponding to ν. We write $\mathfrak{g} = \theta\mathfrak{n}_\mathfrak{p} \oplus \mathfrak{m}_\mathfrak{p} \oplus \mathfrak{a}_\mathfrak{p} \oplus \mathfrak{n}_\mathfrak{p}$, express members of $U(\mathfrak{g}^\mathbb{C})$ in the corresponding basis given by the Poincaré–Birkhoff–Witt Theorem, and apply an element to v. Since $\mathfrak{n}_\mathfrak{p}$ pushes restricted weights up and $\mathfrak{a}_\mathfrak{p}$ acts by scalars in V_ν and $\theta\mathfrak{n}_\mathfrak{p}$ pushes weights down, we see from the irreducibility of π on V that $U(\mathfrak{m}_\mathfrak{p}^\mathbb{C})v = V_\nu$. Since v is an arbitrary nonzero member of V_ν, $\mathfrak{m}_\mathfrak{p}$ acts irreducibly on V_ν.

Theorem 8.49 (Helgason). Let the reductive Lie group G be semisimple and have a complexification $G^\mathbb{C}$. For an irreducible finite-dimensional representation π of G, the following statements are equivalent:

(a) π has a nonzero K fixed vector,
(b) $M_\mathfrak{p}$ acts by the 1-dimensional trivial representation in the highest restricted-weight space of π,
(c) the highest weight $\widetilde{\nu}$ of π vanishes on $\mathfrak{t}_\mathfrak{p}$, and the restriction ν of $\widetilde{\nu}$ to $\mathfrak{a}_\mathfrak{p}$ is such that $\langle \nu, \beta \rangle / |\beta|^2$ is an integer for every restricted root β.

Conversely any dominant $\nu \in \mathfrak{a}_\mathfrak{p}^*$ such that $\langle \nu, \beta \rangle / |\beta|^2$ is an integer for every restricted root β is the highest restricted weight of some irreducible finite-dimensional π with a nonzero K fixed vector.

PROOF. For the proofs that (a) through (c) are equivalent, there is no loss in generality in assuming that $G^{\mathbb{C}}$ is simply connected, as we may otherwise take a simply connected cover of $G^{\mathbb{C}}$ and replace G by the analytic subgroup of this cover with Lie algebra $\mathfrak{g}$. With $G^{\mathbb{C}}$ simply connected, the representation π of G yields a representation of $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, then of $\mathfrak{g}^{\mathbb{C}}$, and then of the compact form $\mathfrak{u} = \mathfrak{k} \oplus i\mathfrak{p}$. Since $G^{\mathbb{C}}$ is simply connected, so is the analytic subgroup U with Lie algebra $\mathfrak{u}$ (Theorem 6.31). The representation π therefore lifts from $\mathfrak{u}$ to U. By Proposition 4.6 we can introduce a Hermitian inner product on the representation space so that U acts by unitary operators. Then it follows that K acts by unitary operators and $i\mathfrak{t}_{\mathfrak{p}} \oplus \mathfrak{a}_{\mathfrak{p}}$ acts by Hermitian operators. In particular, distinct weight spaces are orthogonal, and so are distinct restricted-weight spaces.

(a) $\Rightarrow$ (b). Let ϕ_ν be a nonzero highest restricted-weight vector, and let ϕ_K be a nonzero K fixed vector. Since $\mathfrak{n}_{\mathfrak{p}}$ pushes restricted weights up and since the exponential map carries $\mathfrak{n}_{\mathfrak{p}}$ onto $N_{\mathfrak{p}}$ (Theorem 1.127), $\pi(n)\phi_\nu = \phi_\nu$ for $n \in N_{\mathfrak{p}}$. Therefore

$$(\pi(kan)\phi_\nu, \phi_K) = (\pi(a)\phi_\nu, \pi(k)^{-1}\phi_K) = e^{\nu \log a}(\phi_\nu, \phi_K).$$

By the irreducibility of π and the fact that $G = KA_{\mathfrak{p}}N_{\mathfrak{p}}$, the left side cannot be identically 0, and hence (ϕ_ν, ϕ_K) on the right side is nonzero. The inner product with ϕ_K is then an everywhere-nonzero linear functional on the highest restricted-weight space, and the highest restricted-weight space must be 1-dimensional. If ϕ_ν is a nonzero vector of norm 1 in this space, then $(\phi_K, \phi_\nu)\phi_\nu$ is the orthogonal projection of ϕ_K into this space. Since $M_{\mathfrak{p}}$ commutes with $\mathfrak{a}_{\mathfrak{p}}$, the action by $M_{\mathfrak{p}}$ commutes with this projection. But $M_{\mathfrak{p}}$ acts trivially on ϕ_K since $M_{\mathfrak{p}} \subseteq K$, and therefore $M_{\mathfrak{p}}$ acts trivially on ϕ_ν.

(b) $\Rightarrow$ (a). Let $v \neq 0$ be in the highest restricted-weight space, with restricted weight ν. Then $\int_K \pi(k)v\,dk$ is obviously fixed by K, and the problem is to see that it is not 0. Since v is assumed to be fixed by $M_{\mathfrak{p}}$, $k \mapsto \pi(k)v$ is a function on K right invariant under $M_{\mathfrak{p}}$. By Proposition 8.46,

$$\int_K \pi(k)v\,dk = \int_{N_{\mathfrak{p}}^-} \pi(\kappa(\bar{n}))ve^{-2\rho_{A_{\mathfrak{p}}}H(\bar{n})}\,d\bar{n} = \int_{N_{\mathfrak{p}}^-} \pi(\bar{n})ve^{(-\nu-2\rho_{A_{\mathfrak{p}}})H(\bar{n})}\,d\bar{n}.$$

Here $e^{(-\nu-2\rho_{A_{\mathfrak{p}}})H(\bar{n})}$ is everywhere positive since ν is real, and $(\pi(\bar{n})v, v) = |v|^2$ since the exponential map carries $\theta\mathfrak{n}_{\mathfrak{p}}$ onto $N_{\mathfrak{p}}^-$, $\theta\mathfrak{n}_{\mathfrak{p}}$ lowers restricted weights, and the different restricted-weight spaces are orthogonal. Therefore $\left(\int_K \pi(k)v\,dk, v\right)$ is positive, and $\int_K \pi(k)v\,dk$ is not 0.

(b) $\Rightarrow$ (c). Since $(M_{\mathfrak{p}})_0$ acts trivially, it follows immediately that $\widetilde{\nu}$ vanishes on $\mathfrak{t}_{\mathfrak{p}}$. For each restricted root β, define $\gamma_\beta = \exp 2\pi i|\beta|^{-2}H_\beta$ as in (7.57). This element is in $M_{\mathfrak{p}}$ by (7.58). Since $G^{\mathbb{C}}$ is simply connected, π extends to a holomorphic representation of $G^{\mathbb{C}}$. Then we can compute $\pi(\gamma_\beta)$ on a vector v of restricted weight ν as

$$\pi(\gamma_\beta)v = \pi(\exp(2\pi i|\beta|^{-2}H_\beta))v = e^{2\pi i\langle\nu,\beta\rangle/|\beta|^2}v. \tag{8.50}$$

Since the left side equals v by (b), $\langle\nu,\beta\rangle/|\beta|^2$ must be an integer.

(c) $\Rightarrow$ (b). The action of $(M_{\mathfrak{p}})_0$ on the highest restricted-weight space is irreducible by Lemma 8.48. Since $\widetilde{\nu}$ vanishes on $\mathfrak{t}_{\mathfrak{p}}$, the highest weight of this representation of $(M_{\mathfrak{p}})_0$ is 0. Thus $(M_{\mathfrak{p}})_0$ acts trivially, and the space is 1-dimensional. The calculation (8.50), in the presence of (c), shows that each γ_β acts trivially. Since the γ_β that come from real roots generate F (by Theorem 7.55) and since $M_{\mathfrak{p}} = (F)(M_{\mathfrak{p}})_0$ (by Corollary 7.52), $M_{\mathfrak{p}}$ acts trivially.

We are left with the converse statement. Suppose $\nu \in \mathfrak{a}_{\mathfrak{p}}^*$ is such that $\langle\nu,\beta\rangle/|\beta|^2$ is an integer ≥ 0 for all $\beta \in \Sigma^+$. Define $\widetilde{\nu}$ to be ν on $\mathfrak{a}_{\mathfrak{p}}$ and 0 on $\mathfrak{t}_{\mathfrak{p}}$. We are to prove that $\widetilde{\nu}$ is the highest weight of an irreducible finite-dimensional representation of G with a K fixed vector. The form $\widetilde{\nu}$ is dominant. If it is algebraically integral, then Theorem 5.5 gives us a complex-linear representation π of $\mathfrak{g}^{\mathbb{C}}$ with highest weight $\widetilde{\nu}$. Some finite covering group $\widetilde{G}$ of G will have a simply connected complexification, and then π lifts to $\widetilde{G}$. By the implication (c) $\Rightarrow$ (a), π has a nonzero $\widetilde{K}$ fixed vector. Since the kernel of $\widetilde{G} \to G$ is in $\widetilde{K}$ and since such elements must then act trivially, π descends to a representation of G with a nonzero K fixed vector. In other words, it is enough to prove that $\widetilde{\nu}$ is algebraically integral.

Let α be a root, and let β be its restriction to $\mathfrak{a}_{\mathfrak{p}}$. Since $\langle\widetilde{\nu},\alpha\rangle = \langle\nu,\beta\rangle$, we may assume that $\beta \neq 0$. Let $|\alpha|^2 = C|\beta|^2$. Then

$$\frac{2\langle\widetilde{\nu},\alpha\rangle}{|\alpha|^2} = \frac{2\langle\nu,\beta\rangle}{C|\beta|^2},$$

and it is enough to show that either

$$2/C \text{ is an integer} \tag{8.51a}$$

or

$$|2/C| = \tfrac{1}{2} \text{ and } \langle\nu,\beta\rangle/|\beta|^2 \text{ is even.} \tag{8.51b}$$

Write $\alpha = \beta + \varepsilon$ with $\varepsilon \in i\mathfrak{t}_{\mathfrak{p}}^*$. Then $\theta\alpha$ is the root $\theta\alpha = -\beta + \varepsilon$. Thus $-\theta\alpha = \beta - \varepsilon$ is a root with the same length as α.

If α and $-\theta\alpha$ are multiples of one another, then $\varepsilon = 0$ and $C = 1$, so that $2/C$ is an integer. If α and $-\theta\alpha$ are not multiples of one another, then the Schwarz inequality gives

$$
\begin{aligned}
(-1 \text{ or } 0 \text{ or } +1) &= \frac{2\langle \alpha, -\theta\alpha\rangle}{|\alpha|^2} = \frac{2\langle \beta+\varepsilon, \beta-\varepsilon\rangle}{|\alpha|^2} \\
&= \frac{2(|\beta|^2 - |\varepsilon|^2)}{|\alpha|^2} = \frac{2(2|\beta|^2 - |\alpha|^2)}{|\alpha|^2} = \frac{4}{C} - 2.
\end{aligned}
\tag{8.52}
$$

If the left side of (8.52) is -1, then $2/C = \frac{1}{2}$. Since the left side of (8.52) is -1, $\alpha - \theta\alpha = 2\beta$ is a root, hence also a restricted root. By assumption, $\langle \nu, 2\beta\rangle/|2\beta|^2$ is an integer; hence $\langle \nu, \beta\rangle/|\beta|^2$ is even. Thus (8.51b) holds. If the left side of (8.52) is 0, then $2/C = 1$ and (8.51a) holds.

To complete the proof, we show that the left side of (8.52) cannot be $+1$. If it is $+1$, then $\alpha - (-\theta\alpha) = 2\varepsilon$ is a root vanishing on $\mathfrak{a}_{\mathfrak{p}}$, and hence any root vector for it is in $\mathfrak{m}_{\mathfrak{p}}^{\mathbb{C}} \subseteq \mathfrak{k}^{\mathbb{C}}$. However this root is also equal to $\alpha + \theta\alpha$, and $[X_\alpha, \theta X_\alpha]$ must be a root vector. Since $\theta[X_\alpha, \theta X_\alpha] = -[X_\alpha, \theta X_\alpha]$, $[X_\alpha, \theta X_\alpha]$ is in $\mathfrak{p}^{\mathbb{C}}$. Thus the root vector is in $\mathfrak{k}^{\mathbb{C}} \cap \mathfrak{p}^{\mathbb{C}} = 0$, and we have a contradiction.

5. Weyl Integration Formula

The original Weyl Integration Formula tells how to integrate over a compact connected Lie group by first integrating over each conjugacy class and then integrating over the set of conjugacy classes. Let G be a compact connected Lie group, let T be a maximal torus, and let $\mathfrak{g}_0$ and $\mathfrak{t}_0$ be the respective Lie algebras. Let $m = \dim G$ and $l = \dim T$. As in §VII.8, an element g of G is **regular** if the eigenspace of $\operatorname{Ad}(g)$ for eigenvalue 1 has dimension l. Let G' and T' be the sets of regular elements in G and T; these are open subsets of G and T, respectively.

Theorem 4.36 implies that the smooth map $G \times T \to G$ given by $\psi(g, t) = gtg^{-1}$ is onto G. Fix $g \in G$ and $t \in T$. If we identify tangent spaces at g, t, and gtg^{-1} with $\mathfrak{g}_0$, $\mathfrak{t}_0$, and $\mathfrak{g}_0$ by left translation, then (4.45) computes the differential of ψ at (g, t) as

$$d\psi(X, H) = \operatorname{Ad}(g)((\operatorname{Ad}(t^{-1}) - 1)X + H) \qquad \text{for } X \in \mathfrak{g}_0,\ H \in \mathfrak{t}_0.$$

The map ψ descends to $G/T \times T \to G$, and we call the descended map ψ also. We may identify the tangent space of G/T with an orthogonal complement $\mathfrak{t}_0^\perp$ to $\mathfrak{t}_0$ in $\mathfrak{g}_0$ (relative to an invariant inner product). The space $\mathfrak{t}_0^\perp$ is invariant under $\mathrm{Ad}(t^{-1}) - 1$, and we can write

$$d\psi(X, H) = \mathrm{Ad}(g)((\mathrm{Ad}(t^{-1}) - 1)X + H) \qquad \text{for } X \in \mathfrak{t}_0^\perp,\ H \in \mathfrak{t}_0.$$

Now $d\psi$ at (g, t) is essentially a map of $\mathfrak{g}_0$ to itself, with matrix

$$(d\psi)_{(g,t)} = \mathrm{Ad}(g) \begin{array}{c} \begin{matrix} \mathfrak{t}_0 & \qquad \mathfrak{t}_0^\perp \end{matrix} \\ \begin{pmatrix} 1 & 0 \\ 0 & \mathrm{Ad}(t^{-1}) - 1 \end{pmatrix} \end{array}.$$

Since $\det \mathrm{Ad}(g) = 1$ by compactness and connectedness of G,

$$\det(d\psi)_{(g,t)} = \det((\mathrm{Ad}(t^{-1}) - 1)|_{\mathfrak{t}_0^\perp}). \tag{8.53}$$

We can think of building a left-invariant $(m - l)$ form on G/T from the duals of the X's in $\mathfrak{t}_0^\perp$ and a left-invariant l form on T from the duals of the H's in $\mathfrak{t}_0$. We may think of a left-invariant m form on G as the wedge of these forms. Referring to Proposition 8.19 and (8.7b) and taking (8.53) into account, we at first expect an integral formula

(8.54a)
$$\int_G f(x)\, dx \overset{?}{=} \int_T \Big[\int_{G/T} f(gtg^{-1})\, d(gT) \Big] \big| \det(\mathrm{Ad}(t^{-1}) - 1)|_{\mathfrak{t}_0^\perp} \big|\, dt$$

if the measures are normalized so that

$$\int_G f(x)\, dx = \int_{G/T} \Big[\int_T f(xt)\, dt \Big] d(xT). \tag{8.54b}$$

But Proposition 8.19 fails to be applicable in two ways. One is that the onto map $\psi : G/T \times T \to G$ has differential of determinant 0 at some points, and the other is that ψ is not one-one even if we exclude points of the domain where the differential has determinant 0.

From (8.53) we can exclude the points where the differential has determinant 0 if we restrict ψ to a map $\psi : G/T \times T' \to G'$. To understand T', consider $\mathrm{Ad}(t^{-1}) - 1$ as a linear map of the complexification $\mathfrak{g}$ to itself. If $\Delta = \Delta(\mathfrak{g}, \mathfrak{t})$ is the set of roots, then $\mathrm{Ad}(t^{-1}) - 1$ is diagonable with eigenvalues 0 with multiplicity l and also $\xi_\alpha(t^{-1}) - 1$ with multiplicity 1 each. Hence $\left|\det(\mathrm{Ad}(t^{-1})-1)|_{\mathfrak{t}_0^\perp}\right| = \left|\prod_{\alpha\in\Delta}(\xi_\alpha(t^{-1})-1)\right|$. If we fix a positive system Δ^+ and recognize that $\xi_\alpha(t^{-1}) = \overline{\xi_{-\alpha}(t^{-1})}$, then we see that

$$\left|\det(\mathrm{Ad}(t^{-1})-1)|_{\mathfrak{t}_0^\perp}\right| = \prod_{\alpha\in\Delta^+}|\xi_\alpha(t^{-1})-1|^2. \tag{8.55}$$

Putting $t = \exp iH$ with $iH \in \mathfrak{t}_0$, we have $\xi_\alpha(t^{-1}) = e^{-i\alpha(H)}$. Thus the set in the torus where (8.55) is 0 is a countable union of lower-dimensional sets and is a lower-dimensional set. By (8.25) the singular set in T has dt measure 0. The singular set in G is the smooth image of the product of G/T and the singular set in T, hence is lower dimensional and is of measure 0 for $d\mu(gT)$. Therefore we may disregard the singular set and consider ψ as a map $G/T \times T' \to G'$.

The map $\psi : G/T \times T' \to G'$ is not, however, one-one. If w is in $N_G(T)$, then

$$\psi(gwT, w^{-1}tw) = \psi(gT, t). \tag{8.56}$$

Since $gwT \neq gT$ when w is not in $Z_G(T) = T$, each member of G' has at least $|W(G,T)|$ preimages.

Lemma 8.57. Each member of G' has exactly $|W(G,T)|$ preimages under the map $\psi : G/T \times T' \to G'$.

PROOF. Let us call two members of $G/T \times T'$ equivalent, written $\sim$, if they are related by a member w of $N_G(T)$ as in (8.56), namely

$$(gwT, w^{-1}tw) \sim (gT, t).$$

Each equivalence class has exactly $|W(G,T)|$ members.

Now suppose that $\psi(gT, s) = \psi(hT, t)$ with s and t regular. We shall show that

$$(gT, s) \sim (hT, t), \tag{8.58}$$

and then the lemma will follow. The given equality $\psi(gT, s) = \psi(hT, t)$ means that $gsg^{-1} = hth^{-1}$. Proposition 4.53 shows that s and t are conjugate via $N_G(T)$. Say $s = w^{-1}tw$. Then $hth^{-1} = gw^{-1}twg^{-1}$, and $wg^{-1}h$ centralizes the element t. Since t is regular and G has a complexification, Corollary 7.106 shows that $wg^{-1}h$ is in $N_G(T)$, say $wg^{-1}h = w'$. Then $h = gw^{-1}w'$, and we have

$$\begin{aligned}(hT, t) &= (gw^{-1}w'T, t) \\ &= (gw^{-1}w'T, w'^{-1}tw') \\ &\sim (gw^{-1}T, t) \\ &\sim (gT, w^{-1}tw) \\ &= (gT, s).\end{aligned}$$

This proves (8.58) and the lemma.

Now we look at Proposition 8.19 again. Instead of assuming that $\Phi : M \to N$ is an orientation-preserving diffeomorphism, we assume for some n that Φ is an everywhere regular n-to-1 map of M onto N with $\dim M = \dim N$. Then the proof of Proposition 8.19 applies with easy modifications to give

$$n \int_N f\omega = \int_M (f \circ \Phi)\Phi^*\omega. \tag{8.59}$$

Therefore we have the following result in place of (8.54).

Theorem 8.60 (Weyl Integration Formula). Let T be a maximal torus of the compact connected Lie group G, and let invariant measures on G, T, and G/T be normalized so that

$$\int_G f(x)\,dx = \int_{G/T} \Big[\int_T f(xt)\,dt \Big] d(xT)$$

for all continuous f on G. Then every Borel function $F \geq 0$ on G satisfies

$$\int_G F(x)\,dx = \frac{1}{|W(G,T)|} \int_T \Big[\int_{G/T} F(gtg^{-1})\,d(gT) \Big] |D(t)|^2\,dt,$$

where
$$|D(t)|^2 = \prod_{\alpha \in \Delta^+} |1 - \xi_\alpha(t^{-1})|^2.$$

The integration formula in Theorem 8.60 is a starting point for an analytic treatment of parts of representation theory for compact connected Lie groups. For a given such group for which δ is analytically integral, let us sketch how the theorem leads simultaneously to a construction of an irreducible representation with given dominant analytically integral highest weight and to a proof of the Weyl Character Formula.

Define

$$D(t) = \xi_\delta(t) \prod_{\alpha\in\Delta^+} (1 - \xi_{-\alpha}(t)), \tag{8.61}$$

so that Theorem 8.60 for any Borel function f constant on conjugacy classes and either nonnegative or integrable reduces to

$$\int_G f(x)\,dx = \frac{1}{|W(G,T)|} \int_T f(t)|D(t)|^2\,dt \tag{8.62}$$

if we take dx, dt, and $d(gT)$ to have total mass one. For $\lambda \in \mathfrak{t}^*$ dominant and analytically integral, define

$$\chi_\lambda(t) = \frac{\sum_{s\in W(G,T)} \varepsilon(s)\xi_{s(\lambda+\delta)}(t)}{D(t)}.$$

Then χ_λ is invariant under $W(G,T)$, and Proposition 4.53 shows that $\chi_\lambda(t)$ extends to a function χ_λ on G constant on conjugacy classes. Applying (8.62) with $f = |\chi_\lambda|^2$, we see that

$$\int_G |\chi_\lambda|^2\,dx = 1. \tag{8.63a}$$

Applying (8.62) with $f = \chi_\lambda\overline{\chi_{\lambda'}}$, we see that

$$\int_G \chi_\lambda(x)\overline{\chi_{\lambda'}(x)}\,dx = 0 \qquad \text{if } \lambda \neq \lambda'. \tag{8.63b}$$

Let χ be the character of an irreducible finite-dimensional representation of G. On T, $\chi(t)$ must be of the form $\sum_\mu \xi_\mu(t)$, where the μ's are the weights repeated according to their multiplicities. Also $\chi(t)$ is even under $W(G,T)$. Then $D(t)\chi(t)$ is odd under $W(G,T)$ and is of the form $\sum_\nu n_\nu\xi_\nu(t)$ with each n_ν in $\mathbb{Z}$. Focusing on the dominant ν's and seeing that the ν's orthogonal to a root must drop out, we find that $\chi(t) = \sum_\lambda a_\lambda\chi_\lambda(t)$ with $a_\lambda \in \mathbb{Z}$. By (8.63),

$$\int_G |\chi(x)|^2\,dx = \sum_\lambda |a_\lambda|^2.$$

For an irreducible character Corollary 4.16 shows that the left side is 1. So one a_λ is ± 1 and the others are 0. Since $\chi(t)$ is of the form $\sum_\mu \xi_\mu(t)$, we readily find that $a_\lambda = +1$ for some λ. Hence every irreducible character is of the form $\chi = \chi_\lambda$ for some λ. This proves the Weyl Character Formula. Using the Peter–Weyl Theorem (Theorem 4.20), we readily see that no L^2 function on G that is constant on conjugacy classes can be orthogonal to all irreducible characters. Then it follows from (8.63b) that every χ_λ is an irreducible character. This proves the existence of an irreducible representation corresponding to a given dominant analytically integral form as highest weight.

For reductive Lie groups that are not necessarily compact, there is a formula analogous to Theorem 8.60. This formula is a starting point for the analytic treatment of representation theory on such groups. We state the result as Theorem 8.64 but omit the proof. The proof makes use of Theorem 7.108 and of other variants of results that we applied in the compact case.

Theorem 8.64 (Harish-Chandra). Let G be a reductive Lie group, let $(\mathfrak{h}_1)_0, \ldots, (\mathfrak{h}_r)_0$ be a maximal set of nonconjugate θ stable Cartan subalgebras of $\mathfrak{g}_0$, and let $H_1, \ldots, H_r$ be the corresponding Cartan subgroups. Let the invariant measures on each H_j and G/H_j be normalized so that

$$\int_G f(x)\,dx = \int_{G/H_j}\Big[\int_{H_j} f(gh)\,dh\Big]\,d(gH_j) \qquad \text{for all } f \in C_{\text{com}}(G).$$

Then every Borel function $F \geq 0$ on G satisfies

$$\int_G F(x)\,dx = \sum_{j=1}^{r} \frac{1}{|W(G, H_j)|}\int_{H_j}\Big[\int_{G/H_j} F(ghg^{-1})\,d(gH_j)\Big]|D_{H_j}(h)|^2\,dh,$$

where

$$|D_{H_j}(h)|^2 = \prod_{\alpha \in \Delta(\mathfrak{g}, \mathfrak{h}_j)} |1 - \xi_\alpha(h^{-1})|.$$

6. Problems

1. Prove that if M is an oriented m-dimensional manifold, then M admits a nowhere-vanishing smooth m form.

2. Prove that the zero locus of a nonzero real analytic function on a cube in $\mathbb{R}^n$ has Lebesgue measure 0.

3. Let G be the group of all real matrices $\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$ with $a > 0$. Show that $a^{-2}\, da\, db$ is a left Haar measure and that $a^{-1}\, da\, db$ is a right Haar measure.

4. Let G be a noncompact semisimple Lie group with finite center, and let $M_\mathfrak{p} A_\mathfrak{p} N_\mathfrak{p}$ be a minimal parabolic subgroup. Prove that $G/M_\mathfrak{p} A_\mathfrak{p} N_\mathfrak{p}$ has no nonzero G invariant Borel measure.

5. Prove that the complement of the set of regular points in a reductive Lie group G is a closed set of Haar measure 0.

Problems 6–8 concern Haar measure on $GL(n, \mathbb{R})$.

6. Why is Haar measure on $GL(n, \mathbb{R})$ two-sided invariant?

7. Regard $\mathfrak{gl}(n, \mathbb{R})$ as an n^2-dimensional vector space over $\mathbb{R}$. For each x in $GL(n, \mathbb{R})$, let L_x denote left multiplication by x. Prove that $\det L_x = (\det x)^n$.

8. Let E_{ij} be the matrix that is 1 in the $(i, j)^{\text{th}}$ place and is 0 elsewhere. Regard $\{E_{ij}\}$ as the standard basis of $\mathfrak{gl}(n, \mathbb{R})$, and introduce Lebesgue measure accordingly.
   (a) Why does $\{x \in \mathfrak{gl}(n, \mathbb{R}) \mid \det x = 0\}$ have Lebesgue measure 0?
   (b) Deduce from Problem 7 that $|\det y|^{-n}\, dy$ is a Haar measure for $GL(n, \mathbb{R})$.

Problems 9–12 concern the function $e^{\nu H_\mathfrak{p}(x)}$ for a semisimple Lie group G with a complexification $G^\mathbb{C}$. Here it is assumed that $G = K A_\mathfrak{p} N_\mathfrak{p}$ is an Iwasawa decomposition of G and that elements decompose as $x = \kappa(g) \exp H_\mathfrak{p}(x)\, n$. Let $\mathfrak{a}_\mathfrak{p}$ be the Lie algebra of $A_\mathfrak{p}$, and let ν be in $\mathfrak{a}_\mathfrak{p}^*$.

9. Let π be an irreducible finite-dimensional representation of G on V, and introduce a Hermitian inner product in V as in the proof of Theorem 8.49. If π has highest restricted weight ν and if v is in the restricted-weight space for ν, prove that $\|\pi(x)v\|^2 = e^{2\nu H_\mathfrak{p}(x)}\|v\|^2$.

10. In $G = SL(3, \mathbb{R})$, let $K = SO(3)$ and let $M_\mathfrak{p} A_\mathfrak{p} N_\mathfrak{p}$ be upper-triangular. Introduce parameters for $N_\mathfrak{p}^-$ by writing $N_\mathfrak{p}^- = \left\{ \bar{n} = \begin{pmatrix} 1 & 0 & 0 \\ x & 1 & 0 \\ z & y & 1 \end{pmatrix} \right\}$. Let $f_1 - f_2$, $f_2 - f_3$, and $f_1 - f_3$ be the positive restricted roots as usual, and let $\rho_\mathfrak{p}$ denote half their sum (namely $f_1 - f_3$).
    (a) Show that $e^{2 f_1 H_\mathfrak{p}(\bar{n})} = 1 + x^2 + z^2$ and $e^{2(f_1+f_2) H_\mathfrak{p}(\bar{n})} = 1 + y^2 + (z - xy)^2$ for $\bar{n} \in N_\mathfrak{p}^-$.
    (b) Deduce that $e^{2\rho_\mathfrak{p} H_\mathfrak{p}(\bar{n})} = (1 + x^2 + z^2)(1 + y^2 + (z - xy)^2)$ for $\bar{n} \in N_\mathfrak{p}^-$.

11. In $G = SO(n,1)_0$, let $K = SO(n) \times \{1\}$ and $\mathfrak{a}_\mathfrak{p} = \mathbb{R}(E_{1,n+1} + E_{n+1,1})$, with E_{ij} as in Problem 8. If $\lambda(E_{1,n+1} + E_{n+1,1}) > 0$, say that $\lambda \in \mathfrak{a}_\mathfrak{p}^*$ is positive, and obtain $G = KA_\mathfrak{p}N_\mathfrak{p}$ accordingly.
    (a) Using the standard representation of $SO(n,1)_0$, compute $e^{2\lambda H_\mathfrak{p}(x)}$ for a suitable λ and all $x \in G$.
    (b) Deduce a formula for $e^{2\rho_\mathfrak{p} H_\mathfrak{p}(x)}$ from the result of (a). Here $\rho_\mathfrak{p}$ is half the sum of the positive restricted roots repeated according to their multiplicities.

12. In $G = SU(n,1)$, let $K = S(U(n) \times U(1))$, and let $\mathfrak{a}_\mathfrak{p}$ and positivity be as in Problem 11. Repeat the two parts of Problem 11 for this group.