Appendix A. Tensors, Filtrations, and Gradings, 639-658

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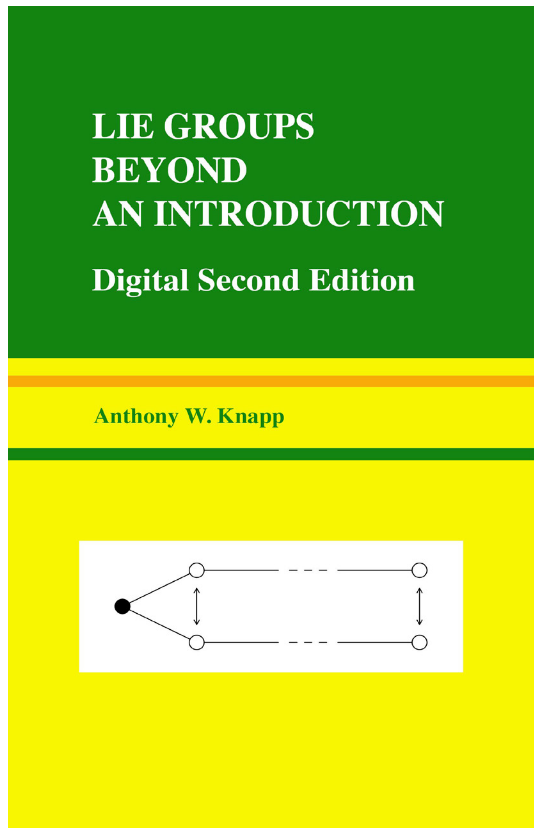

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Lie Groups Beyond an Introduction, Digital Second Edition

Pages vii–xviii and 1–812 are the same in the digital and printed second editions. A list of corrections as of June 2023 has been included as pages 813–820 of the digital second edition. The corrections have not been implemented in the text.

Cover: Vogan diagram of $\mathfrak{sl}(2n, \mathbb{R})$. See page 399.

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APPENDIX A

Tensors, Filtrations, and Gradings

Abstract. If E is a vector space, the tensor algebra $T(E)$ of E is the direct sum over $n \geq 0$ of the n-fold tensor product of E with itself. This is an associative algebra with a universal mapping property relative to any linear mapping of E into an associative algebra A with identity: the linear map extends to an algebra homomorphism of $T(E)$ into A carrying 1 into 1. Also any linear map of E into $T(E)$ extends to a derivation of $T(E)$.

The symmetric algebra $S(E)$ is a quotient of $T(E)$ with the following universal mapping property: any linear mapping of E into a commutative associative algebra A with identity extends to an algebra homomorphism of $S(E)$ into A carrying 1 into 1. The symmetric algebra is commutative.

Similarly the exterior algebra $\bigwedge(E)$ is a quotient of $T(E)$ with this universal mapping property: any linear mapping l of E into an associative algebra A with identity such that $l(v)^2 = 0$ for all $v \in E$ extends to an algebra homomorphism of $\bigwedge(E)$ into A carrying 1 into 1.

The tensor algebra, the symmetric algebra, and the exterior algebra are all examples of graded associative algebras. A more general notion than a graded algebra is that of a filtered algebra. A filtered associative algebra has an associated graded algebra. The notions of gradings and filtrations make sense in the context of vector spaces, and a linear map between filtered vector spaces that respects the filtration induces an associated graded map between the associated graded vector spaces. If the associated graded map is an isomorphism, then the original map is an isomorphism.

A ring with identity is left Noetherian if its left ideals satisfy the ascending chain condition. If a filtered algebra is given and if the associated graded algebra is left Noetherian, then the filtered algebra itself is left Noetherian.

1. Tensor Algebra

Just as polynomial rings are often used in the construction of more general commutative rings, so tensor algebras are often used in the construction of more general rings that may not be commutative. In this section we construct the tensor algebra of a vector space as a direct sum of iterated tensor products of the vector space with itself, and we establish its properties. We shall proceed with care, in order to provide a complete proof of the associativity of the multiplication.

Fix a field $\Bbbk$. Let E and F be vector spaces over the field $\Bbbk$. A **tensor product** V of E and F is a pair (V, ι) consisting of a vector space V over $\Bbbk$ together with a bilinear map $\iota : E \times F \to V$, with the following universal mapping property: Whenever b is a bilinear mapping of $E \times F$ into a vector space U over $\Bbbk$, then there exists a unique linear mapping B of V into U such that the diagram

$$\text{(A.1)}\qquad \begin{array}{ccc} & V \ (= \text{tensor product}) & \\ {}^{\iota}\nearrow & & \searrow^{B} \\ E \times F & \xrightarrow[b]{\qquad\qquad} & U \end{array}$$

commutes. We call B the **linear extension** of b to the tensor product.

It is well known that a tensor product of E and F exists and is unique up to canonical isomorphism, and we shall not repeat the proof. One feature of the proof is that it gives an explicit construction of a vector space that has the required property.

A tensor product of E and F is denoted $E \otimes_{\Bbbk} F$, and the associated bilinear map ι is written $(e, f) \mapsto e \otimes f$. The elements $e \otimes f$ generate $E \otimes_{\Bbbk} F$, as a consequence of a second feature of the proof of existence of a tensor product.

There is a canonical isomorphism

$$E \otimes_{\Bbbk} F \cong F \otimes_{\Bbbk} E \tag{A.2}$$

given by taking the linear extension of $(e, f) \mapsto f \otimes e$ as the map from left to right. The linear extension of $(f, e) \mapsto e \otimes f$ gives a two-sided inverse.

Another canonical isomorphism of interest is

$$E \otimes_{\Bbbk} \Bbbk \cong E. \tag{A.3}$$

Here the map from left to right is the linear extension of $(e, c) \mapsto ce$, while the map from right to left is $e \mapsto e \otimes 1$. In view of (A.2) we have $\Bbbk \otimes_{\Bbbk} E \cong E$ also.

Tensor product distributes over direct sums, even infinite direct sums:

$$E \otimes_{\Bbbk} \Big(\bigoplus_{\alpha} F_{\alpha} \Big) \cong \bigoplus_{\alpha} (E \otimes_{\Bbbk} F_{\alpha}). \tag{A.4}$$

The map from left to right is the linear extension of the bilinear map $(e, \sum f_{\alpha}) \mapsto \sum (e \otimes f_{\alpha})$. To define the inverse, we have only to define it on each $E \otimes_{\Bbbk} F_\alpha$, where it is the linear extension of $(e,\ f_\alpha) \mapsto e \otimes (i_\alpha(f_\alpha))$; here $i_\alpha : F_\alpha \to \bigoplus F_\beta$ is the injection corresponding to α. It follows from (A.3) and (A.4) that if $\{x_i\}$ is a basis of E and $\{y_j\}$ is a basis of F, then $\{x_i \otimes y_j\}$ is a basis of $E \otimes_{\Bbbk} F$. Consequently

$$\text{(A.5)} \qquad \dim(E \otimes_{\Bbbk} F) = (\dim E)(\dim F).$$

Let $\operatorname{Hom}_{\Bbbk}(E, F)$ be the vector space of $\Bbbk$ linear maps from E into F. One special case is $E = \Bbbk$, and we have

$$\text{(A.6)} \qquad \operatorname{Hom}_{\Bbbk}(\Bbbk, F) \cong F.$$

The map from left to right sends φ into $\varphi(1)$, while the map from right to left sends f into φ with $\varphi(c) = cf$. Another special case of interest occurs when $F = \Bbbk$. Then $\operatorname{Hom}(E, \Bbbk) = E^*$ is just the vector space **dual** of E.

We can use $\otimes_{\Bbbk}$ to construct new linear mappings. Let E_1, F_1, E_2 and F_2 be vector spaces, Suppose that L_1 is in $\operatorname{Hom}_{\Bbbk}(E_1, F_1)$ and L_2 is in $\operatorname{Hom}_{\Bbbk}(E_2, F_2)$. Then we can define

$$\text{(A.7)} \qquad L_1 \otimes L_2 \quad \text{in} \quad \operatorname{Hom}_{\Bbbk}(E_1 \otimes_{\Bbbk} E_2,\ F_1 \otimes_{\Bbbk} F_2)$$

as follows: The map $(e_1, e_2) \mapsto L_1(e_1) \otimes L_2(e_2)$ is bilinear from $E_1 \times E_2$ into $F_1 \otimes_{\Bbbk} F_2$, and we let $L_1 \otimes L_2$ be its linear extension to $E_1 \otimes_{\Bbbk} E_2$. The uniqueness in the universal mapping property allows us to conclude that

$$\text{(A.8)} \qquad (L_1 \otimes L_2)(M_1 \otimes M_2) = L_1 M_1 \otimes L_2 M_2$$

when the domains and ranges match in the obvious way.

Let A, B, and C be vector spaces over $\Bbbk$. A **triple tensor product** $V = A \otimes_{\Bbbk} B \otimes_{\Bbbk} C$ is a vector space over $\Bbbk$ with a trilinear map $\iota : A \times B \times C \to V$ having the following universal mapping property: Whenever t is a trilinear mapping of $A \times B \times C$ into a vector space U over $\Bbbk$, then there exists a linear mapping T of V into U such that the diagram

$$\text{(A.9)} \qquad \begin{array}{ccccc} & & V \ (= \text{triple tensor product}) & & \\ & \overset{\iota}{\nearrow} & & \overset{T}{\searrow} & \\ A \times B \times C & & \xrightarrow[t]{\hspace{3em}} & & U \end{array}$$

commutes. It is clear that there is at most one triple tensor product up to canonical isomorphism, and one can give an explicit construction just as for ordinary tensor products $E \otimes_{\Bbbk} F$. We shall use triple tensor products to establish an associativity formula for ordinary tensor products.

Proposition A.10.

(a) $(A \otimes_{\Bbbk} B) \otimes_{\Bbbk} C$ and $A \otimes_{\Bbbk} (B \otimes_{\Bbbk} C)$ are triple tensor products.

(b) There exists a unique isomorphism Φ from left to right in

$$(A \otimes_{\Bbbk} B) \otimes_{\Bbbk} C \cong A \otimes_{\Bbbk} (B \otimes_{\Bbbk} C) \tag{A.11}$$

such that $\Phi((a \otimes b) \otimes c) = a \otimes (b \otimes c)$ for all $a \in A$, $b \in B$, and $c \in C$.

PROOF.

(a) Consider $(A \otimes_{\Bbbk} B) \otimes_{\Bbbk} C$. Let $t : A \times B \times C \to U$ be trilinear. For $c \in C$, define $t_c : A \times B \to U$ by $t_c(a, b) = t(a, b, c)$. Then t_c is bilinear and hence extends to a linear $T_c : A \otimes_{\Bbbk} B \to U$. Since t is trilinear, $t_{c_1+c_2} = t_{c_1} + t_{c_2}$ and $t_{xc} = x t_c$ for scalar x; thus uniqueness of the linear extension forces $T_{c_1+c_2} = T_{c_1} + T_{c_2}$ and $T_{xc} = x T_c$. Consequently

$$t' : (A \otimes_{\Bbbk} B) \times C \to U$$

given by $t'(d, c) = T_c(d)$ is bilinear and hence extends to a linear $T : (A \otimes_{\Bbbk} B) \otimes_{\Bbbk} C \to U$. This T proves existence of the linear extension of the given t. Uniqueness is trivial, since the elements $(a \otimes b) \otimes c$ generate $(A \otimes_{\Bbbk} B) \otimes_{\Bbbk} C$. So $(A \otimes_{\Bbbk} B) \otimes_{\Bbbk} C$ is a triple tensor product. In a similar fashion, $A \otimes_{\Bbbk} (B \otimes_{\Bbbk} C)$ is a triple tensor product.

(b) In (A.9) take $V = (A \otimes_{\Bbbk} B) \otimes_{\Bbbk} C$, $U = A \otimes_{\Bbbk} (B \otimes_{\Bbbk} C)$, and $t(a, b, c) = a \otimes (b \otimes c)$. We have just seen in (a) that V is a triple tensor product with $\iota(a, b, c) = (a \otimes b) \otimes c$. Thus there exists a linear $T : V \to U$ with $T\iota(a, b, c) = t(a, b, c)$. This equation means that $T((a \otimes b) \otimes c) = a \otimes (b \otimes c)$. Interchanging the roles of $(A \otimes_{\Bbbk} B) \otimes_{\Bbbk} C$ and $A \otimes_{\Bbbk} (B \otimes_{\Bbbk} C)$, we obtain a two-sided inverse for T. Thus T will serve as Φ in (b), and existence is proved. Uniqueness is trivial, since the elements $(a \otimes b) \otimes c$ generate $(A \otimes_{\Bbbk} B) \otimes_{\Bbbk} C$.

When this proposition is used, it is often necessary to know that the isomorphism Φ is compatible with maps $A \to A'$, $B \to B'$, and $C \to C'$. This property is called **naturality** in the variables A, B, and C, and we make it precise in the next proposition.

Proposition A.12. Let A, B, C, A', B', and C' be vector spaces over $\Bbbk$, and let $L_A : A \to A'$, $L_B : B \to B'$, and $L_C : C \to C'$ be linear maps. Then the isomorphism Φ of Proposition A.10b is natural in the sense that the diagram

$$\begin{array}{ccc} (A \otimes_{\Bbbk} B) \otimes_{\Bbbk} C & \xrightarrow{\ \Phi\ } & A \otimes_{\Bbbk} (B \otimes_{\Bbbk} C) \\ {\scriptstyle (L_A \otimes L_B) \otimes L_C}\downarrow & & \downarrow{\scriptstyle L_A \otimes (L_B \otimes L_C)} \\ (A' \otimes_{\Bbbk} B') \otimes_{\Bbbk} C' & \xrightarrow{\ \Phi\ } & A' \otimes_{\Bbbk} (B' \otimes_{\Bbbk} C') \end{array}$$

commutes.

PROOF. We have

$$\begin{aligned} ((L_A \otimes (L_B \otimes L_C)) \circ \Phi)((a \otimes b) \otimes c) & \\ &= (L_A \otimes (L_B \otimes L_C))(a \otimes (b \otimes c)) \\ &= L_A a \otimes (L_B \otimes L_C)(b \otimes c) \\ &= \Phi((L_A a \otimes L_B b) \otimes L_C c) \\ &= \Phi((L_A \otimes L_B)(a \otimes b) \otimes L_C c) \\ &= (\Phi \circ ((L_A \otimes L_B) \otimes L_C))((a \otimes b) \otimes c), \end{aligned}$$

and the proposition follows.

There is no difficulty in generalizing matters to n-fold tensor products by induction. An n-**fold tensor product** is to be universal for n-multilinear maps. It is clearly unique up to canonical isomorphism. A direct construction is possible. Another such tensor product is the $(n-1)$-fold tensor product of the first $n-1$ spaces, tensored with the n^{th} space. Proposition A.10b allows us to regroup parentheses (inductively) in any fashion we choose, and iterated application of Proposition A.12 shows that we get a well defined notion of the tensor product of n linear maps.

Fix a vector space E over $\Bbbk$, and let $T^n(E)$ be the n-fold tensor product of E with itself. In the case $n = 0$, we let $T^0(E)$ be the field $\Bbbk$. Define, initially as a vector space, $T(E)$ to be the direct sum

$$T(E) = \bigoplus_{n=0}^{\infty} T^n(E) \tag{A.13}$$

The elements that lie in one or another $T^n(E)$ are called **homogeneous**. We define a bilinear multiplication on homogeneous elements

$$T^m(E) \times T^n(E) \to T^{m+n}(E)$$

to be the restriction of the above canonical isomorphism

$$T^m(E) \otimes_{\Bbbk} T^n(E) \to T^{m+n}(E).$$

This multiplication is associative because the restriction of the isomorphism

$$T^l(E) \otimes_{\Bbbk} (T^m(E) \otimes_{\Bbbk} T^n(E)) \to (T^l(E) \otimes_{\Bbbk} T^m(E)) \otimes_{\Bbbk} T^n(E)$$

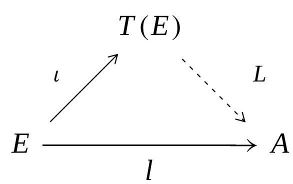
to $T^l(E) \times (T^m(E) \times T^n(E))$ factors through the map

$$T^l(E) \times (T^m(E) \times T^n(E)) \to (T^l(E) \times T^m(E)) \times T^n(E)$$

given by $(r, (s, t)) \mapsto ((r, s), t)$. Thus $T(E)$ becomes an associative algebra with identity and is known as the **tensor algebra** of E. The algebra $T(E)$ has the universal mapping properties given in the following two propositions.

Proposition A.14. $T(E)$ has the following universal mapping property: Let ι be the map that embeds E as $T^1(E) \subseteq T(E)$. If $l : E \to A$ is any linear map of E into an associative algebra with identity, then there exists a unique associative algebra homomorphism $L : T(E) \to A$ with $L(1) = 1$ such that the diagram

(A.15)



commutes.

PROOF. Uniqueness is clear, since E and 1 generate $T(E)$ as an algebra. For existence we define $L^{(n)}$ on $T^n(E)$ to be the linear extension of the n-multilinear map

$$(v_1, v_2, \dots, v_n) \mapsto l(v_1)l(v_2)\cdots l(v_n),$$

and we let $L = \bigoplus L^{(n)}$ in obvious notation. Let $u_1 \otimes \cdots \otimes u_m$ be in $T^m(E)$ and $v_1 \otimes \cdots \otimes v_n$ be in $T^n(E)$. Then we have

$$L^{(m)}(u_1 \otimes \cdots \otimes u_m) = l(u_1)\cdots l(u_m)$$

$$L^{(n)}(v_1 \otimes \cdots \otimes v_n) = l(v_1)\cdots l(v_n)$$

$$L^{(m+n)}(u_1 \otimes \cdots \otimes u_m \otimes v_1 \otimes \cdots \otimes v_n) = l(u_1)\cdots l(u_m)l(v_1)\cdots l(v_n).$$

Hence

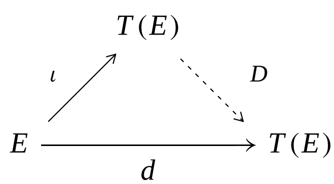
$$L^{(m)}(u_1 \otimes \cdots \otimes u_m)L^{(n)}(v_1 \otimes \cdots \otimes v_n) = L^{(m+n)}(u_1 \otimes \cdots \otimes u_m \otimes v_1 \otimes \cdots \otimes v_n).$$

Taking linear combinations, we see that L is a homomorphism.

A **derivation** $D : A \to A$ of an associative algebra with identity is a linear mapping such that $D(uv) = (Du)v + u(Dv)$ for all u and v in A. A derivation automatically satisfies $D(1) = 0$.

Proposition A.16. $T(E)$ has the following universal mapping property: Let ι be the map that embeds E as $T^1(E) \subseteq T(E)$. If $d : E \to T(E)$ is any linear map of E into $T(E)$, then there exists a unique derivation $D : T(E) \to T(E)$ such that the diagram

(A.17)



commutes.

PROOF. Uniqueness is clear, since E and 1 generate $T(E)$ as an algebra. For existence we define $D^{(n)}$ on $T^n(E)$ to be the linear extension of the n-multilinear map

$$(v_1, v_2, \ldots, v_n) \mapsto$$
$$(dv_1) \otimes v_2 \otimes \cdots \otimes v_n + v_1 \otimes (dv_2) \otimes v_3 \otimes \cdots \otimes v_n + v_1 \otimes \cdots \otimes v_{n-1} \otimes (dv_n),$$

and we let $D = \bigoplus D^{(n)}$ in obvious notation. Then we argue in the same way as in the proof of Proposition A.14 that D is the required derivation of $T(E)$.

2. Symmetric Algebra

We continue to allow $\Bbbk$ to be an arbitrary field. Let E be a vector space over $\Bbbk$, and let $T(E)$ be the tensor algebra. We begin by defining the symmetric algebra $S(E)$. The elements of $S(E)$ are to be all the symmetric tensors, and so we want to force $u \otimes v = v \otimes u$. Thus we define the **symmetric algebra** by

$$S(E) = T(E)/I, \tag{A.18a}$$

where

$$I = \begin{pmatrix} \text{two-sided ideal generated by all} \\ u \otimes v - v \otimes u \text{ with } u \text{ and } v \\ \text{in } T^1(E) \end{pmatrix}. \tag{A.18b}$$

Then $S(E)$ is an associative algebra with identity.

Since the generators of I are homogeneous elements (all in $T^2(E)$), it is clear that the ideal I satisfies

$$I = \bigoplus_{n=0}^{\infty} (I \cap T^n(E)).$$

An ideal with this property is said to be **homogeneous**. Since I is homogeneous,

$$S(E) = \bigoplus_{n=0}^{\infty} T^n(E)/(I \cap T^n(E)).$$

We write $S^n(E)$ for the n^{th} summand on the right side, so that

$$S(E) = \bigoplus_{n=0}^{\infty} S^n(E). \tag{A.19}$$

Since $I \cap T^1(E) = 0$, the map of E into first-order elements $S^1(E)$ is one-one onto. The product operation in $S(E)$ is written without a product sign, the image in $S^n(E)$ of $v_1 \otimes \cdots \otimes v_n$ in $T^n(E)$ being denoted $v_1 \cdots v_n$. If a is in $S^m(E)$ and b is in $S^n(E)$, then ab is in $S^{m+n}(E)$. Moreover $S^n(E)$ is generated by elements $v_1 \cdots v_n$ with all v_j in $S^1(E) \cong E$, since $T^n(E)$ is generated by corresponding elements $v_1 \otimes \cdots \otimes v_n$. The defining relations for $S(E)$ make $v_i v_j = v_j v_i$ for v_i and v_j in $S^1(E)$, and it follows that $S(E)$ is commutative.

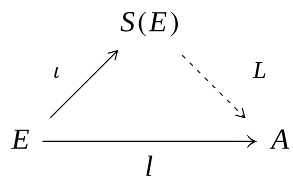
Proposition A.20.

(a) $S^n(E)$ has the following universal mapping property: Let ι be the map $\iota(v_1, \ldots, v_n) = v_1 \cdots v_n$ of $E \times \cdots \times E$ into $S^n(E)$. If l is any symmetric n-multilinear map of $E \times \cdots \times E$ into a vector space U, then there exists a unique linear map $L : S^n(E) \to U$ such that the diagram

$$\begin{array}{ccc} & S^n(E) & \\ & {}^{\iota}\!\nearrow \qquad \searrow^{L} & \\ E \times \cdots \times E & \xrightarrow[\quad l \quad]{} & U \end{array}$$

commutes.

(b) $S(E)$ has the following universal mapping property: Let ι be the map that embeds E as $S^1(E) \subseteq S(E)$. If l is any linear map of E into a commutative associative algebra A with identity, then there exists a unique algebra homomorphism $L : S(E) \to A$ with $L(1) = 1$ such that the diagram



commutes.

PROOF. In both cases uniqueness is trivial. For existence we use the universal mapping properties of $T^n(E)$ and $T(E)$ to produce $\widetilde{L}$ on $T^n(E)$ or $T(E)$. If we can show that $\widetilde{L}$ annihilates the appropriate subspace so as to descend to $S^n(E)$ or $S(E)$, then the resulting map can be taken as L, and we are done. For (a) we have $\widetilde{L} : T^n(E) \to U$, and we are to show that $\widetilde{L}(T^n(E) \cap I) = 0$, where I is generated by all $u \otimes v - v \otimes u$ with u and v in $T^1(E)$. A member of $T^n(E) \cap I$ is thus of the form $\sum a_i \otimes (u_i \otimes v_i - v_i \otimes u_i) \otimes b_i$ with each term in $T^n(E)$. Each term here is a sum of pure tensors

(A.21)
$$x_1 \otimes \cdots \otimes x_r \otimes u_i \otimes v_i \otimes y_1 \otimes \cdots \otimes y_s - x_1 \otimes \cdots \otimes x_r \otimes v_i \otimes u_i \otimes y_1 \otimes \cdots \otimes y_s$$

with $r + 2 + s = n$. Since l by assumption takes equal values on

$$x_1 \times \cdots \times x_r \times u_i \times v_i \times y_1 \times \cdots \times y_s$$

and
$$x_1 \times \cdots \times x_r \times v_i \times u_i \times y_1 \times \cdots \times y_s,$$

$\widetilde{L}$ vanishes on (A.21), and it follows that $\widetilde{L}(T^n(E) \cap I) = 0$.

For (b) we are to show that $\widetilde{L} : T(E) \to A$ vanishes on I. Since $\ker \widetilde{L}$ is an ideal, it is enough to check that $\widetilde{L}$ vanishes on the generators of I. But $\widetilde{L}(u \otimes v - v \otimes u) = l(u)l(v) - l(v)l(u) = 0$ by the commutativity of A, and thus $L(I) = 0$.

Corollary A.22. If E and F are vector spaces over $\mathbb{k}$, then the vector space $\mathrm{Hom}_{\mathbb{k}}(S^n(E), F)$ is canonically isomorphic (via restriction to pure tensors) to the vector space of F valued symmetric n-multilinear functions on $E \times \cdots \times E$.

PROOF. Restriction is linear and one-one. It is onto by Proposition A.20a.

Next we shall identify a basis for $S^n(E)$ as a vector space. The union of such bases as n varies will then be a basis of $S(E)$. Let $\{u_i\}_{i\in A}$ be a basis of E. A **simple ordering** on the index set A is a partial ordering in which every pair of elements is comparable.

Proposition A.23. Let E be a vector space over $\mathbb{k}$, let $\{u_i\}_{i\in A}$ be a basis of E, and suppose that a simple ordering has been imposed on the index set A. Then the set of all monomials $u_{i_1}^{j_1}\cdots u_{i_k}^{j_k}$ with $i_1 < \cdots < i_k$ and $\sum_m j_m = n$ is a basis of $S^n(E)$.

REMARK. In particular if E is finite dimensional with ordered basis $u_1, \dots, u_N$, then the monomials $u_1^{j_1}\cdots u_N^{j_N}$ of total degree n form a basis of $S^n(E)$.

PROOF. Since $S(E)$ is commutative and since monomials span $T^n(E)$, the indicated set spans $S^n(E)$. Let us see independence. The map $\sum c_i u_i \mapsto \sum c_i X_i$ of E into the polynomial algebra $\mathbb{k}[\{X_i\}_{i\in A}]$ is linear into a commutative algebra with identity. Its extension via Proposition A.20b maps our spanning set for $S^n(E)$ to distinct monomials in $\mathbb{k}[\{X_i\}_{i\in A}]$, which are necessarily linearly independent. Hence our spanning set is a basis.

The proof of Proposition A.23 may suggest that $S(E)$ is just polynomials in disguise, but this suggestion is misleading, even if E is finite dimensional. The isomorphism with $\mathbb{k}[\{X_i\}_{i\in A}]$ in the proof depended on choosing a basis of E. The canonical isomorphism is between $S(E^*)$ and polynomials on E. Part (b) of the corollary below goes in the direction of establishing such an isomorphism.

Corollary A.24. Let E be a finite-dimensional vector space over $\mathbb{k}$ of dimension N. Then

(a) $\dim S^n(E) = \binom{n+N-1}{N-1}$ for $0 \le n < \infty$,

(b) $S^n(E^*)$ is canonically isomorphic to $S^n(E)^*$ by

$$(f_1\cdots f_n)(w_1, \dots, w_n) = \sum_{\tau\in\mathfrak{S}_n} \prod_{j=1}^n f_j(w_{\tau(j)}),$$

where $\mathfrak{S}_n$ is the symmetric group on n letters.

PROOF.

(a) A basis has been described in Proposition A.23. To see its cardinality, we recognize that picking out $N-1$ objects from $n+N-1$ to label as dividers is a way of assigning exponents to the u_j's in an ordered basis; thus the cardinality of the indicated basis is $\binom{n+N-1}{N-1}$.

(b) Let $f_1, \dots, f_n$ be in E^*, and define

$$l_{f_1,\dots,f_n}(w_1, \dots, w_n) = \sum_{\tau\in\mathfrak{S}_n} \prod_{j=1}^{n} f_j(w_{\tau(j)}).$$

Then $l_{f_1,\dots,f_n}$ is symmetric n-multilinear from $E \times \cdots \times E$ into $\mathbb{k}$ and extends by Proposition A.20a to a linear $L_{f_1,\dots,f_n} : S^n(E) \to \mathbb{k}$. Thus $l(f_1, \dots, f_n) = L_{f_1,\dots,f_n}$ defines a symmetric n-multilinear map of $E^* \times \cdots \times E^*$ into $S^n(E^*)$. Its linear extension L maps $S^n(E^*)$ into $S^n(E)^*$.

To complete the proof, we shall show that L carries basis to basis. Let $u_1, \dots, u_N$ be an ordered basis of E, and let $u_1^*, \dots, u_N^*$ be the dual basis. Part (a) shows that the elements $(u_1^*)^{j_1} \cdots (u_N^*)^{j_N}$ with $\sum_m j_m = n$ form a basis of $S^n(E^*)$ and that the elements $(u_1)^{k_1} \cdots (u_N)^{k_N}$ with $\sum_m k_m = n$ form a basis of $S^n(E)$. We show that L of the basis of $S^n(E^*)$ is the dual basis of the basis of $S^n(E)$, except for nonzero scalar factors. Thus let $f_1, \dots, f_{j_1}$ all be u_1^*, let $f_{j_1+1}, \dots, f_{j_1+j_2}$ all be u_2^*, and so on. Similarly let $w_1, \dots, w_{k_1}$ all be u_1, let $w_{k_1+1}, \dots, w_{k_1+k_2}$ all be u_2, and so on. Then

$$\begin{aligned} L((u_1^*)^{j_1} \cdots (u_N^*)^{j_N})((u_1)^{k_1} \cdots (u_N)^{k_N}) &= L(f_1 \cdots f_n)(w_1 \cdots w_n) \\ &= l(f_1, \dots, f_n)(w_1 \cdots w_n) \\ &= \sum_{\tau\in\mathfrak{S}_n} \prod_{i=1}^{n} f_i(w_{\tau(i)}). \end{aligned}$$

For given τ, the product on the right side is 0 unless, for each index i, an inequality $j_{m-1} + 1 \le i \le j_m$ implies that $k_{m-1} + 1 \le \tau(i) \le k_m$. In this case the product is 1; so the right side counts the number of such τ's. For given τ, getting product nonzero forces $k_m = j_m$ for all m. And when $k_m = j_m$ for all m, the choice $\tau = 1$ does lead to product 1. Hence the members of L of the basis are nonzero multiples of the members of the dual basis, as asserted.

Now let us suppose that $\mathbb{k}$ has characteristic 0. We define an n-multilinear function from $E \times \cdots \times E$ into $T^n(E)$ by

$$(v_1, \dots, v_n) \mapsto \frac{1}{n!} \sum_{\tau\in\mathfrak{S}_n} v_{\tau(1)} \otimes \cdots \otimes v_{\tau(n)},$$

and let $\sigma : T^n(E) \to T^n(E)$ be its linear extension. We call σ the **symmetrizer** operator. The image of σ is denoted $\widetilde{S}^n(E)$, and the members of this subspace are called **symmetrized** tensors.

Corollary A.25. Let $\Bbbk$ have characteristic 0, and let E be a vector space over $\Bbbk$. Then the symmetrizer operator σ satisfies $\sigma^2 = \sigma$. The kernel of σ is exactly $T^n(E) \cap I$, and therefore

$$T^n(E) = \widetilde{S}^n(E) \oplus (T^n(E) \cap I).$$

REMARK. In view of this corollary, the quotient map $T^n(E) \to S^n(E)$ carries $\widetilde{S}^n(E)$ one-one onto $S^n(E)$. Thus $\widetilde{S}^n(E)$ can be viewed as a copy of $S^n(E)$ embedded as a direct summand of $T^n(E)$.

PROOF. We have

$$\begin{aligned}
\sigma^2(v_1 \otimes \cdots \otimes v_n) &= \frac{1}{(n!)^2} \sum_{\rho,\tau \in \mathfrak{S}_n} v_{\rho\tau(1)} \otimes \cdots \otimes v_{\rho\tau(n)} \\
&= \frac{1}{(n!)^2} \sum_{\rho \in \mathfrak{S}_n} \sum_{\substack{\omega \in \mathfrak{S}_n, \\ (\omega = \rho\tau)}} v_{\omega(1)} \otimes \cdots \otimes v_{\omega(n)} \\
&= \frac{1}{n!} \sum_{\rho \in \mathfrak{S}_n} \sigma(v_1 \otimes \cdots \otimes v_n) \\
&= \sigma(v_1 \otimes \cdots \otimes v_n).
\end{aligned}$$

Hence $\sigma^2 = \sigma$. Consequently $T^n(E)$ is the direct sum of image σ and $\ker \sigma$. We thus are left with identifying $\ker \sigma$ as $T^n(E) \cap I$.

The subspace $T^n(E) \cap I$ is spanned by elements

$$x_1 \otimes \cdots \otimes x_r \otimes u \otimes v \otimes y_1 \otimes \cdots \otimes y_s - x_1 \otimes \cdots \otimes x_r \otimes v \otimes u \otimes y_1 \otimes \cdots \otimes y_s$$

with $r + 2 + s = n$, and it is clear that σ vanishes on such elements. Hence $T^n(E) \cap I \subseteq \ker \sigma$. Suppose that the inclusion is strict, say with t in $\ker \sigma$ but t not in $T^n(E) \cap I$. Let q be the quotient map $T^n(E) \to S^n(E)$. The kernel of q is $T^n(E) \cap I$, and thus $q(t) \neq 0$. From Proposition A.23 it is clear that q carries $\widetilde{S}^n(E) = \text{image}\,\sigma$ onto $S^n(E)$. Thus choose $t' \in \widetilde{S}^n(E)$ with $q(t') = q(t)$. Then $t' - t$ is in $\ker q = T^n(E) \cap I \subseteq \ker \sigma$. Since $\sigma(t) = 0$, we see that $\sigma(t') = 0$. Consequently t' is in $\ker \sigma \cap \text{image}\,\sigma = 0$, and we obtain $t' = 0$ and $q(t) = q(t') = 0$, contradiction.

3. Exterior Algebra

We turn to a discussion of the exterior algebra. Let $\Bbbk$ be an arbitrary field, and let E be a vector space over $\Bbbk$. The construction, results, and proofs for the exterior algebra $\bigwedge(E)$ are similar to those for the symmetric algebra $S(E)$. The elements of $\bigwedge(E)$ are to be all the alternating tensors (= skew-symmetric if $\Bbbk$ has characteristic $\neq 2$), and so we want to force $v \otimes v = 0$. Thus we define the **exterior algebra** by

$$\bigwedge(E) = T(E)/I', \tag{A.26a}$$

where

$$I' = \begin{pmatrix} \text{two-sided ideal generated by all} \\ v \otimes v \text{ with } v \text{ in } T^1(E) \end{pmatrix}. \tag{A.26b}$$

Then $\bigwedge(E)$ is an associative algebra with identity.

It is clear that I' is homogeneous: $I' = \bigoplus_{n=0}^{\infty} (I' \cap T^n(E))$. Thus we can write

$$\bigwedge(E) = \bigoplus_{n=0}^{\infty} T^n(E)/(I' \cap T^n(E)).$$

We write $\bigwedge^n(E)$ for the n^{th} summand on the right side, so that

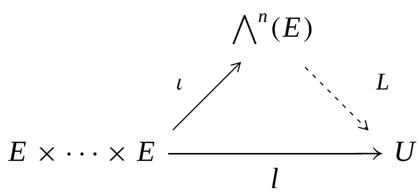
$$\bigwedge(E) = \bigoplus_{n=0}^{\infty} \bigwedge^n(E). \tag{A.27}$$

Since $I' \cap T^1(E) = 0$, the map of E into first-order elements $\bigwedge^1(E)$ is one-one onto. The product operation in $\bigwedge(E)$ is denoted $\wedge$ rather than $\otimes$, the image in $\bigwedge^n(E)$ of $v_1 \otimes \cdots v_n$ in $T^n(E)$ being denoted $v_1 \wedge \cdots \wedge v_n$. If a is in $\bigwedge^m(E)$ and b is in $\bigwedge^n(E)$, then $a \wedge b$ is in $\bigwedge^{m+n}(E)$. Moreover $\bigwedge^n(E)$ is generated by elements $v_1 \wedge \cdots \wedge v_n$ with all v_j in $\bigwedge^1(E) \cong E$, since $T^n(E)$ is generated by corresponding elements $v_1 \otimes \cdots \otimes v_n$. The defining relations for $\bigwedge(E)$ make $v_i \wedge v_j = -v_j \wedge v_i$ for v_i and v_j in $\bigwedge^1(E)$, and it follows that

$$a \wedge b = (-1)^{mn} b \wedge a \qquad \text{if } a \in \bigwedge^m(E) \text{ and } b \in \bigwedge^n(E). \tag{A.28}$$

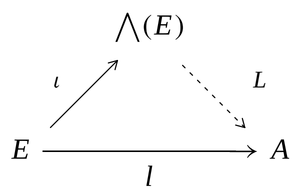
Proposition A.29.

(a) $\bigwedge^n(E)$ has the following universal mapping property: Let ι be the map $\iota(v_1, \ldots, v_n) = v_1 \wedge \cdots \wedge v_n$ of $E \times \cdots \times E$ into $\bigwedge^n(E)$. If l is any alternating n-multilinear map of $E \times \cdots \times E$ into a vector space U, then there exists a unique linear map $L : \bigwedge^n(E) \to U$ such that the diagram



commutes.

(b) $\bigwedge(E)$ has the following universal mapping property: Let ι be the map that embeds E as $\bigwedge^1(E) \subseteq \bigwedge(E)$. If l is any linear map of E into an associative algebra A with identity such that $l(v)^2 = 0$ for all $v \in E$, then there exists a unique algebra homomorphism $L : \bigwedge(E) \to A$ with $L(1) = 1$ such that the diagram



commutes.

PROOF. The proof is completely analogous to the proof of Proposition A.20.

Corollary A.30. If E and F are vector spaces over $\Bbbk$, then the vector space $\operatorname{Hom}_{\Bbbk}(\bigwedge^n(E), F)$ is canonically isomorphic (via restriction to pure tensors) to the vector space of F valued alternating n-multilinear functions on $E \times \cdots \times E$.

PROOF. Restriction is linear and one-one. It is onto by Proposition A.29a.

Next we shall identify a basis for $\bigwedge^n(E)$ as a vector space. The union of such bases as n varies will then be a basis of $\bigwedge(E)$.

Proposition A.31. Let E be a vector space over $\Bbbk$, let $\{u_i\}_{i \in A}$ be a basis of E, and suppose that a simple ordering has been imposed on the index set A. Then the set of all monomials $u_{i_1} \wedge \cdots \wedge u_{i_n}$ with $i_1 < \cdots < i_n$ is a basis of $\bigwedge^n(E)$.

PROOF. Since multiplication in $\bigwedge(E)$ satisfies (A.28) and since monomials span $T^n(E)$, the indicated set spans $\bigwedge^n(E)$. Let us see independence.

For $i \in A$, let u_i^* be the member of E^* with $u_i^*(u_j)$ equal to 1 for $j = i$ and equal to 0 for $j \neq i$. Fix $r_1 < \cdots < r_n$, and define

$$l(w_1, \dots, w_n) = \det\{u_{r_i}^*(w_j)\} \qquad \text{for } w_1, \dots, w_n \text{ in } E.$$

Then l is alternating n-multilinear from $E \times \cdots \times E$ into $\mathbb{k}$ and extends by Proposition A.29a to $L : \bigwedge^n(E) \to \mathbb{k}$. If $k_1 < \cdots < k_n$, then

$$L(u_{k_1} \wedge \cdots \wedge u_{k_n}) = l(u_{k_1}, \dots, u_{k_n}) = \det\{u_{r_i}^*(u_{k_j})\},$$

and the right side is 0 unless $r_1 = k_1, \dots, r_n = k_n$, in which case it is 1. This proves that the $u_{r_1} \wedge \cdots \wedge u_{r_n}$ are linearly independent in $\bigwedge^n(E)$.

Corollary A.32. Let E be a finite-dimensional vector space over $\mathbb{k}$ of dimension N. Then

(a) $\dim \bigwedge^n(E) = \binom{N}{n}$ for $0 \le n \le N$ and $= 0$ for $n > N$,

(b) $\bigwedge^n(E^*)$ is canonically isomorphic to $\bigwedge^n(E)^*$ by

$$(f_1 \wedge \cdots \wedge f_n)(w_1, \dots, w_n) = \det\{f_i(w_j)\}.$$

PROOF. Part (a) is an immediate consequence of Proposition A.31, and (b) is proved in the same way as Corollary A.24b, using Proposition A.29a as a tool.

Now let us suppose that $\mathbb{k}$ has characteristic 0. We define an n-multilinear function from $E \times \cdots \times E$ into $T^n(E)$ by

$$(v_1, \dots, v_n) \mapsto \frac{1}{n!} \sum_{\tau \in \mathfrak{S}_n} (\operatorname{sgn} \tau) v_{\tau(1)} \otimes \cdots \otimes v_{\tau(n)},$$

and let $\sigma' : T^n(E) \to T^n(E)$ be its linear extension. We call σ' the **antisymmetrizer** operator. The image of σ' is denoted $\widetilde{\bigwedge}^n(E)$, and the members of this subspace are called **antisymmetrized** tensors.

Corollary A.33. Let $\mathbb{k}$ have characteristic 0, and let E be a vector space over $\mathbb{k}$. Then the antisymmetrizer operator σ' satisfies $\sigma'^2 = \sigma'$. The kernel of σ' is exactly $T^n(E) \cap I'$, and therefore

$$T^n(E) = \widetilde{\bigwedge}^n(E) \oplus (T^n(E) \cap I').$$

REMARK. In view of this corollary, the quotient map $T^n(E) \to \bigwedge^n(E)$ carries $\widetilde{\bigwedge}^n(E)$ one-one onto $\bigwedge^n(E)$. Thus $\widetilde{\bigwedge}^n(E)$ can be viewed as a copy of $\bigwedge^n(E)$ embedded as a direct summand of $T^n(E)$.

PROOF. We have

$$
\begin{aligned}
\sigma'^2(v_1\otimes\cdots\otimes v_n) &= \frac{1}{(n!)^2}\sum_{\rho,\tau\in\mathfrak{S}_n}(\operatorname{sgn}\rho\tau)v_{\rho\tau(1)}\otimes\cdots\otimes v_{\rho\tau(n)}\\
&= \frac{1}{(n!)^2}\sum_{\rho\in\mathfrak{S}_n}\sum_{\substack{\omega\in\mathfrak{S}_n,\\(\omega=\rho\tau)}}(\operatorname{sgn}\omega)v_{\omega(1)}\otimes\cdots\otimes v_{\omega(n)}\\
&= \frac{1}{n!}\sum_{\rho\in\mathfrak{S}_n}\sigma'(v_1\otimes\cdots\otimes v_n)\\
&= \sigma'(v_1\otimes\cdots\otimes v_n).
\end{aligned}
$$

Hence $\sigma'^2=\sigma'$. Consequently $T^n(E)$ is the direct sum of image σ' and $\ker\sigma'$. We thus are left with identifying $\ker\sigma'$ as $T^n(E)\cap I'$.

The subspace $T^n(E)\cap I'$ is spanned by elements

$$x_1\otimes\cdots\otimes x_r\otimes v\otimes v\otimes y_1\otimes\cdots\otimes y_s$$

with $r+2+s=n$, and it is clear that σ' vanishes on such elements. Hence $T^n(E)\cap I'\subseteq\ker\sigma'$. Suppose that the inclusion is strict, say with t in $\ker\sigma'$ but t not in $T^n(E)\cap I'$. Let q be the quotient map $T^n(E)\to\bigwedge^n(E)$. The kernel of q is $T^n(E)\cap I'$, and thus $q(t)\neq 0$. From Proposition A.31 it is clear that q carries $\widetilde{\bigwedge}^n(E)=\operatorname{image}\sigma'$ onto $\bigwedge^n(E)$. Thus choose $t'\in\widetilde{\bigwedge}^n(E)$ with $q(t')=q(t)$. Then $t'-t$ is in $\ker q=T^n(E)\cap I'\subseteq\ker\sigma'$. Since $\sigma'(t)=0$, we see that $\sigma'(t')=0$. Consequently t' is in $\ker\sigma'\cap\operatorname{image}\sigma'=0$, and we obtain $t'=0$ and $q(t)=q(t')=0$, contradiction.

4. Filtrations and Gradings

Let $\Bbbk$ be any field. A vector space V over $\Bbbk$ will be said to be **filtered** if there is a specified increasing sequence of subspaces

$$V_0\subseteq V_1\subseteq V_2\subseteq\cdots \tag{A.34}$$

with union V. In this case we put $V_{-1}=0$ by convention. We shall say that V is **graded** if there is a specified sequence of subspaces $V^0, V^1, V^2, \ldots$ such that

$$V=\bigoplus_{n=0}^{\infty}V^n. \tag{A.35}$$

When V is graded, there is a natural filtration of V given by

$$V_n = \bigoplus_{k=0}^{n} V^k. \tag{A.36}$$

When E is a vector space, the tensor algebra $V = T(E)$ is graded as a vector space, and the same thing is true of the symmetric algebra $S(E)$ and the exterior algebra $\bigwedge(E)$. In each case the n^{th} subspace of the grading consists of the subspace of tensors that are homogeneous of degree n.

When V is a filtered vector space as in (A.34), the **associated graded vector space** is

$$\operatorname{gr} V = \bigoplus_{n=0}^{\infty} V_n/V_{n-1}. \tag{A.37}$$

In the case that V is graded and its filtration is the natural one given in (A.36), $\operatorname{gr} V$ recovers the given grading on V, i.e., $\operatorname{gr} V$ is canonically isomorphic with V in a way that preserves the grading.

Let V and V' be two filtered vector spaces, and let φ be a linear map between them such that $\varphi(V_n) \subseteq V'_n$ for all n. Since the restriction of φ to V_n carries V_{n-1} into V'_{n-1}, this restriction induces a linear map $\operatorname{gr}^n \varphi : (V_n/V_{n-1}) \to (V'_n/V'_{n-1})$. The direct sum of these linear maps is then a linear map

$$\operatorname{gr}\varphi : \operatorname{gr} V \to \operatorname{gr} V' \tag{A.38}$$

called the **associated graded map** for φ.

Proposition A.39. Let V and V' be two filtered vector spaces, and let φ be a linear map between them such that $\varphi(V_n) \subseteq V'_n$ for all n. If $\operatorname{gr}\varphi$ is an isomorphism, then φ is an isomorphism.

PROOF. It is enough to prove that $\varphi|_{V_n} : V_n \to V'_n$ is an isomorphism for every n. We establish this property by induction on n, the trivial case for the induction being $n = -1$. Suppose that

$$\varphi|_{V_{n-1}} : V_{n-1} \to V'_{n-1} \quad \text{is an isomorphism.} \tag{A.40}$$

By assumption

$$\operatorname{gr}^n \varphi : (V_n/V_{n-1}) \to (V'_n/V'_{n-1}) \quad \text{is an isomorphism.} \tag{A.41}$$

If v is in $\ker(\varphi|_{V_n})$, then $(\operatorname{gr}^n \varphi)(v + V_{n-1}) = 0 + V'_{n-1}$, and (A.41) shows that v is in V_{n-1}. By (A.40), $v = 0$. Thus $\varphi|_{V_n}$ is one-one. Next suppose that v' is in V'_n. By (A.41) there exists v_n in V_n such that $(\operatorname{gr}^n \varphi)(v_n + V_{n-1}) = v' + V'_{n-1}$. Write $\varphi(v_n) = v' + v'_{n-1}$ with v'_{n-1} in V'_{n-1}. By (A.40) there exists v_{n-1} in V_{n-1} with $\varphi(v_{n-1}) = v'_{n-1}$. Then $\varphi(v_n - v_{n-1}) = v'$, and thus $\varphi|_{V_n}$ is onto. This completes the induction.

Now let A be an associative algebra over $\mathbb{k}$ with identity. If A has a filtration $A_0, A_1, \ldots$ of vector subspaces with $1 \in A_0$ such that $A_m A_n \subseteq A_{m+n}$ for all m and n, then we say that A is a **filtered associative algebra**. Similarly if A is graded as $A = \bigoplus_{n=0}^{\infty} A^n$ in such a way that $A^m A^n \subseteq A^{m+n}$ for all m and n, then we say that A is a **graded associative algebra**.

Proposition A.42. If A is a filtered associative algebra with identity, then the graded vector space gr A acquires a multiplication in a natural way making it into a graded associative algebra with identity.

PROOF. We define a product

$$(A_m/A_{m-1}) \times (A_n/A_{n-1}) \to A_{m+n}/A_{m+n-1}$$

by
$$(a_m + A_{m-1})(a_n + A_{n-1}) = a_m a_n + A_{m+n-1}.$$

This is well defined since $a_m A_{n-1}$, $A_{m-1}a_n$, and $A_{m-1}A_{n-1}$ are all contained in A_{m+n-1}. It is clear that this multiplication is distributive and associative as far as it is defined. We extend the definition of multiplication to all of gr A by taking sums of products of homogeneous elements, and the result is an associative algebra. The identity is the element $1 + A_{-1}$ of A_0/A_{-1}.

5. Left Noetherian Rings

The first part of this section works with an arbitrary ring A with identity. All left A modules are understood to be **unital** in the sense that 1 acts as 1. Later in the section we specialize to the case that A is an associative algebra with identity over a field $\mathbb{k}$.

Let M be a left A module. We say that M satisfies the **ascending chain condition** as a left A module if whenever $M_1 \subseteq M_2 \subseteq \cdots$ is an infinite ascending sequence of left A submodules of M, then there exists an integer n such that $M_i = M_n$ for $i \geq n$. We say that M satisfies the **maximum condition** as a left A module if every nonempty collection of left A submodules of M has a maximal element under inclusion.

Proposition A.43. The left A module M satisfies the ascending chain condition if and only if it satisfies the maximum condition, if and only if every left A submodule of M is finitely generated.

PROOF. If M satisfies the ascending chain condition, we argue by contradiction that M satisfies the maximum condition. Let a nonempty collection $\{M_\alpha\}$ of left A submodules be given for which the maximum condition fails. Let M_1 be any M_α. Since M_1 is not maximal, choose M_2 as an M_α properly containing M_1. Since M_2 is not maximal, choose M_3 as an M_α properly containing M_2. Continuing in this way results in a properly ascending infinite chain, in contradiction to the hypothesis that M satisfies the ascending chain condition.

If M satisfies the maximum condition and N is a left A submodule, define a left A submodule $N_F = \sum_{m \in F} Am$ of N for every finite subset F of N. The maximum condition yields an F_0 with $N_F \subseteq N_{F_0}$ for all F, and we must have $N_{F_0} = N$. Then F_0 generates N.

If every left A submodule of M is finitely generated and if an ascending chain $M_1 \subseteq M_2 \subseteq \cdots$ is given, let $\{m_\alpha\}$ be a finite set of generators for $\bigcup_{j=1}^\infty M_j$. Then all m_α are in some M_n, and it follows that $M_i = M_n$ for $i \geq n$.

We say that the ring A is **left Noetherian** if A, as a left A module, satisfies the ascending chain condition, i.e., if the left ideals of A satisfy the ascending chain condition.

Proposition A.44. The ring A is left Noetherian if and only if every left ideal is finitely generated.

PROOF. This follows from Proposition A.43.

Theorem A.45 (Hilbert Basis Theorem). If A is a commutative Noetherian ring with identity, then the polynomial ring $A[X]$ in one indeterminate is Noetherian.

REFERENCE. Zariski–Samuel [1958], p. 201.

EXAMPLES. Any field $\mathbb{k}$ is Noetherian, having only the two ideals 0 and $\mathbb{k}$. Iterated application of Theorem A.45 shows that any polynomial ring $\mathbb{k}[X_1, \ldots, X_n]$ is Noetherian. If E is an n-dimensional vector space over $\mathbb{k}$, then $S(E)$ is noncanonically isomorphic as a ring to $\mathbb{k}[X_1, \ldots, X_n]$ as a consequence of Corollary A.24b, and $S(E)$ is therefore Noetherian.

Now let A be a filtered associative algebra over $\mathbb{k}$, in the sense of the previous section, and let $\operatorname{gr} A$ be the corresponding graded associative algebra. Let a be an element of A, and suppose that a is in A_n but not A_{n-1}. The member $\bar{a} = a + A_{n-1}$ of $A^n \subseteq \operatorname{gr} A$ is called the **leading term** of a. In the case of the 0 element of A, we define the leading term to be the 0 element of gr A.

Lemma A.46. Let A be a filtered associative algebra, and let gr A be the corresponding graded associative algebra. If I is a left ideal in A, then the set $\bar{I}$ of finite sums of leading terms of members of I is a left ideal of gr A that is homogeneous in the sense that $\bar{I} = \bigoplus_{n=0}^{\infty}(\bar{I} \cap A^n)$.

PROOF. Every leading term other than 0 lies in some A^n, and therefore $\bar{I}$ is homogeneous. Let $\bar{x}$ be homogeneous in gr A, and let $\bar{y}$ be a leading term in $\bar{I}$, arising from some $y \in I$. We are to prove that $\bar{x}\bar{y}$ is in $\bar{I}$. From the definition of gr A, $\bar{x}$ has to be the leading term of some $x \in A$. Then xy is in I, and $\overline{xy}$ is in $\bar{I}$. From the rule for multiplication in gr A and the requirement that $A_m A_n \subseteq A_{m+n}$ in A, either $\bar{x}\bar{y} = \overline{xy}$ or $\bar{x}\bar{y} = 0$. In either case, $\bar{x}\bar{y}$ is in $\bar{I}$.

Proposition A.47. Let A be a filtered associative algebra, and let gr A be the corresponding graded associative algebra. If gr A is left Noetherian, then A is left Noetherian.

PROOF. By Proposition A.44 every left ideal of gr A is finitely generated, and we are to prove that A has the same property. Suppose I is a left ideal in A, and form $\bar{I}$. By Lemma A.46, $\bar{I}$ is a homogeneous left ideal, and thus it has finitely many generators $\bar{a}_1, \dots, \bar{a}_r$. Without loss of generality we may assume that each $\bar{a}_j$ is homogeneous and is the leading term of some a_j in I.

The claim is that $a_1, \dots, a_r$ is a finite set of generators for I. We prove by induction on n that each element a whose leading term $\bar{a}$ has degree n can be written as $a = \sum_{i=1}^{r} c_i a_i$ with c_i in A, and then the claim follows. The claim is trivial for $n = 0$. Thus assume the claim for elements with leading term of degree $< n$. Let a be given with leading term $\bar{a} = \sum_{i=1}^{r} \bar{c}_i \bar{a}_i$, $\bar{c}_i \in$ gr A. Equating homogeneous parts, we may assume that each $\bar{c}_i$ is homogeneous and that each $\bar{c}_i \bar{a}_i$ is homogeneous of degree n. Then $\bar{c}_i$ is the leading term for some c_i, and the leading term of $\sum_{i=1}^{r} c_i a_i$ is $\bar{a}$. Hence $a - \sum_{i=1}^{r} c_i a_i$ is in A_{n-1} and by inductive hypothesis is in the left ideal generated by the a_i. Hence a is in the left ideal generated by the a_i. This completes the induction and the proof of the proposition.