

I. Smooth Manifolds, 1-55

DOI: [10.3792/euclid/9781429799881-1](https://doi.org/10.3792/euclid/9781429799881-1)

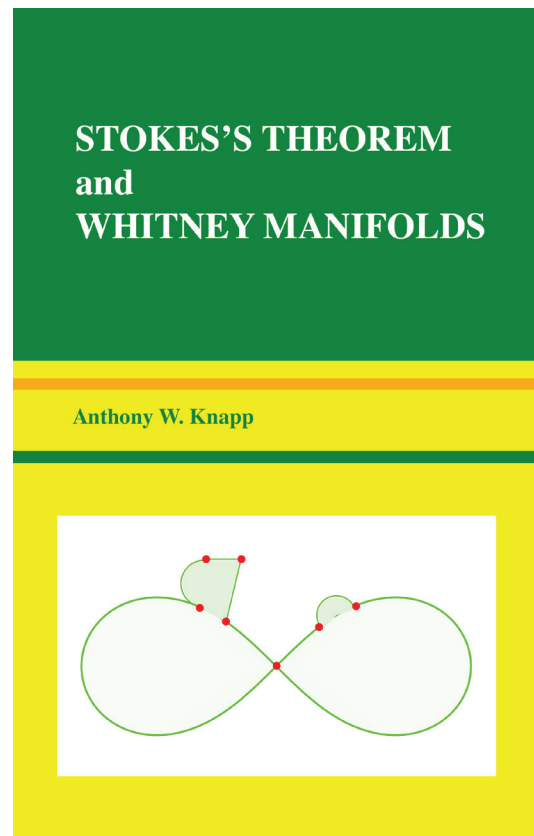
from

***Stokes's Theorem and
Whitney Manifolds:***
A Sequel to
Basic Real Analysis

Anthony W. Knapp

Full Book DOI: [10.3792/euclid/9781429799881](https://doi.org/10.3792/euclid/9781429799881)

ISBN: 978-1-4297-9988-1



Anthony W. Knapp
81 Upper Sheep Pasture Road
East Setauket, N.Y. 11733–1729, U.S.A.
Email to: aknapp@math.stonybrook.edu
Homepage: www.math.stonybrook.edu/~aknapp

Title: Stokes’s Theorem and Whitney Manifolds. A Sequel to *Basic Real Analysis*.

Cover: An example of a Whitney domain in two-dimensional space. The green portion is a manifold-with-boundary for which Stokes’s Theorem applies routinely. The red dots indicate exceptional points of the boundary where a Whitney condition applies that says Stokes’s Theorem extends to the whole domain.

Mathematics Subject Classification (2020): 58–01, 58A05, 58A35, 58C35, 26B20.

Original Digital Edition, not to be sold, no ISBN

©2021 Anthony W. Knapp

Published by the Author

All rights reserved. This file is the original digital edition of the above named book. The text, images, and other data contained in this file, which is in portable document format (PDF), are proprietary to the author, and the author retains all rights, including copyright, in them. The use in this file of trade names, trademarks, service marks, and similar items, even if they are not identified as such, is not to be taken as an expression of opinion whether they are subject to proprietary rights.

The file is made available for limited noncommercial use for purposes of education, scholarship, and research, and for these purposes only, or for fair use as understood in the United States copyright law. Users may freely download this file for their own use and may store it, post it online, and transmit it digitally for purposes of education, scholarship, and research. They may not convert it from PDF to any other format (e.g., EPUB), they may not edit it, and they may not do reverse engineering with it. In transmitting the file to others or posting it online, users must charge no fee, nor may they include the file in any collection of files for which a fee is charged.

Each user of this file is granted permission to make one printed copy of the entire book for personal use toward education, scholarship, and research, and for these purposes only. This personal printed copy must not be sold in whole or part. Any exception to these rules concerning the file and its printing requires written permission from the author.

Except as provided by fair use provisions of the United States copyright law, no extracts or quotations from this file may be used that do not consist of whole pages unless permission has been granted by the author.

The permission granted for use of the whole file and the prohibition against charging fees extend to any partial file that contains only whole pages from this file, except that the copyright notice on this page must be included in any partial file that does not consist exclusively of the front cover page. Such a partial file shall not be included in any derivative work unless permission has been granted by the author.

CHAPTER I

Smooth Manifolds

Abstract. This chapter introduces just enough differential topology to serve as a suitable framework for Stokes's Theorem. The subject matter is the elementary structure of smooth manifolds, which is a topic in real analysis that sits at the intersection of algebraic topology and differential geometry.

Section 1 presents the beginning definitions and results about smooth manifolds, tangent vectors and vector fields, cotangent vectors and differential 1 forms, derivatives of smooth mappings, and differentials.

Section 2 defines the exterior algebra of a finite dimensional real vector space. Tensor algebras, which are discussed in Chapter VI of the author's *Basic Algebra*, are taken as known.

Section 3 introduces differential forms and their pullbacks under smooth maps. It shows how to compute pullbacks, and it establishes some properties of them.

Section 4 introduces the exterior derivative, which is the differentiation operator to be used with differential forms, and shows that it satisfies a number of properties.

Section 5 contains the construction of a smooth partition of unity, which is a device making it unnecessary in many cases to cut manifolds into pieces when treating integration problems.

Section 6 introduces the notion of an oriented smooth manifold and integration of top-degree differential forms on it. The section shows also the relationship between integration and pullback.

1. Smooth Manifolds, Vector Fields, Derivatives, and Differentials

This section introduces smooth manifolds, and it briefly develops the notions of smooth function, tangent and cotangent space, vector field, derivative, differential 1 form, and differential. For a more thorough presentation of this material, the reader may wish to consult the author's *Advanced Real Analysis*, particularly Sections VIII.1–4.

“Manifolds” in our treatment are built from “charts,” each manifold has a uniform dimension, and each manifold will be assumed to be separable in the sense of having a countable base for its topology. The term “smooth” is used interchangeably with the term C^∞ . The prototype for a manifold is the surface of a sphere in three dimensions. Let us discuss this case informally first and then return to develop the formal mathematics.

In the real world one describes the surface of the earth by means of “charts,” with each chart containing a likeness of part of the earth's surface and with all

the charts together describing the whole surface. The collection of charts is an “atlas.” The sense in which a chart contains a likeness of part of the surface is that there is an understood one-one function (“map”) from the one onto the other. In mathematics this function goes from a part of the surface into a likeness; in the real world it tends to go in the opposite direction, namely from the likeness into the surface.

Let M be a separable¹ Hausdorff topological space, and fix an integer $m \geq 0$. A **chart** (M_α, α) on M of **dimension** m is a homeomorphism $\alpha : M_\alpha \rightarrow \alpha(M_\alpha)$ of a nonempty open subset M_α of M onto an open subset $\alpha(M_\alpha)$ of \mathbb{R}^m ; the chart is said to be **about** a point p in M if p is in the domain M_α of α . We say that M is a **manifold** if there is an integer $m \geq 0$ such that each point of M has a chart of dimension m about it.

A **smooth structure** of dimension m on a manifold M is a family \mathcal{F} of m dimensional charts with the following three properties:

- (i) any two charts (M_α, α) and (M_β, β) in \mathcal{F} are smoothly **compatible** in the sense that $\beta \circ \alpha^{-1}$, as a mapping of the open subset $\alpha(M_\alpha \cap M_\beta)$ of \mathbb{R}^m to the open subset $\beta(M_\alpha \cap M_\beta)$ of \mathbb{R}^m , is smooth and has a smooth inverse,
- (ii) the system of compatible charts (M_α, α) is an **atlas** in the sense that the domains M_α together cover M ,
- (iii) \mathcal{F} is maximal among families of compatible charts on M .

A **smooth manifold** of **dimension** m is a manifold together with a smooth structure of dimension m . In the presence of an understood atlas, a chart will be said to be **compatible** if it is compatible with all the members of the atlas.

Once we have an atlas of compatible m dimensional charts for a manifold M , i.e., once (i) and (ii) are satisfied, then the family of all compatible charts satisfies (i) and (iii), as well as (ii), and therefore is a smooth structure. In other words, an atlas of compatible charts determines one and only one smooth structure. As a practical matter we can thus construct a smooth structure for a manifold by finding an atlas satisfying (i) and (ii), and the extension of the atlas for (iii) to hold is automatic. Particularly in discussing orientability in Section 6, it will be convenient to work with atlases that are not maximal.

EXAMPLE. The unit sphere $M = S^n$ in \mathbb{R}^{n+1} , the set of vectors of Euclidean norm 1, can be made into a smooth manifold of dimension n by using two charts defined as follows. One of these charts is (M_φ, φ) with

$$\varphi(x_1, \dots, x_{n+1}) = \left(\frac{x_1}{1 - x_{n+1}}, \dots, \frac{x_n}{1 - x_{n+1}} \right)$$

¹The treatment in Sections VIII.1–4 of *Advanced Real Analysis* does not insist on separability of manifolds.

and with domain $M_\varphi = S^n - \{(0, \dots, 0, 1)\}$, and the other is (M_ψ, ψ) with

$$\psi(x_1, \dots, x_{n+1}) = \left(\frac{x_1}{1+x_{n+1}}, \dots, \frac{x_n}{1+x_{n+1}} \right)$$

and with domain $M_\psi = S^n - \{(0, \dots, 0, -1)\}$. We need to check that the two charts are smoothly compatible. The set $M_\varphi \cap M_\psi$ is $S^n - \{(0, \dots, 0, \pm 1)\}$, and the image of this under φ and ψ is $\mathbb{R}^n - \{(0, \dots, 0)\}$. Put $y_j = x_j/(1-x_{n+1})$, so that $\varphi^{-1}(y_1, \dots, y_n) = (x_1, \dots, x_{n+1})$. Then $\psi \circ \varphi^{-1}(y_1, \dots, y_n)$ is

$$\begin{aligned} &= (x_1/(1+x_{n+1}), \dots, x_n/(1+x_{n+1})) \\ &= (y_1(1-x_{n+1})/(1+x_{n+1}), \dots, y_n(1-x_{n+1})/(1+x_{n+1})). \end{aligned}$$

To compute $(1-x_{n+1})/(1+x_{n+1})$, we take $\sum_{j=1}^{n+1} x_j^2 = 1$ into account and write $1 = \sum_{j=1}^{n+1} x_j^2 = x_{n+1}^2 + \sum_{j=1}^n y_j^2(1-x_{n+1})^2$. Then $\sum_{j=1}^n y_j^2 = (1-x_{n+1}^2)/(1-x_{n+1})^2 = (1+x_{n+1})/(1-x_{n+1})$, and

$$\psi \circ \varphi^{-1}(y_1, \dots, y_n) = \left(y_1 / \sum_{j=1}^n y_j^2, \dots, y_n / \sum_{j=1}^n y_j^2 \right).$$

The entries on the right are smooth functions of y since $y \neq 0$. Similarly if we put $z_j = x_j/(1+x_{n+1})$, we calculate that

$$\varphi \circ \psi^{-1}(z_1, \dots, z_n) = \left(z_1 / \sum_{j=1}^n z_j^2, \dots, z_n / \sum_{j=1}^n z_j^2 \right).$$

Again the entries on the right are smooth functions of z since $z \neq 0$. Thus the two charts are smoothly compatible, and S^n is a smooth manifold.

Euclidean space \mathbb{R}^m itself is of course a smooth manifold of dimension m , with an atlas consisting of the single chart $(\mathbb{R}^m, 1)$, where 1 is the identity function on \mathbb{R}^m . Real projective spaces, which are defined in Problem 3 at the end of the chapter, give further straightforward examples. A number of interesting manifolds arise as a part of the space of simultaneous solutions of some equations, often polynomial equations in several variables. The technical device that shows the solution space to be part of a smooth manifold is normally the Implicit Function Theorem (Theorem 3.16 of *Basic Real Analysis*), as is explained in Problem 30 at the end of the chapter.

Another simple example of a smooth manifold M of dimension m is any nonempty open subset U of M . The subset U becomes a smooth manifold

of dimension m if we define an atlas for it to consist of all restrictions $(U \cap M_\alpha, \alpha|_{U \cap M_\alpha})$ of members of the atlas $\{(M_\alpha, \alpha)\}$ for M ; then we must discard occurrences of the empty set. We shall often use this observation without special notice, in effect making definitions and deducing conclusions for nonempty open subsets of a manifold M from the corresponding definitions and conclusions about all manifolds.

Most manifolds, however, are constructed globally out of other manifolds or are pieced together from local data. The Hausdorff condition often has to be checked, is often subtle, and is always important. The first place that the Hausdorff condition plays a role is in Lemma 1.2 below.

Any manifold is a locally compact Hausdorff space. The separability implies that there exists an **exhausting sequence** in M , i.e., an increasing sequence of compact sets with union all of M and with each set contained in the interior of the next member of the sequence. This is Proposition 10.25 of *Basic Real Analysis*.

Let us mention that because of the separability and Theorem 10.45 of *Basic Real Analysis*, the topology of a manifold can always be realized by a metric; this fact turns out to be more comforting than useful.

Although manifolds have a global definition, it is often convenient to work with them by referring matters to local coordinates. If p is a point of the smooth manifold M of dimension m , then a compatible chart (M_α, α) about p can be viewed as giving a **local coordinate system** near p . Specifically if the Euclidean coordinates in $\alpha(M_\alpha)$ are (u_1, \dots, u_m) , then $q = \alpha^{-1}(u_1, \dots, u_m)$ is a general point of M_α , and we define m real-valued functions $q \mapsto x_j(q)$ on M_α by $x_j(q) = u_j$, $1 \leq j \leq m$. Then $\alpha = (x_1, \dots, x_m)$. To refer the functions x_j to Euclidean space \mathbb{R}^m , we use $x_j \circ \alpha^{-1}$, which carries (u_1, \dots, u_m) to u_j .

The way that the functions x_j are referred to Euclidean space mirrors how a more general real-valued function on an open subset of M may be referred to Euclidean space, and then we can define a real-valued function on M to be smooth if it is smooth in the sense of Euclidean differential calculus when referred to Euclidean space.

Therefore a **smooth function** $f : M \rightarrow \mathbb{R}$ on the smooth manifold M is by definition a function such that for each $p \in M$ and each compatible chart (M_α, α) about p , the function $f \circ \alpha^{-1}$ is smooth as a function from the open subset $\alpha(M_\alpha)$ of \mathbb{R}^m into \mathbb{R} . A smooth function is necessarily continuous.

In verifying that a real-valued function f on M is smooth, it is sufficient, for each point in M , to check smoothness within only one compatible chart about that point. The reason is the compatibility of the charts: if (M_α, α) and (M_β, β) are two compatible charts about p , then $f \circ \beta^{-1}$ is the composition of the smooth function $\alpha \circ \beta^{-1}$ followed by $f \circ \alpha^{-1}$.

The space of smooth real-valued functions on the nonempty open set U of M will be denoted by $C^\infty(U)$. The space $C^\infty(U)$ is an associative algebra over \mathbb{R}

under the pointwise operations, and it contains the constants. The **support** of a real-valued function is the *closure* of the set where the function is nonzero. We write $C_{\text{com}}^{\infty}(U)$ for the subset of $C^{\infty}(U)$ of functions whose support is a compact subset of U .

The space $C_{\text{com}}^{\infty}(U)$ is not 0. This fact is a consequence of the following result for Euclidean space that appeared as Proposition 8.12 in *Basic Real Analysis*.

Lemma 1.1. If K and U are subsets of \mathbb{R}^m with K compact, U open, and $K \subseteq U$, then there exists $\varphi \in C_{\text{com}}^{\infty}(U)$ with values in $[0, 1]$ such that φ is identically 1 on K .

Lemma 1.2. If U is a nonempty open subset of a smooth manifold M and if f is in $C_{\text{com}}^{\infty}(U)$, then the function F defined on M so as to equal f on U and to equal 0 off U is in $C_{\text{com}}^{\infty}(M)$ and has support contained in U ,

PROOF. The set $S = \text{support}(f)$ is a compact subset of U and is compact as a subset of M since the fact that U gets the relative topology means that the inclusion of U into M is continuous. Since M is Hausdorff, S is closed in M . The function F is smooth at all points of U and in particular at all points of S , and we need to prove that it is smooth at all points of the open complement V of S in M . If p is in V , we can find a compatible chart (M_{α}, α) about p with $M_{\alpha} \subseteq V$. The function F is 0 on $M_{\alpha} \cap U \subseteq V \cap U = S^c \cap U$ because it equals f on U and f is 0 on the complement of S in U . The function F is 0 on $M_{\alpha} \cap U^c$ since it is 0 everywhere on U^c . Therefore F is identically 0 on M_{α} and is exhibited as smooth in a neighborhood of p . Thus F is smooth. \square

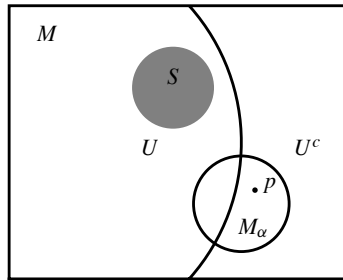


FIGURE 1.1. Diagram for Lemma 1.2 with p shown outside $M_{\alpha} \cap U$.

Lemma 1.3. Suppose that p is a point in a smooth manifold M , that (M_{α}, α) is a compatible chart about p , and that K is a compact subset of M_{α} containing p . Then there is a smooth function $f : M \rightarrow \mathbb{R}$ with compact support contained in M_{α} such that f has values in $[0, 1]$ and f is identically 1 on K .

PROOF. The set $\alpha(K)$ is a compact subset of the open subset $\alpha(M_\alpha)$ of Euclidean space, and Lemma 1.1 produces a smooth function g in $C_{\text{com}}^\infty(\alpha(M_\alpha))$ with values in $[0, 1]$ that is identically 1 on $\alpha(K)$. If f is defined to be $g \circ \alpha$ on M_α , then f is in $C_{\text{com}}^\infty(M_\alpha)$. Extending f to be 0 on the complement of M_α in M and applying Lemma 1.2, we see that the extended f satisfies the required conditions. \square

Proposition 1.4. Let p be a point of a smooth manifold M , let U be an open neighborhood of p , and let f be in $C^\infty(U)$. Then there is a function g in $C^\infty(M)$ such that $g = f$ in a neighborhood of p .

PROOF. Possibly by shrinking U , we may assume that U is the domain of some compatible chart (M_α, α) about p . Let K be a compact neighborhood of p contained in U , and use Lemma 1.3 to find h in $C^\infty(M)$ with compact support in U such that h is identically 1 on K . Define g to be the pointwise product hf on U and to be 0 off U . Then g equals f on the neighborhood K of p , and Lemma 1.2 shows that g is everywhere smooth. \square

In the same way that we defined smoothness of real-valued functions on smooth manifolds by means of local coordinates, we define smoothness for a *continuous* function from an m dimensional manifold M into an n dimensional manifold N . Namely let p be in M , so that $F(p)$ is in N . Assuming that F is continuous at p , let a local coordinate system be given at $F(p)$ by means of a chart (N_β, β) , and choose a local coordinate system at p given by a chart (M_α, α) such that $F(M_\alpha) \subseteq N_\beta$. The local version of F is the function $\beta \circ F \circ \alpha^{-1}$, which carries $\alpha(M_\alpha)$ into $\beta(N_\beta)$. If we write $\alpha = (x_1, \dots, x_m)$ and $\beta = (y_1, \dots, y_n)$, then we obtain an expression of the form

$$(y_1, \dots, y_n) = \beta \circ F \circ \alpha^{-1}(x_1, \dots, x_m),$$

and we see that $\beta \circ F \circ \alpha^{-1}$ is the local function in the Euclidean setting that corresponds to F in the manifold setting. The function $F : M \rightarrow N$ is said to be **smooth** if it is continuous and all the functions $\beta \circ F \circ \alpha^{-1}$ are smooth, more precisely if it is continuous and for each p in E and each compatible chart β about $F(p)$, there is some compatible chart α about p such that $\beta \circ F \circ \alpha^{-1}$ is defined and smooth. In this case we often call F a **smooth map**. A smooth function between smooth manifolds with a smooth inverse is called a **diffeomorphism**.

In this way all questions about smoothness of functions in the manifold setting can be translated into questions about smoothness of functions in the Euclidean setting. One consequence, by means of the Inverse Function Theorem,² is that the dimension of a smooth manifold is well defined. More specifically the same underlying topological space cannot have two compatible atlases of distinct dimensions.

²Theorem 3.17 of *Basic Real Analysis*.

We turn to a discussion of tangent spaces and vector fields. Let M be a smooth manifold of dimension m . The idea is that the tangent space to M at p is the space of all first-order derivatives at p . To make this notion precise, one introduces the space of **germs** $\mathcal{C}_p(M)$ of smooth functions at p . These are equivalence classes formed from pairs (f, U) , each pair consisting of an open set U containing p and a smooth real-valued function f defined on that open set, two such being equivalent if their restrictions are equal on some subneighborhood of p . The set $\mathcal{C}_p(M)$ of equivalence classes inherits arithmetic operations that make it into an associative algebra over \mathbb{R} . Evaluation at p is a well defined linear functional e on $\mathcal{C}_p(M)$. A **derivation** of $\mathcal{C}_p(M)$ is a linear function $L : \mathcal{C}_p(M) \rightarrow \mathbb{R}$ such that $L(fg) = L(f)e(g) + e(f)L(g)$. Each such L annihilates constant functions because

$$L(1) = L(1 \cdot 1) = L(1)e(1) + e(1)L(1) = 2L(1)$$

forces $L(1) = 0$. The set of derivations of $\mathcal{C}_p(M)$ forms a real vector space that is denoted by $T_p(M)$ and is called the **tangent space** of M at p . If a local coordinate system at p is given by means of a chart (M_α, α) with $\alpha = (x_1, \dots, x_m)$, then m examples of members of $T_p(M)$ are given by the derivations $\left[\frac{\partial}{\partial x_j}\right]_p$ defined by

$$\left[\frac{\partial f}{\partial x_j}\right]_p = \frac{\partial(f \circ \alpha^{-1})}{\partial u_j} \Big|_{(u_1, \dots, u_m) = (x_1(p), \dots, x_m(p))} \quad \text{for } j = 1, \dots, m.$$

These derivations satisfy

$$\left[\frac{\partial x_i}{\partial x_j}\right]_p = \frac{\partial u_i}{\partial u_j} \Big|_{(u_1, \dots, u_m) = (x_1(p), \dots, x_m(p))} = \delta_{ij},$$

where δ_{ij} is the Kronecker delta. It follows that the m derivations $\left[\frac{\partial}{\partial x_j}\right]_p$ are linearly independent. Actually these m derivations form a vector-space basis of $T_p(M)$, as is shown in the following proposition. Spanning follows from an expansion formula established by the proposition for all members of $T_p(M)$.

Proposition 1.5. If M is a smooth manifold and if a compatible chart (M_α, α) about a point p in M is given by $\alpha = (x_1, \dots, x_m)$, then each member L of $T_p(M)$ is given on $\mathcal{C}_p(M)$ by

$$L = \sum_{j=1}^m L(x_j) \left[\frac{\partial}{\partial x_j}\right]_p.$$

Consequently the m derivations $\left[\frac{\partial}{\partial x_j}\right]_p$ form a vector-space basis of $T_p(M)$.

PROOF. Let L be a derivation of $\mathcal{C}_p(M)$, and let (f, U) represent a member of $\mathcal{C}_p(M)$, U being an open neighborhood of p in M . Without loss of generality, we may assume that $U \subseteq M_\alpha$ and that $\alpha(U)$ is an open ball in \mathbb{R}^m . Put $u_0 = (u_{0,1}, \dots, u_{0,m}) = \alpha(p)$, let q be a variable point in U , and define $u = (u_1, \dots, u_n) = \alpha(q)$. Taylor's Theorem³ applied to $f \circ \alpha^{-1}$ on $\alpha(U)$ gives

$$\begin{aligned} f \circ \alpha^{-1}(u) &= f \circ \alpha^{-1}(u_0) + \sum_{j=1}^m (u_j - u_{0,j}) \frac{\partial(f \circ \alpha^{-1})}{\partial u_j}(u_0) \\ &\quad + \sum_{i,j} (u_i - u_{0,i})(u_j - u_{0,j}) R_{ij}(u) \end{aligned}$$

with each R_{ij} in $C^\infty(\alpha(U))$. Referring this formula to M , we obtain

$$\begin{aligned} f(q) &= f(p) + \sum_{j=1}^m (x_j(q) - x_j(p)) \left[\frac{\partial f}{\partial x_j} \right]_p \\ &\quad + \sum_{i,j} (x_i(q) - x_i(p))(x_j(q) - x_j(p)) r_{ij}(q) \end{aligned}$$

on U , where $r_{ij} = R_{ij} \circ \alpha$ on U . Because L annihilates constant functions and has the derivation property and satisfies $e(x_j) = x_j(p)$ for $1 \leq j \leq m$, application of L yields

$$\begin{aligned} L(f) &= \sum_{j=1}^m L(x_j) \left[\frac{\partial f}{\partial x_j} \right]_p + \sum_{i,j} (L(x_i)(e(x_j) - x_j(p))e(r_{ij}) \\ &\quad + (e(x_i) - x_i(p))L(x_j)e(r_{ij}) + (e(x_i) - x_i(p))(e(x_j) - x_j(p))L(r_{ij})) \\ &= \sum_{j=1}^m L(x_j) \left[\frac{\partial f}{\partial x_j} \right]_p, \end{aligned}$$

as asserted. □

Still with M as a smooth manifold, form the set $T(M)$ of all pairs (p, L) such that p is in M and L is in $T_p(M)$. The set $T(M)$ can be topologized and given a smooth manifold structure in a natural way, and then the pair consisting of $T(M)$ together with the projection-to-the-first-component function is called the **tangent bundle** of M . For current purposes we do not need to know what the topology and manifold structure on $T(M)$ are, and we shall ignore them.⁴ A **vector field** X on M is a function from M into $T(M)$ that selects a member of $T_p(M)$ for each p in M ; in other words, a vector field is any right inverse to the projection-to-the-first-component function under composition.⁵ An immediate consequence of Proposition 1.5 is the following expansion of any vector field.

³In the form of Theorem 3.11 of *Basic Real Analysis*.

⁴The construction is a chore to carry out. Not needing it, we skip the details. The reader who would like to see a careful construction of the tangent bundle may wish to look at Proposition 8.14 and the remarks after it in Section VIII.4 of *Advanced Real Analysis*.

⁵In the terminology of tangent bundles, a vector field is any section of the tangent bundle.

Corollary 1.6. Let M be a smooth manifold of dimension m . If (M_α, α) is any compatible chart for M , say with $\alpha = (x_1, \dots, x_m)$, and if X is a vector field on M_α , then

$$Xf(p) = \sum_{i=1}^m \frac{\partial f}{\partial x_i}(p) (Xx_i)(p)$$

for all p in M_α and f in $C^\infty(M_\alpha)$.

For vector fields we satisfy ourselves with the following definition of smoothness: the vector field X on M is defined to be **smooth** on M if Xx_i is smooth for each coordinate function x_i of each compatible chart⁶ on M . From Corollary 1.6 it is apparent that the set of smooth vector fields on M is closed under addition and scalar multiplication and is also closed under multiplication by members of $C^\infty(M)$. It is therefore a $C^\infty(M)$ module.

Next we discuss derivatives. Let $F : M \rightarrow N$ be a smooth function from a smooth manifold M of dimension m into a smooth manifold N of dimension n . For any p in M , the function F carries any tangent vector L in $T_p(M)$ into a tangent vector $(DF)_p(L)$ in $T_{F(p)}(N)$ by the formula $(DF)_p(L)(g) = L(g \circ F)$ for g in the space $\mathcal{C}_{F(p)}(N)$ on which a tangent vector in $T_{F(p)}(N)$ operates. The result is a linear function $(DF)_p : T_p(M) \rightarrow T_{F(p)}(N)$ called the **derivative** of F at p . The name “derivative” and the notation $(DF)_p$ are a change from *Advanced Real Analysis*.⁷

Proposition 1.7. Let M and N be smooth manifolds of respective dimensions m and n , and let $F : M \rightarrow N$ be a smooth function. Fix p in M , let $\alpha = (x_1, \dots, x_m)$ be a compatible chart in M about p , and let $\beta = (y_1, \dots, y_n)$ be a compatible chart in N about $F(p)$. Define $F_i = y_i \circ F$ for $1 \leq i \leq n$. Relative to the bases $\left[\frac{\partial}{\partial x_j} \right]_p$ of $T_p(M)$ and $\left[\frac{\partial}{\partial y_i} \right]_{F(p)}$ of $T_{F(p)}(N)$, the matrix of the linear function $(DF)_p : T_p(M) \rightarrow T_{F(p)}(N)$ has size n by m , and its (i, j) th entry is $\left[\frac{\partial F_i}{\partial u_j} \right]_{(u_1, \dots, u_m) = (x_1(p), \dots, x_m(p))}$.

⁶Section VIII.4 of *Advanced Real Analysis* shows that the smooth vector fields are exactly the sections of the tangent bundle that are smooth. We never need to use this fact.

⁷In *Advanced Real Analysis* the name was “differential,” and the notation was $(dF)_p$. The need for a change may be seen in the case that F is a real-valued function, i.e., $N = \mathbb{R}$. In this case, $(DF)_p$ is a member of $\text{Hom}_{\mathbb{R}}(T_p(M), T_{F(p)}(\mathbb{R}))$, and it is being called the “derivative.” The word “differential” will acquire a different standard meaning later in this section in such a way that $(dF)_p$ is a member of $\text{Hom}_{\mathbb{R}}(T_p(M), \mathbb{R})$. The two range spaces, $T_{F(p)}(\mathbb{R})$ and \mathbb{R} , are isomorphic, but confusion easily arises when the isomorphism is not made explicit. Some authors use the term “push-forward” in referring to what is being called the the derivative here.

REMARKS. In other words the matrix in question is the usual derivative matrix or Jacobian matrix of the set of coordinate functions of the function obtained by referring F to Euclidean space. Hence the derivative at a point is the object for smooth manifolds that generalizes the multivariable derivative at a point for Euclidean space. Accordingly, let us make the definition

$$\left[\frac{\partial F_i}{\partial x_j} \right]_p = \left[\frac{\partial F_i}{\partial u_j} \Big|_{(u_1, \dots, u_n) = (x_1(p), \dots, x_n(p))} \right].$$

PROOF. Application of the definitions gives

$$\begin{aligned} (DF)_p \left(\left[\frac{\partial}{\partial x_j} \right]_p \right) (y_i) &= \left[\frac{\partial}{\partial x_j} \right]_p (y_i \circ F) \\ &= \frac{\partial (y_i \circ F \circ \alpha^{-1})}{\partial u_j} (x_1(p), \dots, x_n(p)) \\ &= \frac{\partial F_i}{\partial u_j} \Big|_{(u_1, \dots, u_m) = (x_1(p), \dots, x_m(p))}. \end{aligned}$$

The formula in Proposition 1.5 allows us to express any member of $T_{F(p)}(N)$ in terms of its values on the local coordinate functions y_i , and therefore

$$(DF)_p \left(\left[\frac{\partial}{\partial x_j} \right]_p \right) = \sum_{i=1}^n \frac{\partial F_i}{\partial u_j} \Big|_{(u_1, \dots, u_m) = (x_1(p), \dots, x_m(p))} \left[\frac{\partial}{\partial y_i} \right]_p \quad \text{for } 1 \leq j \leq m.$$

Thus the matrix is as asserted. \square

Proposition 1.8 (chain rule). Let M , N , and R be smooth manifolds, and let $F : M \rightarrow N$ and $G : N \rightarrow R$ be smooth functions. If p is in M , then

$$(D(G \circ F))_p = (DG)_{F(p)} \circ (DF)_p.$$

PROOF. If L is in $T_p(M)$ and h is in $\mathcal{C}_{G(F(p))}(R)$, then the definitions give

$$\begin{aligned} (D(G \circ F))_p(L)(h) &= L(h \circ G \circ F) \\ &= (DF)_p(L)(h \circ G) = (DG)_{F(p)}(DF)_p(L)(h), \end{aligned}$$

as asserted. \square

Finally we discuss differential 1 forms and differentials. Still with M as smooth manifold, for each $p \in M$, let $T_p^*(M)$ be the dual vector space of $T_p(M)$, i.e., the real vector space of all linear functionals on $T_p(M)$. Members of $T_p^*(M)$ are called **cotangent vectors** at p . Consider the set $T^*(M)$ of all pairs (p, ℓ) such that

p is in M and ℓ is in $T_p^*(M)$. As with $T(M)$, the set $T^*(M)$ can be topologized and given a smooth manifold structure in a natural way,⁸ and then the pair consisting of $T^*(M)$ together with the projection-to-the-first-component function is called the **cotangent bundle** of M . Once again we do not need to know what the topology and manifold structure are, and we shall ignore them. A **differential 1 form** on M is a function from M into $T^*(M)$ that selects a member of $T_p^*(M)$ for each p in M ; in other words, a differential 1 form is any right inverse to the projection-to-the-first-component function under composition.⁹

To get some first examples of differential 1 forms, fix $p \in M$ and let f be any member of $C^\infty(M)$. Then f carries any germ L in $\mathcal{C}_p(M)$ into the germ $(Df)_p(L)$ in $\mathcal{C}_{f(p)}(\mathbb{R})$ by the formula

$$(Df)_p(L)(g) = L(g \circ f) \quad \text{for all } g \in \mathcal{C}_{f(p)}(\mathbb{R}).$$

Let us take g to be the identity function $g_0(t) = t$ on \mathbb{R} , no matter what f is. For this choice of g , the formula reduces to $(Df)_p(L)(g_0) = Lf$ for all L in $T_p(M)$. If we suppress g_0 in this formula and write $(df)_p(L)$ for the left side, the formula becomes

$$(df)_p(L) = Lf \quad \text{for all } L \in T_p(M),$$

and we obtain a linear functional on $T_p(M)$. This linear functional $(df)_p$ is called the **differential** of f at p . As p varies, the result is a differential 1 form df on M .

Let us look more closely at this construction for a moment. For f in $C^\infty(M)$, we passed from $(Df)_p$, which is a member of $\text{Hom}_{\mathbb{R}}(T_p(M), T_{f(p)}(\mathbb{R}))$, to $(df)_p$, which is a member of $\text{Hom}_{\mathbb{R}}(T_p(M), \mathbb{R})$. We did so, in effect, by following the member of $\text{Hom}_{\mathbb{R}}(T_p(M), T_{f(p)}(\mathbb{R}))$ by a particular isomorphism of $T_{f(p)}(\mathbb{R})$ with \mathbb{R} .

We just saw that the differentials at p of members of $C^\infty(M)$ are examples of members of $T_p^*(M)$. The proposition below identifies all members of $T_p^*(M)$.

Proposition 1.9. Let M be a smooth manifold of dimension m , fix p in M , and let (M_α, α) be a compatible chart about p with $\alpha = (x_1, \dots, x_m)$. Then the differentials $(dx_1)_p, \dots, (dx_m)_p$ form the dual basis in $T_p^*(M)$ to the basis $[\frac{\partial}{\partial x_1}]_p, \dots, [\frac{\partial}{\partial x_m}]_p$ of $T_p(M)$. Also if $f : M \rightarrow \mathbb{R}$ is any smooth function on M , then

$$(df)_p = \sum_{i=1}^m \left(\frac{\partial f}{\partial x_i} \right)_p (dx_i)_p \quad \text{for all } p \in M_\alpha.$$

⁸The details appear in Section VIII.4 of *Advanced Real Analysis*.

⁹In the terminology of cotangent bundles, a differential 1 form is any section of the cotangent bundle.

PROOF. Taking $f = x_i$ and $L = \left[\frac{\partial}{\partial x_j}\right]_p$ in the formula $(df)_p(L) = Lf$, we obtain $(dx_i)_p\left(\left[\frac{\partial}{\partial x_j}\right]_p\right) = \left(\frac{\partial x_i}{\partial x_j}\right)_p = \delta_{ij}$. Hence $(dx_1)_p, \dots, (dx_m)_p$ indeed forms the dual basis to the basis $\left[\frac{\partial}{\partial x_1}\right]_p, \dots, \left[\frac{\partial}{\partial x_m}\right]_p$ of $T_p(M)$.

To prove the displayed equality in the proposition, it is enough to prove that equality is maintained when both sides are applied to each basis vector $\left[\frac{\partial}{\partial x_j}\right]_p$ of $T_p(M)$. On the left side we have

$$(df)_p \left[\frac{\partial}{\partial x_j}\right]_p = \left(\frac{\partial f}{\partial x_j}\right)_p,$$

and on the right side we have

$$\sum_{i=1}^m \left(\frac{\partial f}{\partial x_i}\right)_p (dx_i)_p \left(\left[\frac{\partial}{\partial x_j}\right]_p\right) = \sum_{i=1}^m \left(\frac{\partial f}{\partial x_i}\right)_p \delta_{ij} = \left(\frac{\partial f}{\partial x_j}\right)_p.$$

These are equal, and the proof is complete. \square

According to Proposition 1.9, any differential 1 form $\omega(p)$ on the smooth manifold M expands as

$$\omega(p) = \sum_{i=1}^m a_i(p)(dx_i)_p$$

in each compatible chart (M_α, α) with $\alpha = (x_1, \dots, x_m)$. We say that the differential 1 form ω is **smooth** if all coefficient functions a_i for all compatible charts are smooth functions.¹⁰ Part of the content of Proposition 1.9 is that every differential 1 form df with $f \in C^\infty(M)$ is smooth.¹¹

2. Properties of Exterior Algebras

If one looks carefully at the classical integration theorems stated in the Introduction, one sees that minus signs play an important role in the theory. Why is it that the right side of the Fundamental Theorem of Calculus reads $F(b) - F(a)$ and not $F(a) - F(b)$? Why is it in Green's Theorem that the region is to lie on the left of the boundary curve as the curve is traced out? And what are these "important questions of orientations" that need to be sorted out in the Divergence Theorem?

¹⁰Section VIII.4 of *Advanced Real Analysis* shows that the smooth sections of the cotangent bundle are exactly the differential 1 forms that are smooth.

¹¹It turns out that not every smooth differential form on a smooth manifold M need be given as df for some smooth f . See Problem 8 at the end of the chapter for an example.

It turns out that all such questions can be resolved by augmenting the heuristic interpretation of dx that one is often taught. Instead of its being an element of numerical length, it is to be a one dimensional vector element of length, with both a magnitude (the usual notion of length) and direction (its sign). In two dimensions similarly, $dx dy$ is to be thought of as incorporating information about the angle between the vector dx and the vector dy , thus akin to the area of the parallelogram spanned by the two vectors in \mathbb{R}^2 , namely the product of their magnitudes by the sine of the angle between them. As soon as one makes this adjustment, one is led to think of dx and dy not as commuting objects but as anticommuting objects.¹² This section takes up the algebraic preliminaries for dealing with a multiplication that is anticommutative but is still associative.

Chapter VI of *Basic Algebra* defines the **tensor algebra** $T(V)$ of a vector space V over \mathbb{R} to be the direct sum over $n \geq 0$ of the n -fold tensor product $T^n(V)$ of V with itself, the 0-fold tensor product being understood to consist just of the scalars \mathbb{R} . The operation of multiplication is written as \otimes . The space $T^n(V)$ is a vector space with a universal mapping property relative to n -linear functions on V . The full tensor algebra $T(V)$ is an associative algebra with a universal mapping property relative to any linear mapping of V into an associative algebra A with identity: the linear map extends uniquely to an algebra homomorphism of $T(V)$ into A carrying 1 into 1. We take all this as known.

Chapter VI of *Basic Algebra* speaks also of multilinear forms that are **alternating** in the sense that their value is 0 whenever two of the arguments are equal. Alternating forms are skew symmetric in the sense that if two of the arguments are interchanged, then the value of the form is multiplied by -1 . Alternating forms will play an important role in what follows.

We shall introduce “exterior algebras” over the field \mathbb{R} . If E is a vector space over \mathbb{R} , the exterior algebra $\wedge(E)$ is to be an associative algebra, and the elements of $\wedge(E)$ are to include the members of \mathbb{R} and all the members of E itself. The algebra $\wedge(E)$ will be defined as a quotient of the tensor algebra $T(E)$, with all those members of $T(E)$ mapped to 0 that are to represent 0 in the quotient. Its product operation is written as \wedge . To force skew symmetry (i.e., $y \wedge x = -x \wedge y$) for multiplication in the quotient of the embedded members of E , we require that $v \otimes v$ maps to 0 in $\wedge(E)$ whenever v is in $T^1(E)$. To arrange that the quotient algebra is as large as possible, we factor out nothing more than is necessary from $T(E)$. Thus we define the **exterior algebra**¹³ of E by the formula

$$\wedge(E) = T(E)/I',$$

where $I' = \left(\begin{array}{l} \text{two-sided ideal in } T(E) \text{ generated} \\ \text{by all } v \otimes v \text{ with } v \text{ in } T^1(E) \end{array} \right)$.

¹²With this thought in mind, we shall be writing $dx \wedge dy$ instead of $dx dy$. The notation with the symbol \wedge was already used in the Introduction.

¹³Sometimes known as **Grassmann algebras** for historical reasons.

Then $\bigwedge(E)$ is an associative algebra with identity.

It is clear that I' is homogeneous in the sense that $I' = \bigoplus_{n=0}^{\infty} (I' \cap T^n(E))$. Consequently we can write

$$\bigwedge(E) = \bigoplus_{n=0}^{\infty} T^n(E)/(I' \cap T^n(E)).$$

We write $\bigwedge^n(E)$ for the n^{th} summand on the right side, so that

$$\bigwedge(E) = \bigoplus_{n=0}^{\infty} \bigwedge^n(E).$$

Since $I' \cap T^0(E) = 0$, $\bigwedge^0(E)$ consists of just the scalar multiples of the identity. Since $I' \cap T^1(E) = 0$, the map of E into first-order elements $\bigwedge^1(E)$ is one-one onto and is just a copy of E . The product operation in $\bigwedge(E)$ is called the **exterior product** or **wedge product** and is denoted by \wedge rather than \otimes . Thus the image in $\bigwedge^n(E)$ of the element $v_1 \otimes \cdots \otimes v_n$ of $T^n(E)$ can be written as $v_1 \wedge \cdots \wedge v_n$. If a is in $\bigwedge^m(E)$ and b is in $\bigwedge^n(E)$, then $a \wedge b$ is in $\bigwedge^{m+n}(E)$. Moreover, $\bigwedge^n(E)$ is generated by elements $v_1 \wedge \cdots \wedge v_n$ with all v_j in $\bigwedge^1(E) \cong E$, since $T^n(E)$ is generated by corresponding elements $v_1 \otimes \cdots \otimes v_n$. The defining relations for $\bigwedge(E)$ force the condition of **skew symmetry**, $v_i \wedge v_j = -v_j \wedge v_i$ for v_i and v_j in $\bigwedge^1(E)$. Writing members of $\bigwedge(E)$ as linear combinations of monomials and making repeated use of the skew symmetry of multiplication for members of $\bigwedge^1(E)$, we obtain the following result.

Proposition 1.10. If E is a vector space over \mathbb{R} , then

$$a \wedge b = (-1)^{mn} b \wedge a \quad \text{for } a \in \bigwedge^m(E) \text{ and } b \in \bigwedge^n(E).$$

PROOF. By linearity in each variable in wedge product, it is enough to prove the conclusion when a and b are monomials, say $a = a_1 \wedge \cdots \wedge a_m$ and $b = b_1 \wedge \cdots \wedge b_n$. We induct on m , the base case for the induction being $m = 1$. With $a \in \bigwedge^1(E)$, the skew symmetry allows us to start from $a \wedge b$ and commute a to the right one step at a time, until a is on the right side of b . Then we are introducing n sign changes, and the base case is established. In the general case we write $a = a' \wedge a_m$ with $a' \in \bigwedge^{m-1}(E)$ and $a_m \in \bigwedge^1(E)$. Applying the base case and then the induction hypothesis, we obtain

$$a \wedge b = a' \wedge a_m \wedge b = (-1)^n a' \wedge b \wedge a_m = (-1)^n (-1)^{(m-1)n} b \wedge a' \wedge a_m,$$

and $a \wedge b = (-1)^{mn} b \wedge a$ as required. \square

Proposition 1.11. Let E be a real vector space.

(a) Let ι be the n -multilinear function $\iota(v_1, \dots, v_n) = v_1 \wedge \dots \wedge v_n$ of $E \times \dots \times E$ into $\bigwedge^n(E)$. Then $(\bigwedge^n(E), \iota)$ has the following **universal mapping property**: whenever l is any alternating n -multilinear map of $E \times \dots \times E$ into a vector space U , then there exists a unique linear map $L : \bigwedge^n(E) \rightarrow U$ such that the diagram

$$\begin{array}{ccc} E \times \dots \times E & \xrightarrow{l} & U \\ \downarrow \iota & \nearrow L & \\ \bigwedge^n(E) & & \end{array}$$

commutes.

(b) Let ι be the function that embeds E as $\bigwedge^1(E) \subseteq \bigwedge(E)$. Then $(\bigwedge(E), \iota)$ has the following **universal mapping property**: whenever l is any linear map of E into an associative algebra A with identity such that $l(v)^2 = 0$ for all $v \in E$, then there exists a unique algebra homomorphism $L : \bigwedge(E) \rightarrow A$ with $L(1) = 1$ such that the diagram

$$\begin{array}{ccc} E & \xrightarrow{l} & A \\ \downarrow \iota & \nearrow L & \\ \bigwedge(E) & & \end{array}$$

commutes.

PROOF. In both cases uniqueness is trivial. For existence we use the universal mapping properties of $T^n(E)$ and $T(E)$ to produce \tilde{L} on $T^n(E)$ or $T(E)$. If we can show that \tilde{L} annihilates the appropriate subspace so as to descend to $\bigwedge^n(E)$ or $\bigwedge(E)$, then the resulting map can be taken as L , and we are done. For (a), we have $\tilde{L} : T^n(E) \rightarrow U$, and we are to show that $\tilde{L}(T^n(E) \cap I') = 0$, where I' is generated by all $v \otimes v$ with v in $T^1(E)$. A member of $T^n(E) \cap I'$ is thus of the form $\sum a_i \otimes (v_i \otimes v_i) \otimes b_i$ with each term in $T^n(E)$. Each term here is a sum of pure tensors

$$x_1 \otimes \dots \otimes x_r \otimes v_i \otimes v_i \otimes y_1 \otimes \dots \otimes y_s \tag{*}$$

with $r + 2 + s = n$. Since l by assumption takes the value 0 on

$$x_1 \times \dots \times x_r \times v_i \times v_i \times y_1 \times \dots \times y_s,$$

\tilde{L} vanishes on (*), and it follows that $\tilde{L}(T^n(E) \cap I') = 0$.

For (b) we are to show that $\tilde{L} : T(E) \rightarrow A$ vanishes on I' . Since $\ker \tilde{L}$ is an ideal, it is enough to check that \tilde{L} vanishes on the generators of I' . But $\tilde{L}(v \otimes v) = l(v)l(v)$, and the right side is 0 by hypothesis. Thus $L(I') = 0$. \square

Corollary 1.12. If E and F are real vector spaces, then the vector space $\text{Hom}_{\mathbb{R}}(\bigwedge^n(E), F)$ of linear mappings from $\bigwedge^n(E)$ into F is canonically isomorphic (via restriction to pure tensors) to the vector space of all F -valued alternating n -multilinear functions on $E \times \cdots \times E$.

PROOF. Restriction is linear and one-one. It is onto by Proposition 1.10a. \square

Corollary 1.13. If E is a real vector space, then the vector space dual $(\bigwedge^n(E))'$ of $\bigwedge^n(E)$ is canonically isomorphic (via restriction to pure tensors) to the real vector space of alternating n -multilinear forms on $E \times \cdots \times E$.

PROOF. This is the special case $F = \mathbb{R}$ of Corollary 1.12. \square

Up until now, it has been immaterial whether E is finite dimensional or infinite dimensional. That circumstance now changes.

Proposition 1.14. Let E be a real vector space of finite dimension N , and let n be an integer ≥ 0 . Then

- (a) $\dim \bigwedge^n(E) = \binom{N}{n}$ for $0 \leq n \leq N$ and $= 0$ for $n > N$,
- (b) for each integer n with $1 \leq n \leq N$, there is a canonical linear mapping $L : \bigwedge^n(E') \rightarrow \bigwedge^n(E)'$ such that $(f_1 \wedge \cdots \wedge f_n)(w_1 \wedge \cdots \wedge w_n) = \det\{f_i(w_j)\}_{i,j=1}^n$ for all $f_i \in E'$ and $w_j \in E$,
- (c) whenever u_1, \dots, u_N is a basis of E , then the monomials $u_{i_1} \wedge \cdots \wedge u_{i_n}$ with $1 \leq i_1 < \cdots < i_n \leq N$ form a basis of $\bigwedge^n(E)$,
- (d) the linear mapping $L : \bigwedge^n(E') \rightarrow \bigwedge^n(E)'$ of (c) is an isomorphism onto,
- (e) if u_1, \dots, u_N is a basis of E and u'_1, \dots, u'_N is the dual basis of E' , then the dual basis for $\bigwedge^n(E')$ to the basis of monomials $u_{i_1} \wedge \cdots \wedge u_{i_n}$ with $1 \leq i_1 < \cdots < i_n \leq N$ as in (b) is the basis of monomials $u'_{i_1} \wedge \cdots \wedge u'_{i_n}$ with $1 \leq i_1 < \cdots < i_n \leq N$.

REMARK. A version of some parts of this proposition remains valid even if E is infinite dimensional, but we shall not pursue the details.

PROOF. Let u_1, \dots, u_N be a basis of E . For $n = 0$, $\bigwedge^0(E)$ consists of the scalar multiples of the identity, and $\dim \bigwedge^0(E) = 1$. We may assume therefore that $n > 0$. The monomials of degree n in the u_j 's span $T^n(E)$, and the same thing is therefore true of the quotient $\bigwedge^n(E)$. Any such monomial in $\bigwedge^n(E)$ with two equal factors is 0 by the alternating condition and can be disregarded. For the remaining monomials we can permute the factors, using the identity $b \wedge a = -a \wedge b$ valid for members of $\bigwedge^1(E)$, to arrange that the indices on the factors of the monomial are in increasing order. As a result we see the monomials of degree n in u_1, \dots, u_N whose indices are in strictly increasing order span $\bigwedge^n(E)$. If $n > N$, there are no such monomials, and $\bigwedge^n(E) = 0$. If $0 < n \leq N$,

the number of such monomials in $\binom{N}{n}$. Thus $\dim \bigwedge^n(E) \leq \binom{N}{n}$. This gives part of (a) and allows us to assume that $1 \leq n \leq N$ from now on. Also it proves that the monomials in (c) form a spanning set for $\bigwedge^n(E)$.

For (b), fix f_1, \dots, f_n in E' , let w_1, \dots, w_n be in E , and define

$$l_{f_1, \dots, f_n}(w_1, \dots, w_n) = \det\{f_i(w_j)\}_{i,j=1}^n.$$

Then l_{f_1, \dots, f_n} is an alternating n -multilinear form on $E \times \dots \times E$ and extends by Proposition 1.10a to a linear functional $L_{f_1, \dots, f_n} : \bigwedge^n(E) \rightarrow \mathbb{R}$. Next we let f_1, \dots, f_n vary, and the result is that $l(f_1, \dots, f_n) = L_{f_1, \dots, f_n}$ defines an alternating n -multilinear map of $E' \times \dots \times E'$ into $\bigwedge^n(E)'$. Its linear extension L given by Proposition 1.11a maps $\bigwedge^n(E')$ into $\bigwedge^n(E)'$. This proves (b).

Before proceeding with the remaining parts, let us prove the displayed formula (*) below. Let $\{u_1, \dots, u_N\}$ be a basis of E , and let $\{u'_1, \dots, u'_N\}$ be the dual basis of E' . Suppose that two strictly increasing sets of n -element indices $I = (i_s)_{s=1}^n$ and $J = (j_t)_{t=1}^n$ between 1 and N are given. The claim is that

$$\det\{u'_{i_s}(u_{j_t})\}_{s,t=1}^n = \begin{cases} 1 & \text{if } i_k = j_k \text{ for } 1 \leq k \leq n, \\ 0 & \text{otherwise.} \end{cases} \quad (*)$$

To see this, assume that $i_k \neq j_k$ for some k , and let l be the least such k . If $i_l < j_l$, then $i_l \neq j_t$ for all t and it follows that $u'_{i_l}(u_{j_t}) = 0$ for $1 \leq t \leq n$. The matrix $\{u'_{i_s}(u_{j_t})\}_{s,t=1}^n$ has a row of zeros, and its determinant is 0. On the other hand, if $i_l > j_l$, then the matrix $\{u'_{i_s}(u_{j_t})\}_{s,t=1}^n$ has a column of zeros, and its determinant is 0. The only other possibility is that $i_k = j_k$ for $1 \leq k \leq n$. Then the matrix $\{u'_{i_s}(u_{j_t})\}_{s,t=1}^n$ is the identity, and its determinant is 1. This proves (*).

With the sets of indices $I = (i_s)_{s=1}^n$ and $J = (j_t)_{t=1}^n$ as above, define

$$\begin{aligned} u'_I &= u'_{i_1} \wedge \dots \wedge u'_{i_s} \wedge \dots \wedge u'_{i_n} && \text{as a member of } \bigwedge^n(E'), \\ u_J &= u_{j_1} \wedge \dots \wedge u_{j_s} \wedge \dots \wedge u_{j_n} && \text{as a member of } \bigwedge^n(E). \end{aligned}$$

What (*) says, in terms of the mapping L of conclusion (a), is that $L(u'_I)(u_J) = \delta_{IJ}$. It follows from (*) that the set of all u'_I as I varies through n -element sets of indices is linearly independent in $\bigwedge^n(E')$ and that the set of all u_J as J varies through n -element set of indices is linearly independent in $\bigwedge^n(E)$. This conclusion for $\bigwedge^n(E)$ completes the proof of (a) and (c), spanning having been proved earlier.

In view of (c), the linear mapping in (d) carries a basis to a basis and is therefore an isomorphism. This proves (d). Conclusion (e) is then immediate from (*). \square

In our applications of this algebraic theory to manifolds, we shall be interested in the case that E is a tangent space $T_p(M)$ and its dual is the cotangent space

$T_p^*(M)$. Let ξ and η be typical vector fields, so that ξ_p and η_p are members of the tangent space $T_p(M)$, and let ω and σ be typical differential 1 forms, so that ω_p and σ_p are members of the cotangent space $T_p^*(M)$. Then expressions like $\xi_p \mapsto \omega_p(\xi_p)$ and $\eta_p \mapsto \sigma_p(\eta_p)$ are meaningful, and we can multiply them, obtaining a bilinear form $(\xi_p, \eta_p) \mapsto \omega_p(\xi_p)\sigma_p(\eta_p)$. How is the bilinear form $(\xi_p, \eta_p) \mapsto \omega_p(\xi_p)\sigma_p(\eta_p)$ related to the bilinear form $(\xi_p, \eta_p) \mapsto (\omega_p \wedge \sigma_p)(\xi_p, \eta_p)$? The answer is given by the corollary of the following proposition, which strips away the unnecessary information about manifolds. The corollary will be proved by applying Proposition 1.15 below with V equal to the dual E' of E .

Let V be a finite dimensional real vector space. On $V \times \cdots \times V$, let us define an n -multilinear function with values in $T^n(V)$ by

$$(v_1, \dots, v_n) \mapsto \frac{1}{n!} \sum_{\tau \in \mathfrak{S}_n} (\text{sgn } \tau) v_{\tau(1)} \otimes \cdots \otimes v_{\tau(n)},$$

where \mathfrak{S}_n is the symmetric group on n letters, and let $\mathcal{A} : T^n(E) \rightarrow T^n(E)$ be its linear extension. We shall call \mathcal{A} the **antisymmetrizer** operator. The image of \mathcal{A} in $T^n(V)$ will be denoted by $\tilde{\wedge}^n(V)$, and the members of this subspace will be called **antisymmetrized** tensors.

Proposition 1.15. If V is a finite dimensional real vector space, then the antisymmetrizer operator \mathcal{A} satisfies $\mathcal{A}^2 = \mathcal{A}$. The kernel of \mathcal{A} on $T^n(E)$ is exactly $T^n(E) \cap I'$, where I' is the two-sided ideal of $T(V)$ generated by all elements $v \otimes v$ with $v \in T^1(V)$. Therefore $T^n(V)$ is the vector-space direct sum

$$T^n(V) = \tilde{\wedge}^n(V) \oplus (T^n(V) \cap I').$$

REMARK. In view of this proposition, the quotient map $T^n(V) \rightarrow \wedge^n(V)$ carries $\tilde{\wedge}^n(V)$ one-one onto $\wedge^n(V)$. Thus $\tilde{\wedge}^n(V)$ can be viewed as a copy of $\wedge^n(V)$ embedded as a direct summand of $T^n(V)$.

PROOF. We have

$$\begin{aligned} \mathcal{A}^2(v_1 \otimes \cdots \otimes v_n) &= \frac{1}{(n!)^2} \sum_{\sigma, \tau \in \mathfrak{S}_n} (\text{sgn } \sigma \tau) v_{\sigma\tau(1)} \otimes \cdots \otimes v_{\sigma\tau(n)} \\ &= \frac{1}{(n!)^2} \sum_{\sigma \in \mathfrak{S}_n} \sum_{\substack{\rho \in \mathfrak{S}_n, \\ (\rho = \sigma\tau)}} (\text{sgn } \rho) v_{\rho(1)} \otimes \cdots \otimes v_{\rho(n)} \\ &= \frac{1}{n!} \sum_{\rho \in \mathfrak{S}_n} \mathcal{A}(v_1 \otimes \cdots \otimes v_n) \\ &= \mathcal{A}(v_1 \otimes \cdots \otimes v_n). \end{aligned}$$

Hence $\mathcal{A}^2 = \mathcal{A}$. Consequently $T^n(E)$ is the direct sum of image \mathcal{A} and $\ker \mathcal{A}$, and we are left with identifying $\ker \mathcal{A}$ as $T^n(V) \cap I'$.

The subspace $T^n(V) \cap I'$ is spanned by elements

$$x_1 \otimes \cdots \otimes x_r \otimes v \otimes v \otimes y_1 \otimes \cdots \otimes y_s$$

with $r + 2 + s = n$, and the antisymmetrizer \mathcal{A} certainly vanishes on such elements. Hence $T^n(V) \cap I' \subseteq \ker \mathcal{A}$. Arguing by contradiction, suppose that the inclusion is strict, say with t in $\ker \mathcal{A}$ but t not in $T^n(V) \cap I'$. Let q be the quotient map $T^n(V) \rightarrow \bigwedge^n(V)$. The kernel of q is $T^n(V) \cap I'$, and thus $q(t) \neq 0$. From Proposition 1.14c the monomials $T(V)$ in members of a basis of V that have strictly increasing indices map onto a basis of $\bigwedge(V)$. The antisymmetrized version of each of these monomials has to map to a multiple of the initial monomial, and that multiple has to be nonzero because Proposition 1.14d says that the basis maps to a basis. Consequently q carries $\tilde{\bigwedge}^n(V) = \text{image } \mathcal{A}$ onto $\bigwedge^n(V)$. Thus we can choose $t' \in \tilde{\bigwedge}^n(V)$ with $q(t') = q(t)$. Then $t' - t$ is in $\ker q = T^n(V) \cap I' \subseteq \ker \mathcal{A}$. Since $\mathcal{A}(t) = 0$, we see that $\mathcal{A}(t') = 0$. Consequently t' is in $\ker \mathcal{A} \cap \text{image } \mathcal{A} = 0$, and we obtain $t' = 0$ and $q(t) = q(t') = 0$, contradiction. \square

Corollary 1.16. Let E be a finite dimensional real vector space, and let E' be its dual. If $\omega_1, \dots, \omega_n$ are members of E' and v_1, \dots, v_n are members of E , then

$$(\omega_1 \wedge \cdots \wedge \omega_n)(v_1, \dots, v_n) = \frac{1}{n!} \sum_{\tau \in \mathfrak{S}_n} (\text{sgn } \tau) \omega_1(v_{\tau(1)}) \cdots \omega_n(v_{\tau(n)}).$$

PROOF. Let $\sigma : T(E') \rightarrow \bigwedge(E')$ be the quotient mapping, let I' be the kernel, and let \mathcal{A} be the antisymmetrizer mapping of $T(E')$ into itself. If $\omega_1, \dots, \omega_n$ are in E' , then Proposition 1.15 shows that $\omega_1 \otimes \cdots \otimes \omega_n - \mathcal{A}(\omega_1 \otimes \cdots \otimes \omega_n)$ lies in I' . Since $\sigma(\omega_1 \otimes \cdots \otimes \omega_n) = \omega_1 \wedge \cdots \wedge \omega_n$ and since a similar equality holds for each of the terms in $\mathcal{A}(\omega_1 \otimes \cdots \otimes \omega_n)$, we obtain

$$\omega_1 \wedge \cdots \wedge \omega_n = \frac{1}{n!} \sum_{\tau \in \mathfrak{S}_n} (\text{sgn } \tau) \omega_{\tau(1)} \wedge \cdots \wedge \omega_{\tau(n)}.$$

Restricting to pure tensors, using the isomorphism of Corollary 1.13 with $E = V'$, and making a change a variables in the sum, we can write this conclusion as

$$\begin{aligned} (\omega_1 \wedge \cdots \wedge \omega_n)(v_1, \dots, v_n) &= \frac{1}{n!} \sum_{\tau \in \mathfrak{S}_n} (\text{sgn } \tau) \omega_{\tau(1)}(v_1) \cdots \omega_{\tau(n)}(v_n) \\ &= \frac{1}{n!} \sum_{\tau \in \mathfrak{S}_n} (\text{sgn } \tau) \omega_1(v_{\tau(1)}) \cdots \omega_n(v_{\tau(n)}), \end{aligned}$$

as required. \square

EXAMPLES.

(1) Just before Proposition 1.15, this question was raised: If ξ and η are vector fields and ω and σ are differential 1 forms, how is the bilinear form $(\xi_p, \eta_p) \mapsto \omega_p(\xi_p)\sigma_p(\eta_p)$ related to the bilinear form $(\xi_p, \eta_p) \mapsto (\omega_p \wedge \sigma_p)(\xi_p, \eta_p)$? Corollary 1.16 tells us that the formula is supposed to turn out to be

$$(\omega \wedge \sigma)(\xi, \eta) = \frac{1}{2}(\omega(\xi)\sigma(\eta)) - \frac{1}{2}(\omega(\eta)\sigma(\xi)). \quad (*)$$

On the level of full tensors before passage to the quotient, the formula with ξ and η suppressed is

$$\begin{aligned} \omega \otimes \sigma &= \mathcal{A}(\omega \otimes \sigma) + (\text{error}) \\ &= \frac{1}{2}(\omega \otimes \sigma - \sigma \otimes \omega) + \left(\frac{1}{2}(\omega \otimes \sigma + \sigma \otimes \omega)\right) \\ &= \frac{1}{2}(\omega \otimes \sigma - \sigma \otimes \omega) + \frac{1}{2}\left((\omega + \sigma) \otimes (\omega + \sigma) - (\omega \otimes \omega) - (\sigma \otimes \sigma)\right), \end{aligned}$$

and it is plain that the term called “error” above is in the ideal I' and hence maps to 0 under passage to the quotient. Thus passage to the quotient indeed yields (*).

(2) This example elaborates on the heuristic interpretation near the beginning of this section concerning expressions like dx . With $M = \mathbb{R}^2$ and p equal to $(0, 0)$, let us use Corollary 1.16 to evaluate

$$((dx)_{(0,0)} \wedge dy_{(0,0)})\left(a\left[\frac{\partial}{\partial x}\right]_{(0,0)} + b\left[\frac{\partial}{\partial y}\right]_{(0,0)}, c\left[\frac{\partial}{\partial x}\right]_{(0,0)} + d\left[\frac{\partial}{\partial y}\right]_{(0,0)}\right).$$

The corollary says that this expression is $\frac{1}{2}$ times the sum of two terms, separated by a minus sign, namely that it is

$$\begin{aligned} &= \frac{1}{2}\left(\left((dx)_{(0,0)}\left(a\left[\frac{\partial}{\partial x}\right]_{(0,0)} + b\left[\frac{\partial}{\partial y}\right]_{(0,0)}\right) \times (dy)_{(0,0)}\left(c\left[\frac{\partial}{\partial x}\right]_{(0,0)} + d\left[\frac{\partial}{\partial y}\right]_{(0,0)}\right)\right) \right. \\ &\quad \left. - \left((dx)_{(0,0)}\left(c\left[\frac{\partial}{\partial x}\right]_{(0,0)} + d\left[\frac{\partial}{\partial y}\right]_{(0,0)}\right) \times (dy)_{(0,0)}\left(a\left[\frac{\partial}{\partial x}\right]_{(0,0)} + b\left[\frac{\partial}{\partial y}\right]_{(0,0)}\right)\right)\right). \end{aligned}$$

Since $(dx)_{(0,0)}\left(\left[\frac{\partial}{\partial x}\right]_{(0,0)}\right) = (dy)_{(0,0)}\left(\left[\frac{\partial}{\partial y}\right]_{(0,0)}\right) = 1$ and $(dx)_{(0,0)}\left(\left[\frac{\partial}{\partial y}\right]_{(0,0)}\right) = (dy)_{(0,0)}\left(\left[\frac{\partial}{\partial x}\right]_{(0,0)}\right) = 0$, the expression reduces to

$$= \frac{1}{2}(ad - bc).$$

Except for a sign and the factor $\frac{1}{2}$, this is just the area of the rectangle in \mathbb{R}^2 spanned by the vectors $\begin{pmatrix} a \\ c \end{pmatrix}$ and $\begin{pmatrix} b \\ d \end{pmatrix}$. The factor $\frac{1}{2}$ means that the area is in fact the area of the triangle spanned by the two vectors rather than the rectangle. Thus the expression evaluates as the signed area of the spanned simplex.

(3) This example notes the corresponding calculation for Example 2 when done in \mathbb{R}^n for the point p equal to the origin 0. Here we are to evaluate

$$((dx_1)_0 \wedge \cdots \wedge (dx_n)_0) \left(\sum_{j=1}^n a_{1j} \left[\frac{\partial}{\partial x_j} \right]_0, \dots, \sum_{j=1}^n a_{nj} \left[\frac{\partial}{\partial x_j} \right]_0 \right),$$

and a similar computation shows that the corollary gives $\frac{1}{n!} \det\{a_{ij}\}_{i,j=1}^n$, as will be shown in the proof of Proposition 1.17 below. Again the geometric significance of the coefficient $1/n!$ is that $n!$ is the ratio of the volume of the fundamental parallelepiped to the volume of the fundamental simplex.

3. Differential Forms and Pullbacks

We introduce differential k forms by analogy with how we introduced differential 1 forms. Still with M as smooth manifold, for each $p \in M$, let $\bigwedge^k(T_p^*(M))$ be the k^{th} exterior power of the cotangent space $T_p^*(M)$ at p on M . In view of Proposition 1.14d, we can regard this space as the vector space of all alternating k -linear forms on the product of k copies of $T_p(M)$ with itself. Let $\bigwedge^k T^*(M)$ be the set of all pairs (p, η) such that p is in M and η is in $\bigwedge^k(T_p^*(M))$. As with $T(M)$ and $T^*(M)$, the set $\bigwedge^k T^*(M)$ can be topologized and given a smooth manifold structure in a natural way, and then the pair consisting of $\bigwedge^k T^*(M)$ together with the projection-to-the-first-component function is called the **exterior k bundle** of M . Once again we do not need to know what this manifold structure is, and we shall ignore it. For $k > 0$, a **differential k form** on M is a function from M into $\bigwedge^k T^*(M)$ that selects, for each p in M , a member of $\bigwedge^k T^*(M)$ with first component p ; in other words, a differential k form is any right inverse to the projection-to-the-first-component function under composition.¹⁴ The integer k is called the **degree** of the differential form. The wedge product of any k differential 1 forms is an example of a differential k form. In any compatible chart (M_α, α) with $\alpha = (x_1, \dots, x_m)$, it follows from Propositions 1.9 and 1.14 that any differential k form ω has a unique local expansion

$$\omega(p) = \sum_{1 \leq i_1 < \cdots < i_k \leq m} a_{i_1, \dots, i_k}(p) (dx_{i_1})_p \wedge \cdots \wedge (dx_{i_k})_p.$$

The form¹⁵ is said to be **smooth** on M if all the coefficient functions $p \mapsto a_{i_1, \dots, i_k}(p)$ in all such coordinate systems are smooth. As usual it is enough to

¹⁴In the terminology of vector bundles, a differential k form is any section of the exterior k bundle.

¹⁵The word **form** as a general matter refers to a scalar-valued function of several variables, always multilinear in this book but sometimes quadratic or homogenous of some other kind elsewhere in mathematics. In this book we shall follow the practice of freely using the word “form” as shorthand for “differential form” when there is no chance of ambiguity.

have smoothness relative to a family of compatible charts that covers M . We write $\Omega^k(M)$ for the real vector space of all smooth differential k forms on M . The space $\Omega^k(M)$ is a $C^\infty(M)$ module.

We extend the definition to the case $k = 0$ by saying that a **differential 0 form** on M is simply a real-valued function on M . The differential 0 form is smooth if it is smooth as a real-valued function. We write $\Omega^0(M)$ for the space $C^\infty(M)$ of all smooth differential 0 forms on M .

Referring to the unique local expansion that differential forms have, we see that the wedge product of a member of $\Omega^k(M)$ and a member of $\Omega^l(M)$ is a member of $\Omega^{k+l}(M)$; in particular, the wedge product of two smooth differential forms is smooth. Sometimes we shall consider differential forms of all degrees at once, taking $\Omega(M) = \bigoplus_{k=0}^m \Omega^k(M)$. The space $\Omega(M)$ is a $C^\infty(M)$ module and an associative algebra. As a consequence of Proposition 1.10, wedge product in $\Omega(M)$ has the property that

$$\omega \wedge \sigma = (-1)^{kl} \sigma \wedge \omega$$

whenever ω is in $\Omega^k(M)$ and σ is in $\Omega^l(M)$.

The theory of differential forms makes crucial use of “pullbacks” of differential forms. The formulas for these are akin to, but more general than, certain change-of-variables formulas in advanced calculus. If $\Phi : M \rightarrow N$ is a smooth function between manifolds, we describe how Φ associates to each k form ω on N a certain k form $\Phi^*\omega$ on M that is known as the pullback of ω . In the case $k = 0$, a 0 form on N is nothing more than a real-valued function ω on N , and the **pullback** of the function ω is just the composition $\Phi^*\omega = \omega \circ \Phi$, which is a real-valued function on M .

EXAMPLE 1. Let M be a smooth manifold of dimension m , and let (M_α, α) be a compatible chart. If (u_1, \dots, u_m) are standard coordinates on \mathbb{R}^m , then the coordinates (x_1, \dots, x_m) on M_α given by $x_j = u_j \circ \alpha$ have the property that x_j is the pullback of u_j . In symbols, $x_j = \alpha^*(u_j)$. Similarly just before Proposition 1.5 we defined derivations $\left[\frac{\partial}{\partial x_j}\right]_p$ of $T_p(M)$ by

$$\left[\frac{\partial f}{\partial x_j}\right]_p = \frac{\partial(f \circ \alpha^{-1})}{\partial u_j} \Big|_{(u_1, \dots, u_m) = (x_1(p), \dots, x_m(p))} \quad \text{for } j = 1, \dots, m.$$

In the present terminology, $\frac{\partial f}{\partial x_j}$ is therefore defined as the partial derivative with respect to the j^{th} variable of the pullback function $f \circ \alpha^{-1}$.

Pullback on 0 forms is \mathbb{R} linear and carries smooth 0 forms to smooth 0 forms. If ω is a smooth 0 form and f is in $C^\infty(M)$, then

$$\Phi^*(f\omega) = (f\omega) \circ \Phi = f(\omega \circ \Phi) = f(\Phi^*\omega).$$

Hence pullback on 0 forms is $C^\infty(M)$ linear.

For $k \geq 1$, the notion of pullback involves the derivative of Φ . We start with the case $k = 1$. Let ω be a 1 form on N . The derivative $(D\Phi)_p$ of Φ at a point p of M is a linear function carrying the tangent space $T_p(M)$ at p into the tangent space $T_{\Phi(p)}(N)$ at the point $\Phi(p)$ in N . Thus $(D\Phi)_p(X_p)$ is in $T_{\Phi(p)}(N)$ whenever X_p is in $T_p(M)$. If we apply to this the value $\omega_{\Phi(p)}$ of the given 1 form at $\Phi(p)$, the result is a linear function from $T_p(M)$ into \mathbb{R} , hence a member of $T_p^*(M)$. Letting p move, we thus obtain a 1 form $\Phi^*\omega$ on M from the definition

$$(\Phi^*\omega)_p(X_p) = (\omega_{\Phi(p)})((D\Phi)_p(X_p)).$$

We take $\Phi^*\omega$ as the **pullback** of the 1 form ω from N to M .

Let us observe that the definition depends only on germs at p , specifically on $\Phi(p)$ and $(D\Phi)_p$, not otherwise on the behavior of Φ in a neighborhood of p . To underscore this point, we can introduce a more primitive notion of **pullback** as the linear function $\Phi_p^\# : T_{\Phi(p)}^*(N) \rightarrow T_p^*(M)$ defined by

$$(\Phi_p^\#(\omega_{\Phi(p)}))(X_p) = (\omega_{\Phi(p)})((D\Phi)_p(X_p)).$$

Then the pullback $\Phi^*\omega$ of a differential form ω on N is the differential form on M given by

$$(\Phi^*\omega)_p = \Phi_p^\#(\omega_{\Phi(p)}).$$

The definition of Φ^* via $\Phi^\#$ will play a role in Proposition 1.18 when we assemble a list of properties of pullback.

EXAMPLE 2. Let Φ be a smooth map from an open subset U of \mathbb{R}^m into an open subset V of \mathbb{R}^n . Let us use the standard Euclidean coordinates (x_1, \dots, x_m) in \mathbb{R}^m and (y_1, \dots, y_n) in \mathbb{R}^n , and let us write the entries of Φ as (Φ_1, \dots, Φ_n) . This situation is an instance of the theory where $M = U$, $N = V$, and each of M and N is covered by a single chart. We shall compute the pullback $\Phi^*(dy_i)$ for $1 \leq i \leq n$, obtaining the result that $\Phi^*(dy_i) = d\Phi_i$. Since the set $\{(dy_1)_q, \dots, (dy_n)_q\}$ is a basis of $T_q^*(V)$ for each q in V , we will in essence have computed the pullback of every differential 1 form on V .

By definition, $\Phi^*(dy_i)$ is the 1 form given by

$$(\Phi^*(dy_i))_p(X_p) = (dy_i)_{\Phi(p)}((D\Phi)_p(X_p)) \quad \text{for every vector field } X \text{ on } U.$$

The right side is

$$\begin{aligned} &= (dy_i)_{\Phi(p)}((D\Phi)_p\left(\sum_{j=1}^m (Xx_j)_p \left[\frac{\partial}{\partial x_j}\right]_p\right)) && \text{by Proposition 1.5} \\ &= (dy_i)_{\Phi(p)}\left(\sum_{j=1}^m (Xx_j)_p (D\Phi)_p\left(\left[\frac{\partial}{\partial x_j}\right]_p\right)\right) \end{aligned}$$

$$\begin{aligned}
&= (dy_i)_{\Phi(p)} \left(\sum_{j=1}^m \sum_{k=1}^n (Xx_j)_p \left(\frac{\partial \Phi_k}{\partial x_j} \right)_p \left[\frac{\partial}{\partial y_k} \right]_{\Phi(p)} \right) && \text{by Proposition 1.7} \\
&= \sum_{j=1}^m (Xx_j)_p \left(\frac{\partial \Phi_i}{\partial x_j} \right)_p && \text{since } (dy_i)_{\Phi(p)} \left(\left[\frac{\partial}{\partial y_k} \right]_{\Phi(p)} \right) = \delta_{ik} \\
&= X_p \Phi_i && \text{by Proposition 1.6} \\
&= (d\Phi_i)_p (X_p) && \text{by definition of } (d\Phi_i)_p.
\end{aligned}$$

Therefore $(\Phi^*(dy_i))_p = (d\Phi_i)_p$. In fact, the computation actually showed that $\Phi_p^\#(dy_i)_{\Phi(p)} = (d\Phi_i)_p$. Anyway, the final result is that $\Phi^*(dy_i) = d\Phi_i$.

EXAMPLE 3. Let $\Phi : M \rightarrow N$ be a smooth map from a smooth manifold M of dimension m to a smooth manifold N of dimension n . Let p be in M , and introduce local coordinates (y_1, \dots, y_n) about $\Phi(p)$ and (x_1, \dots, x_m) about p . The understanding is that (N_β, β) is a compatible chart about $\Phi(p)$ with $\beta = (y_1, \dots, y_n)$ and that (M_α, α) is a compatible chart about p with M_α chosen small enough so that $\Phi(M_\alpha) \subseteq N_\beta$. We compute the pullback $\Phi^*(dy_i)$ to M_α of the 1 form dy_i on N_β for $1 \leq i \leq n$.

In fact, once we define $\Phi_i = y_i \circ \Phi$, both the result $\Phi^*(dy_i) = d\Phi_i$ and the computation, step by step, are the same as in Example 2. We have only to take into account the definitions of partial derivatives $\left[\frac{\partial}{\partial x_j} \right]_p$ and $\left[\frac{\partial}{\partial y_i} \right]_{\Phi(p)}$ that were given in Proposition 1.6 and its remark. Observe that as in Example 2, the computation is actually valid on the more primitive level of $\Phi^\#$; we shall use this observation later in this section in connection with Proposition 1.17.

Let us extend the definition of Φ^* from 1 forms to k forms for all positive integers k . We still assume that $\Phi : M \rightarrow N$ is a smooth map from a smooth manifold M of dimension m to a smooth manifold N of dimension n . For fixed p , the map

$$\omega \mapsto \Phi_p^\#(\omega_{\Phi(p)}) = (\Phi^*\omega)_p$$

is linear from $T_{\Phi(p)}^*(N)$ into $T_p^*(M)$, and we can regard it as a linear function ℓ from $T_{\Phi(p)}^*(N)$ into the associative algebra $\bigwedge(T_p^*(M))$ with the property that $\ell(v)^2 = 0$ for all v in $T_{\Phi(p)}^*(N)$. By Proposition 1.11b, ℓ extends uniquely to an algebra homomorphism $L : \bigwedge(T_{\Phi(p)}^*(N)) \rightarrow \bigwedge(T_p^*(M))$ sending 1 into 1 such that the diagram in Proposition 1.11b commutes. The resulting algebra homomorphism is the pullback $\Phi_p^\#$ on the full exterior algebra:

$$\Phi_p^\# : \bigwedge(T_{\Phi(p)}^*(N)) \rightarrow \bigwedge(T_p^*(M)).$$

By the nature of the construction, $\Phi_p^\#$ carries $\bigwedge^k(T_{\Phi(p)}^*(N))$ into $\bigwedge^k(T_p^*(M))$ for each integer $k \geq 0$. Letting p vary, we define $(\Phi^*\omega)_p = \Phi_p^\#(\omega_{\Phi(p)})$ for

$\omega \in \bigwedge^k (T_{\Phi(p)}^*(N))$, and we see that Φ^* carries the space $\Omega^k(N)$ of differential k forms on N into the space $\Omega^k(M)$ of differential k forms on M . For any differential k form ω on N , we call $\Phi^*\omega$ the **pullback** of ω to a differential form on M .

EXAMPLE 4. Let notation for a smooth map $\Phi : M \rightarrow N$ be as in Example 3. As a consequence of Propositions 1.9 and 1.14, any differential k form ω on N has a unique local expansion

$$\omega(\Phi(p)) = \sum_{1 \leq i_1 < \dots < i_k \leq n} a_{i_1, \dots, i_k}(\Phi(p)) (dy_{i_1})_{\Phi(p)} \wedge \dots \wedge (dy_{i_k})_{\Phi(p)}.$$

The pullback operation Φ^* is an algebra homomorphism of exterior algebras, it depends only on germs at p , it sends the function $\omega \circ \Phi$ into ω , and Example 3 shows that its value on $(dy_i)_p$ is $(d\Phi_i)_p$. Therefore

$$(\Phi^*\omega)(p) = \sum_{1 \leq i_1 < \dots < i_k \leq n} (a_{i_1, \dots, i_k} \circ \Phi)(p) (d\Phi_{i_1})_p \wedge \dots \wedge (d\Phi_{i_k})_p.$$

This is a perfectly fine way to write the answer for many purposes. On the other hand, if we want to involve the differentials $(dx_1)_p, \dots, (dx_m)_p$ on the right side, then we can substitute for each $(d\Phi_{i_r})_p$ and use the formula $(d\Phi_{i_r})_p = \sum_{j=1}^m \left[\frac{\partial \Phi_{i_r}}{\partial x_j} \right]_p (dx_j)_p$ to expand out the result in terms of expressions $(dx_{j_1})_p \wedge \dots \wedge (dx_{j_k})_p$. Finally we can simplify. As a general rule, this computation is fairly messy. The following proposition isolates one important case in which the result is tidy.

Proposition 1.17. If Φ is a smooth map from M into N with $\dim M = \dim N = n$, if p is in M , and if (x_1, \dots, x_n) and (y_1, \dots, y_n) are local coordinates about p and $\Phi(p)$, then

$$\Phi^*(dy_1 \wedge \dots \wedge dy_n)_{\Phi(p)} = \det \left\{ \frac{\partial \Phi_i}{\partial x_j} \Big|_p \right\}_{i,j=1, \dots, n} (dx_1)_p \wedge \dots \wedge (dx_n)_p.$$

PROOF. Example 4 above shows that

$$\Phi^*(dy_1 \wedge \dots \wedge dy_n)_{\Phi(p)} = (d\Phi_1)_p \wedge \dots \wedge (d\Phi_n)_p, \quad (*)$$

and Proposition 1.9 shows that

$$(d\Phi_i)_p = \sum_{j=1}^m \left(\frac{\partial \Phi_i}{\partial x_j} \right)_p (dx_j)_p \quad \text{for all } p \in M_\alpha. \quad (**)$$

Successively we use Corollary 1.16 and (**) to calculate that

$$\begin{aligned}
& (d\Phi_1)_p \wedge \cdots \wedge (d\Phi_n)_p \left(\left[\frac{\partial}{\partial x_1} \right]_p, \dots, \left[\frac{\partial}{\partial x_n} \right]_p \right) \\
&= \frac{1}{n!} \sum_{\tau \in \mathfrak{S}_n} (\text{sgn } \tau) (d\Phi_1)_p \left(\left[\frac{\partial}{\partial x_{\tau(1)}} \right]_p \right) \cdots (d\Phi_n)_p \left(\left[\frac{\partial}{\partial x_{\tau(n)}} \right]_p \right) \\
&= \frac{1}{n!} \sum_{\tau \in \mathfrak{S}_n} (\text{sgn } \tau) \left(\frac{\partial \Phi_1}{\partial x_{\tau(1)}} \right)_p \cdots \left(\frac{\partial \Phi_n}{\partial x_{\tau(n)}} \right)_p \\
&= \det \left\{ \frac{\partial \Phi_i}{\partial x_j} \Big|_p \right\}_{i,j=1,\dots,n}. \tag{\dagger}
\end{aligned}$$

Since

$$(dx_1)_p \wedge \cdots \wedge (dx_n)_p \left(\left[\frac{\partial}{\partial x_1} \right]_p, \dots, \left[\frac{\partial}{\partial x_n} \right]_p \right) = 1$$

by Proposition 1.14e and since the space of alternating n -linear forms on $T_p(M)$ is 1 dimensional by Proposition 1.14a, we see from (\dagger) that

$$(d\Phi_1)_p \wedge \cdots \wedge (d\Phi_n)_p = \det \left\{ \frac{\partial \Phi_i}{\partial x_j} \Big|_p \right\}_{i,j=1,\dots,n} (dx_1)_p \wedge \cdots \wedge (dx_n)_p.$$

The proposition then follows from (*). \square

We conclude this section by giving another application of Example 4.

Proposition 1.18. If $\Phi : M \rightarrow N$ is a smooth map between smooth manifolds, then pullbacks to M of differential forms on N have the following properties:

- (a) for $k \geq 0$, $\Phi^*(\omega_1 + \omega_2) = \Phi^*\omega_1 + \Phi^*\omega_2$ whenever ω_1 and ω_2 are differential k forms on N ,
- (b) for $k \geq 0$, $\Phi^*(c\omega) = c\Phi^*(\omega)$ whenever c is in \mathbb{R} and ω is a differential k form on N ,
- (c) for $k \geq 0$, $\Phi^*\omega$ is a smooth differential k form on M whenever ω is a smooth differential k form on N ,
- (d) for $k \geq 0$, $\Phi^*(f\omega) = f\Phi^*\omega$ whenever ω is a differential k form on N and $f : M \rightarrow \mathbb{R}$ is a real-valued function, and $f\omega$ is smooth if f and ω are both smooth,
- (e) for $k \geq 0$ and $l \geq 0$, $\Phi^*(\omega_1 \wedge \omega_2) = \Phi^*\omega_1 \wedge \Phi^*\omega_2$ whenever ω_1 is a differential k form on N and ω_2 is a differential l form on N ,
- (f) $(\Psi \circ \Phi)^*\omega = \Phi^*(\Psi^*\omega)$ whenever $\Psi : N \rightarrow R$ is another smooth map between smooth manifolds and ω is a differential form on R .

PROOF. Conclusions (a), (b), and (e) are immediate consequences of the fact that Φ^* can be defined in terms of $\Phi^\#$, which is an algebra homomorphism. Conclusion (c) follows from the formula for pullback given in Example 4. In (d), the equality $\Phi^*(f\omega) = f\Phi^*\omega$ reflects the linearity over \mathbb{R} of $\Phi^\#$ at each point. The conclusion about smoothness follows from the formula in Example 4. Conclusion (f) follows immediately by tracking down the definitions. \square

4. Exterior Derivative

The exterior derivative is an extension of the operator d , which so far carries smooth functions (i.e., 0 forms) into smooth 1 forms, to an operator sending smooth forms of any degree into smooth forms of the next higher degree. The original motivation for the definition of d on differential forms of degree ≥ 1 was from its appearance in Stokes's Theorem.

Even though we do not yet have Stokes's Theorem at hand, let us elaborate a bit. Recall from elementary calculus that the Fundamental Theorem of Calculus, saying that

$$\int_0^1 f'(x) dx = f(1) - f(0),$$

can be motivated heuristically by the approximations

$$\int_0^1 f(x) dx \approx \sum_{k=1}^n \frac{1}{n} f'(\frac{k}{n}) \approx \sum_{k=1}^n [f(\frac{k}{n}) - f(\frac{k-1}{n})] = f(1) - f(0).$$

Here the first \approx refers to the approximation of the Riemann integral by a Riemann sum, the second \approx uses the Mean Value Theorem and the continuity of f' , and the equality on the right takes into account the telescoping nature of the sum. The equality of the Fundamental Theorem says that the aggregate of the infinitesimal change of f over the interval equals the difference between the values of f at the endpoints.

Nineteenth century mathematicians and physicists used this kind of reasoning in three dimensions to compute the total "flux" of a fluid or radiant energy across a given curve or surface, using an integral to express the aggregate of the infinitesimal flux and an integral in one less dimension to express the total. The infinitesimal changes were written in terms of the differential operators grad, curl, and div. Later it was seen that all three operators were instances of one operator that could be generalized to more dimensions. The relevant versions of Stokes's Theorem appear in the Introduction. The operator in question was the exterior derivative d , and we shall see its relation to grad, curl, and div momentarily.

Because of this convoluted history it would be somewhat artificial to begin with simple geometrically motivated axioms for the general operator d , derive

what d must be, and then prove existence and uniqueness. Instead we shall start with an explicit formula for d in the context of \mathbb{R}^m and its open subsets, derive certain properties of d , and then show how d can be defined on smooth manifolds.

Thus for now we work with the smooth manifold \mathbb{R}^m , which has an atlas consisting of one chart $(\mathbb{R}^m, 1)$, the mapping 1 being the identity mapping on \mathbb{R}^m . We can safely ignore 1 for the time being. The coordinates are (x_1, \dots, x_m) . We saw in Section 3 for $k \geq 0$ that $\Omega^k(\mathbb{R}^m)$ consists exactly of all differential forms

$$\omega(p) = \sum_{1 \leq i_1 < \dots < i_k \leq m} a_{i_1, \dots, i_k}(p) (dx_{i_1})_p \wedge \dots \wedge (dx_{i_k})_p$$

with all coefficients $a_{i_1, \dots, i_k}(p)$ in $C^\infty(\mathbb{R}^m)$ and that the expansion of ω in this way is unique.¹⁶ Let us abbreviate this expansion¹⁷ in obvious fashion as

$$\omega = \sum_I a_I dx_I,$$

the sum running over all strictly increasing sequences I of k integers between 1 and m .

We define an \mathbb{R} linear operator $d : \Omega^k(\mathbb{R}^m) \rightarrow \Omega^{k+1}(\mathbb{R}^m)$ by

$$d\left(\sum_I a_I dx_I\right) = \sum_I (da_I) \wedge dx_I.$$

This operator is called **exterior differentiation**. For the special case $k = 0$, the operator d reduces to the passage from a smooth function f to its differential df as defined in Section 1.

The sum $\Omega(\mathbb{R}^m) = \bigoplus_{k=0}^m \Omega^k(\mathbb{R}^m)$ is the space of all smooth differential forms on \mathbb{R}^m . It is an associative algebra over \mathbb{R} and is also a $C^\infty(\mathbb{R}^m)$ module. When it is convenient to do so, we can regard d as an \mathbb{R} linear function from $\Omega(\mathbb{R}^m)$ into itself.

EXAMPLE 1. In \mathbb{R}^2 , let us write (x, y) for the coordinates. The $C^\infty(\mathbb{R}^2)$ module $\Omega^k(\mathbb{R}^2)$ is nonzero for $k = 0, 1, 2$, and a free basis in the three cases consists of $\{1\}$, $\{dx, dy\}$, and $\{dx \wedge dy\}$. On 0 forms, d acts by $df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$, on 1 forms, d acts by

$$d(p dx + q dy) = \left(\frac{\partial q}{\partial x} - \frac{\partial p}{\partial y}\right) (dx \wedge dy),$$

and on 2 forms, d acts as 0.

¹⁶For $k = 0$, the only such increasing sequence (i_1, \dots, i_m) with $1 \leq i_1 < \dots < i_k \leq m$ is the empty sequence, and in this case the wedge product $(dx_{i_1})_p \wedge \dots \wedge (dx_{i_k})_p$ is understood to be the identity element of $\Omega(\mathbb{R}^m)$.

¹⁷The existence and uniqueness of this expansion means in the terminology of Section VIII.1 of *Basic Algebra*, that $\Omega^k(\mathbb{R}^m)$ is a free $C^\infty(\mathbb{R}^m)$ module with free basis the various dx_I .

EXAMPLE 2. In \mathbb{R}^3 , let us write (x, y, z) for the coordinates. The $C^\infty(\mathbb{R}^3)$ module $\Omega^k(\mathbb{R}^3)$ is nonzero for $k = 0, 1, 2, 3$, and a free basis in the four cases consists of $\{1\}$, $\{dx, dy, dz\}$, $\{dy \wedge dz, dz \wedge dx, dx \wedge dy\}$, and $\{dx \wedge dy \wedge dz\}$. On 0 forms, d acts by $df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz$, and we can identify this with the vector-valued function

$$\text{grad } f = \begin{pmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial z} \end{pmatrix}.$$

On 1 forms, d acts by

$$\begin{aligned} d(p dx + q dy + r dz) \\ = \left(\frac{\partial r}{\partial y} - \frac{\partial q}{\partial z}\right)(dy \wedge dz) + \left(\frac{\partial p}{\partial z} - \frac{\partial r}{\partial x}\right)(dz \wedge dx) + \left(\frac{\partial q}{\partial x} - \frac{\partial p}{\partial y}\right)(dx \wedge dy), \end{aligned}$$

and we can identify this with the vector-valued function

$$\text{curl} \begin{pmatrix} p \\ q \\ r \end{pmatrix} = \begin{pmatrix} \frac{\partial r}{\partial y} - \frac{\partial q}{\partial z} \\ \frac{\partial p}{\partial z} - \frac{\partial r}{\partial x} \\ \frac{\partial q}{\partial x} - \frac{\partial p}{\partial y} \end{pmatrix}.$$

On 2 forms, d acts by

$$d(a dy \wedge dz + b dz \wedge dx + c dx \wedge dy) = \left(\frac{\partial a}{\partial x} + \frac{\partial b}{\partial y} + \frac{\partial c}{\partial z}\right)(dx \wedge dy \wedge dz),$$

and we can identify this with the real-valued function

$$\text{div} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \frac{\partial a}{\partial x} + \frac{\partial b}{\partial y} + \frac{\partial c}{\partial z}.$$

Lemma 1.19. If I is a strictly increasing tuple of k integers from 1 to m and J is a strictly increasing tuple of l integers from 1 to m , then

$$dx_I \wedge dx_J = \begin{cases} 0 & \text{if } I \text{ and } J \text{ have an integer in common} \\ \varepsilon dx_K & \text{if } I \text{ and } J \text{ have no integer in common,} \end{cases}$$

where $\varepsilon = \pm 1$ and K is the union of I and J with the terms rearranged to be strictly increasing.

PROOF. If any factor of dx_I matches a factor of dx_J , then $dx_I \wedge dx_J = 0$ by the alternating property. Otherwise we can interchange individual terms of $dx_I \wedge dx_J$ repeatedly until the indices are in increasing order. Each interchange introduces a minus sign. \square

Proposition 1.20. The operator d on $\Omega(\mathbb{R}^m)$ is an antiderivation in the sense that if ω is in $\Omega^k(\mathbb{R}^m)$ and σ is in $\Omega^l(\mathbb{R}^m)$, then

$$d(\omega \wedge \sigma) = d\omega \wedge \sigma + (-1)^k \omega \wedge d\sigma.$$

PROOF. Since d is \mathbb{R} linear and wedge product is \mathbb{R} linear in each variable, we may assume that $\omega = f_I dx_I$ and $\sigma = g_J dx_J$, where I is a strictly increasing tuple of k integers from 1 to m and J is a strictly increasing tuple of l integers from 1 to m . By Lemma 1.19, $dx_I \wedge dx_J = \varepsilon dx_K$ for some strictly increasing $(k+l)$ -tuple of integers, where ε is 0 or ± 1 . Then we have

$$\begin{aligned} d(f_I dx_I \wedge g_J dx_J) &= d(f_I g_J dx_I \wedge dx_J) \\ &= \varepsilon d(f_I g_J dx_K) && \text{by Lemma 1.19} \\ &= \varepsilon d(f_I g_J) \wedge dx_K && \text{by definition of } d \\ &= \varepsilon g_J df_I \wedge dx_K + \varepsilon f_I dg_J \wedge dx_K && \text{by the product rule} \\ &= g_J df_I \wedge dx_I \wedge dx_J + f_I dg_J \wedge dx_I \wedge dx_J && \text{for derivatives} \\ &= (df_I \wedge dx_I) \wedge g_J dx_J + (-1)^k f_I dx_I \wedge dg_J \wedge dx_J && \text{by Proposition 1.16} \\ &= d(f_I dx_I) \wedge (g_J dx_J) + (-1)^k f_I dx_I \wedge d(g_J dx_J). \quad \square \end{aligned}$$

Lemma 1.21. For $k \geq 1$, whenever u_1, \dots, u_k are members of $C^\infty(\mathbb{R}^m)$, then $d(du_1 \wedge \dots \wedge du_k) = 0$.

PROOF. We induct on k . For $k = 1$, the fact that $u = \sum_j \frac{\partial u}{\partial x_j} dx_j$ means that we have

$$d(du) = \sum_j d\left(\frac{\partial u}{\partial x_j} dx_j\right) = \sum_{j,i} \frac{\partial}{\partial x_i} \frac{\partial u}{\partial x_j} dx_i \wedge dx_j.$$

On the right side the terms with $i = j$ are 0 since $dx_i \wedge dx_i = 0$, and a term with $i < j$ cancels a term with $i > j$ since $\frac{\partial^2 u}{\partial x_i \partial x_j} = \frac{\partial^2 u}{\partial x_j \partial x_i}$ and since $dx_i \wedge dx_j = -dx_j \wedge dx_i$. This proves the lemma for $k = 1$. Inductively assuming the result for $k = r - 1$, we use Proposition 1.20 to write

$$d(du_1 \wedge \dots \wedge du_r) = d(du_1 \wedge \dots \wedge du_{r-1}) \wedge du_r - (du_1 \wedge \dots \wedge du_{r-1}) \wedge d(du_r).$$

The first term on the right side is 0 by the case $k = r - 1$ of the lemma, and the second term on the right side is 0 by the case $k = 1$ of the lemma. This completes the induction and the proof. \square

Theorem 1.22. The operator d on $\Omega(\mathbb{R}^m)$ is independent of coordinates in the following sense: Let (u_1, \dots, u_m) be any other system of coordinates on \mathbb{R}^m related to (x_1, \dots, x_m) by a diffeomorphism of \mathbb{R}^m . For each strictly increasing sequence $I = \{i_1, \dots, i_k\}$ of k integers between 1 and m , let $du_I = du_{i_1} \wedge \dots \wedge du_{i_k}$. If $\omega = \sum_I a_I du_I$ is the expansion of a member ω of $\Omega^k(\mathbb{R}^m)$ for $k \geq 0$ in terms of the forms du_I , then $d\omega$ is given by $d\omega = \sum_I da_I \wedge du_I$.

PROOF. We have

$$\begin{aligned} d\omega &= \sum_I d(a_I du_I) \\ &= \sum_I da_I \wedge du_I + d(du_I) \quad \text{by Proposition 1.20} \\ &= \sum_I da_I \wedge du_I \quad \text{by Lemma 1.21.} \quad \square \end{aligned}$$

The results we have just established for $\Omega(\mathbb{R}^m)$ in Lemma 1.19 through Theorem 1.22 remain valid for any nonempty subset U of \mathbb{R}^m in place of \mathbb{R}^m itself, and the proofs need no changes.

Of particular interest is what Theorem 1.22 is saying for the diffeomorphism that arises between two open subsets of \mathbb{R}^m when two compatible charts of an m dimensional smooth manifold M overlap. Thus let (M_α, α) and (M_β, β) be compatible charts of M with $M_\alpha \cap M_\beta \neq \emptyset$. The compatibility condition is that $\beta \circ \alpha^{-1} : \alpha(M_\alpha \cap M_\beta) \rightarrow \beta(M_\alpha \cap M_\beta)$ is smooth and so is its inverse $\alpha \circ \beta^{-1} : \beta(M_\alpha \cap M_\beta) \rightarrow \alpha(M_\alpha \cap M_\beta)$. Theorem 1.22 says that d takes the same form in the coordinate systems of these two open sets. In other words, d can be consistently defined on M_α and M_β by the usual formula $d(\sum_I a_I dx_I) = \sum_I (da_I \wedge dx_I)$, and d becomes globally defined on M . In short, d extends to an operator on the smooth manifold M , carrying $\Omega^k(M)$ to $\Omega^{k+1}(M)$ for all $k \geq 0$. Let us summarize and collect the properties of d that follow at once.

Proposition 1.23. If M is a smooth manifold, then the exterior derivative operator d is well defined on M and carries $\Omega^k(M)$ into $\Omega^{k+1}(M)$ for all integers $k \geq 0$. It has the properties that

- (a) $d(\omega \wedge \sigma) = d\omega \wedge \sigma + (-1)^k \omega \wedge d\sigma$ whenever ω is in $\Omega^k(M)$ and σ is in $\Omega^l(N)$,
- (b) $d(d\omega) = 0$ whenever ω is in $\Omega^k(M)$.

PROOF. Conclusion (a) is an instance of Proposition 1.20. For conclusion (b), it is enough to consider d^2 of a form $f dx_I$. Conclusion (a) gives

$$d^2(f dx_I) = d(df \wedge dx_I) = d^2 f \wedge dx_I - f \wedge d(dx_I).$$

The term $d^2 f$ was shown to be 0 in the proof of Lemma 1.21, and the term $d(dx_I)$ equals 0 by the conclusion of Lemma 1.21. \square

We shall need one further property of the exterior derivative.

Proposition 1.24. Exterior derivative commutes with pullback in the following sense: if $\Phi : M \rightarrow N$ is a smooth map between smooth manifolds and if ω is in $\Omega^k(N)$ with $k \geq 0$, then

$$d(\Phi^*\omega) = \Phi^*(d\omega).$$

PROOF. Let (x_1, \dots, x_m) and (y_1, \dots, y_n) be local coordinates about p in M and $\Phi(p)$ in N . We begin with the case $k = 0$, for which ω reduces to a member f of $C^\infty(N)$. Then

$$\begin{aligned} (d(\Phi^*f))_p &= d((f \circ \Phi))_p && \text{by definition of } \Phi^* \text{ on functions} \\ &= \sum_j \frac{\partial(f \circ \Phi)(p)}{\partial x_j} (dx_j)_p && \text{by Proposition 1.9} \\ &= \sum_{i,j} \frac{\partial f}{\partial y_i}(\Phi(p)) \frac{\partial \Phi_i}{\partial x_j}(p) (dx_j)_p && \text{by the chain rule} \\ &= \sum_i \frac{\partial f}{\partial y_i}(\Phi(p)) (d\Phi_i)_p && \text{by Proposition 1.9} \\ &= \sum_i \frac{\partial f}{\partial y_i}(\Phi(p)) \Phi_p^\#((dy_i)_{\Phi(p)}) && \text{by Example 2 in Section 3} \\ &= \Phi_p^\# \left(\sum_i \frac{\partial f}{\partial y_i}(\Phi(p)) (dy_i)_{\Phi(p)} \right) && \text{by linearity of } \Phi_p^\# \\ &= \Phi_p^\#((df)_{\Phi(p)}) && \text{by Proposition 1.9} \\ &= \Phi^*(df)_p && \text{by definition of } \Phi^* \text{ on 1 forms} \\ &&& \text{in terms of } \Phi^\#, \end{aligned}$$

and the case $k = 0$ is proved. For general $k \geq 1$, let a member

$$\omega = \sum_{1 \leq i_1 < \dots < i_k \leq n} a_{i_1, \dots, i_k}(q) (dy_{i_1})_q \wedge \dots \wedge (dy_{i_k})_q$$

be given in $\Omega^k(N)$, and abbreviate it as $\omega = \sum_I a_I(q) (dy_I)_q$. Then

$$\begin{aligned} d(\Phi^*\omega)_p &= d\left(\sum_I a_I(\Phi(p)) (d\Phi_I)_p\right) && \text{by Example 4 in Section 3} \\ &= \sum_I (da_I)_{\Phi(p)} \wedge (d\Phi_I)_p && \text{by Proposition 1.23} \\ &= \Phi^*\left(\sum_I (da_I)_p \wedge (dy_I)_p\right) && \text{by definition of } \Phi^* \\ &= \Phi^*(d\omega)_p. \end{aligned}$$

□

5. Smooth Partitions of Unity

A **partition of unity** on a smooth manifold is a system of nonnegative real-valued smooth functions with sum one such that each point has a neighborhood on which only finitely many of the functions are not identically zero. The existence of this neighborhood for each point is a condition that removes all questions about limits from the construction.

Historically partitions of unity arose in an effort to make more flexible the requirement that a topological space be decomposed into disjoint subsets for some purpose. Triangulations of manifolds in the subject of topology were notable examples. A different example from *Basic Real Analysis* is the rendering in Section III.13 of an annulus as the union of four quarters of an annulus in order to be able to apply Green's Theorem. In any event a decomposition into disjoint subsets is in effect a system of indicator functions¹⁸ with sum identically one. By allowing the use of other functions with values between 0 and 1, we get less precision in distinguishing the disjoint sets, but in compensation we are allowed to insist that the functions be smooth and hence enjoy nicer analytic properties.

Theorem 1.25. Let M be a smooth manifold, let K be a nonempty compact subset, and let $\{U_i \mid 1 \leq i \leq r\}$ be a finite open cover of K . Then there exist functions f_i in $C^\infty(M)$ for $1 \leq i \leq r$, taking values between 0 and 1 such that each f_i is identically 0 off a compact subset of U_i and $\sum_{i=1}^r f_i$ is identically 1 on K .

REMARK. The language that is used as shorthand for the conclusion of this theorem is that the set $\{f_i\}$ of functions is a **smooth partition of unity** of M **subordinate** to the finite open cover $\{U_i\}$ of K .

We shall use the following lemma, which was proved for \mathbb{R}^n as Lemma 3.15 of *Basic Real Analysis* but is valid in any locally compact separable metric space with no essential change in proof.¹⁹

Lemma 1.26. In a smooth manifold M ,

- (a) if L is a compact set and U is an open set with $L \subseteq U$, then there exists an open set V with V^{cl} compact and $L \subseteq V \subseteq V^{\text{cl}} \subseteq U$,
- (b) if K is a compact set and $\{U_1, \dots, U_r\}$ is a finite open cover of K , then there exists an open cover $\{V_1, \dots, V_r\}$ of K such that V_i^{cl} is a compact subset of U_i for each i .

¹⁸Indicator functions are real-valued functions taking only the values 0 and 1.

¹⁹A sufficiently large closed ball in the proof of Lemma 3.15 is to be replaced by a member of the exhausting sequence that is sufficiently far along in the sequence.

Lemma 1.27. Let M be a smooth manifold, K be a nonempty compact subset of M , and let U be an open subset of M containing K and having compact closure in M . Then there exists a function f in $C^\infty(M)$ such that f is everywhere positive on K and f vanishes off a compact subset of U .

PROOF. For each point p of K , choose a compatible chart $(M_{\alpha,p}, \alpha_p)$ about p . Without loss of generality, we may assume that $M_{\alpha,p} \subseteq U$ for all p . Then choose an open neighborhood $M'_{\alpha,p}$ of p whose compact closure lies in $M_{\alpha,p}$.

As p varies in K , the sets $M'_{\alpha,p}$ form an open cover of K . By compactness of K , let $\{M'_{\alpha,p_1}, \dots, M'_{\alpha,p_l}\}$ be a finite subcover. Applying Lemma 1.3 to each chart M_{α,p_j} , choose a member f_j of $C^\infty(M)$ that has values in $[0, 1]$, that vanishes off a compact subset of M_{α,p_j} , and that is identically 1 on the compact subset $(M'_{\alpha,p_j})^{\text{cl}}$. Then the sum $f = f_1 + \dots + f_l$ is everywhere positive on the union of the sets $(M'_{\alpha,p_j})^{\text{cl}}$, hence is everywhere positive on K . Each f_j is 0 off a compact subset of M_{α,p_j} , hence is 0 off a compact subset of U . Therefore f is 0 off a compact subset of U . \square

Lemma 1.28. Let M be a smooth manifold, and let K, V, L , and U be distinct nonempty subsets with K and L compact, V and U open, and $K \subseteq V \subseteq L \subseteq U$. Then there exists a function g in $C^\infty(M)$ such that g is identically 0 on K , is everywhere positive on $L - V$, and is compactly supported in U .

PROOF. For each point p of $L - V$, choose a compatible chart $(M_{\alpha,p}, \alpha_p)$ about p . Since p is not in K , we may without loss of generality assume that $M_{\alpha,p}$ does not meet K but is contained in U . Then choose an open neighborhood $M'_{\alpha,p}$ of p whose closure is compact and lies in $M_{\alpha,p}$. As p varies in $L - V$, the sets $M'_{\alpha,p}$ form an open cover of $L - V$. By compactness of $L - V$, let $\{M'_{\alpha,p_1}, \dots, M'_{\alpha,p_l}\}$ be a finite subcover.

Applying Lemma 1.3 to each of the charts $(M_{\alpha,p_j}, \alpha_p)$, choose a member g_j of $C^\infty(M)$ that has values in $[0, 1]$, that vanishes off a compact subset of M_{α,p_j} , and that is identically 1 on the compact set $(M'_{\alpha,p_j})^{\text{cl}}$. Then the sum $g = g_1 + \dots + g_l$ is everywhere positive on the union of the sets $(M'_{\alpha,p_j})^{\text{cl}}$, hence is everywhere positive on $L - V$. Each g_j is 0 off a compact subset of M_{α,p_j} , and thus g is identically 0 on K . Each g_j is compactly supported in M_{α,p_j} and therefore in U . Thus g is compactly supported in U . \square

PROOF OF THEOREM 1.25. Apply Lemma 1.26b to produce an open cover $\{W_1, \dots, W_r\}$ of K such that W_i^{cl} is compact and $W_i^{\text{cl}} \subseteq U_i$ for each i with $1 \leq i \leq r$. Then apply it a second time to produce an open cover $\{V_1, \dots, V_r\}$ of K such that V_i^{cl} is compact and $V_i^{\text{cl}} \subseteq W_i$ for each i . Put $V = V_1 \cup \dots \cup V_r$ and $W = W_1 \cup \dots \cup W_r$. Lemma 1.27 produces a function $h_i \geq 0$ in $C^\infty(M)$ that is everywhere positive on V_i^{cl} and is supported in a compact subset S_i of W_i . Then $h = h_1 + \dots + h_r$ is smooth on M , is everywhere positive on V and hence on K ,

is ≥ 0 everywhere, and is identically 0 off the compact subset $S = S_1 \cup \cdots \cup S_r$ of W .

Put $L = W^{\text{cl}}$. Using an exhausting sequence for M , choose an open set U containing L and having compact closure in M . Application of Lemma 1.28 produces a function g in $C^\infty(M)$ that is identically 0 on K , is everywhere positive on $L - V$, and is compactly supported in U . We wish to define

$$f_i = \begin{cases} h_i/(h + g) & \text{on } W \\ 0 & \text{on } S^c. \end{cases} \quad (*)$$

The denominator $h + g$ is nowhere 0 on W since h is everywhere positive on V and g is everywhere positive on the superset $L - V$ of $W - V$. The two expressions for f_i in $(*)$ are both smooth on their respective open domains W and S^c , and they agree on the overlap $W \cap S^c$ because h_i is identically 0 off S . Finally f_i is defined on all of M by $(*)$ because $S \subseteq W$. Therefore $(*)$ makes f_i into a well defined member of $C^\infty(M)$.

Plainly each f_i is ≥ 0 everywhere and is identically 0 off the compact subset S_i of $W_i \subseteq U_i$. The sum $\sum_{i=1}^r f_i$ equals $h/(h + g)$ on W . Since $W \supseteq K$ and since g vanishes on K , $\sum_{i=1}^r f_i$ is identically 1 on K . Thus the functions f_i have the required properties. \square

Two more general results are possible, but they will not really be needed for our purposes and we shall omit their proofs. They both construct smooth partitions of unity relative to an open cover $\{U_\alpha\}$ of a smooth manifold M with an index set I whose typical member is written as α . The partitions of unity are to be “locally finite” in the sense that each point p of M has an open neighborhood on which only finitely many of the functions are not identically 0. The following two situations are of interest:

- (1) The functions in the partition of unity are indexed by the same set I , and the function f_α with index α has (closed) support contained in U_α .
- (2) Each function in the partition of unity has compact support in some U_α , but the index set for the functions is allowed to be larger than the set I .

The example of $M = \mathbb{R}$ with cover $\{\mathbb{R}\}$ shows that we cannot insist on maintaining the same index set I for the members of the smooth of unity if we insist also on compact support for the functions. But we can insist on either condition (1) or condition (2). That is the combined conclusion of the two more general results.

6. Orientation and Integration on Smooth Manifolds

Let M be a smooth manifold of dimension m ; we emphasize that M need not be connected. Our primary interest in this section will be in integrating smooth differential forms of the top degree m on M , since the content of Stokes's Theorem in that two specific integrals of such differential forms are equal. For this purpose we require an "orientation" on M . The orientation that is chosen can affect the value of the integral. If M has an orientation, we say that M is **orientable**.

Orientation refers eventually to a left vs. right kind of decision, or to a number of such decisions. For a smooth manifold M of dimension $m \geq 0$, the notion of orientation can be defined in a number of equivalent ways,²⁰ and we use a definition that leads to integration as quickly as possible.

Before getting started, let us observe that any manifold is locally connected because each point has arbitrarily small neighborhoods that are homeomorphic with open Euclidean balls, hence connected. Consequently the connected components of a manifold are necessarily open. Charts about a point p are allowed to meet more than one component, but it will often be helpful to think of each chart as small enough so as to be connected and therefore to lie in a single connected component of M .

Let us set aside the special case $m = 0$ for now, returning to it after some examples, since some special remarks are appropriate for it. For M of dimension $m \geq 1$, we say that M is **oriented** if an atlas $\{(M_\alpha, \alpha)\}$ of compatible charts is given with the property that the m -by- m derivative matrix of each coordinate change

$$\beta \circ \alpha^{-1} : \alpha(M_\alpha \cap M_\beta) \rightarrow \beta(M_\alpha \cap M_\beta)$$

has everywhere positive determinant. Proposition 1.30 below will show that M can be oriented if and only if M admits a nowhere vanishing differential m form. Once that proposition is in hand, an "orientation" will be defined to be an equivalence class of such forms, two such being equivalent if the one is an everywhere positive function times the other. But we do not need Proposition 1.30 and the definition of orientation yet.

A smooth manifold that is oriented by some atlas is said to be **orientable**, otherwise **not orientable**. It is often easy to show that a certain manifold is orientable. Showing that a manifold is not orientable tends to be harder. Below we shall see examples of both situations.

²⁰Our definition will be given after four examples below. A frequently used definition elsewhere involves singling out an equivalence class of ordered bases of the tangent space $T_p(M)$ at each p , two such bases being equivalent if the one is carried to the other by a linear function with positive determinant. Orientability means that this process can be carried out in a way that depends continuously on p in M , and an orientation is any such choice of continuously varying equivalence classes for all points of M .

When an atlas $\{(M_\alpha, \alpha)\}$ exhibits M as oriented, a compatible chart (U, φ) is said to be **positive** relative to $\{(M_\alpha, \alpha)\}$ if the derivative matrix of $\varphi \circ \alpha^{-1}$ has everywhere positive determinant for all α . We always have the option of adjoining to the given atlas of charts for an oriented M any or all other compatible charts (U, φ) that are positive relative to all (M_α, α) , and M will still be oriented.

EXAMPLE 1. M equal to \mathbb{R}^m . The **standard atlas** for \mathbb{R}^m has just one chart in it, consisting of the open set \mathbb{R}^m and the identity mapping. The standard atlas makes \mathbb{R}^m oriented, and the orientation is called the **standard orientation**. A compatible chart (U, φ) consists of a nonempty open set U of \mathbb{R}^m and a diffeomorphism φ of U onto an open subset of \mathbb{R}^m . The chart is positive in the sense of the above definition if the Jacobian matrix $\left\{\frac{\partial \varphi_i}{\partial x_j}\right\}$ has everywhere positive determinant.

EXAMPLE 2. M equal to the circle $S_1 = \{(\cos \theta, \sin \theta) \in \mathbb{R}^2 \mid \theta \in \mathbb{R}\}$. The two charts (M_1, φ_1) and (M_2, φ_2) form an atlas under the definitions

$$M_1 = \{(\cos \theta, \sin \theta) \in \mathbb{R}^2 \mid -\pi < \theta < \pi\}, \quad \varphi_1(x, y) = \theta, \quad \varphi_1(M_1) = (-\pi, \pi)$$

$$M_2 = \{(\cos \theta, \sin \theta) \in \mathbb{R}^2 \mid 0 < \theta < 2\pi\}, \quad \varphi_2(x, y) = \theta, \quad \varphi_2(M_2) = (0, 2\pi).$$

With these definitions,

$$M_1 \cap M_2 = \{(\cos \theta, \sin \theta) \in \mathbb{R}^2 \mid -\pi < \theta < 0 \quad \text{or} \quad 0 < \theta < \pi\},$$

$$(\varphi_2 \circ \varphi_1^{-1})(\theta) = \begin{cases} \theta + 2\pi & \text{for } -\pi < \theta < 0 \\ \theta & \text{for } 0 < \theta < \pi. \end{cases}$$

The derivative matrix is everywhere the 1-by-1 matrix (1). Thus this atlas of charts exhibits M as oriented.

EXAMPLE 3. M equal to a **Möbius band** or Möbius strip. This is a noncompact 2 dimensional manifold that can be visualized in \mathbb{R}^3 . We start from a rectangle of paper and start to bend it to be taped into the form of a cylinder, but before the cylinder is taped, we twist one end through half a turn. More precisely the Möbius band can be parametrized in \mathbb{R}^3 by two parameters s and t and the equations

$$x(s, t) = (1 + \frac{t}{2} \cos \frac{s}{2}) \cos s$$

$$y(s, t) = (1 + \frac{t}{2} \cos \frac{s}{2}) \sin s$$

$$z(s, t) = \frac{t}{2} \sin \frac{s}{2}.$$

Here s is to vary over a fixed half open interval $[c, c + 2\pi)$, and t is to vary over the open interval $(-1, 1)$. The equations are periodic in the t variable but with a twist:

$$(x(s + 2\pi, t), y(s + 2\pi, t), z(s + 2\pi, t)) = (x(s, -t), y(s, -t), z(s, -t)).$$

Problem 29 at the end of the chapter shows how to define a smooth manifold by means of two charts from this information, and Proposition 1.33 will lead from there to a proof that the manifold is not orientable. See Figure 1.2.

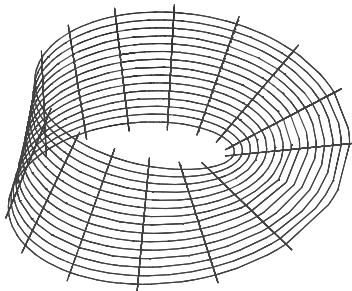


FIGURE 1.2. Möbius band.

EXAMPLE 4. The unit sphere $M = S^n$ in \mathbb{R}^{n+1} . This example was shown to be a smooth manifold in Section 1. It is orientable for $n \geq 1$, as will be deduced in Problem 15 at the end of the chapter. A general method applies for $n \geq 2$, and a special argument is needed for $n = 1$.

This is a good time to return to discuss orientation of a manifold M of dimension 0. In this case M is a discrete set of points, necessarily at most countable because our manifolds are assumed to be separable. The convention is that every smooth manifold of dimension 0 is orientable, being oriented by any atlas, and an **orientation** on it is the assignment of the scalar $+1$ or -1 to each of the points. This case is relevant in seeing how the general version of Stokes's Theorem reduces in one dimension to the Fundamental Theorem of Calculus, the boundary of a finite closed interval of the line being a two-point set. Vacuously every atlas exhibits a manifold of dimension 0 as oriented, and every chart is automatically positive.

Let us turn now to integration on smooth manifolds. In the special case that the manifold is a nonempty open subset U of Euclidean space \mathbb{R}^m , we introduce a notion of integration of smooth m forms. Any such form ω can be written as

$$\omega = F(x_1, \dots, x_m) dx_1 \wedge \cdots \wedge dx_m,$$

with $F(x_1, \dots, x_m)$ equal to some smooth real-valued function of the m variables on U . The integral of this m form, written as $\int_U F(x_1, \dots, x_m) dx_1 \wedge \cdots \wedge dx_m$, is defined simply to be the Lebesgue integral²¹

$$\int_U F(x_1, \dots, x_m) dx_1 \cdots dx_m$$

²¹Alternatively one can use the Riemann integral if the open set U has a sufficiently well behaved topological boundary. If the Lebesgue integral is used, there is no restriction on the topological boundary of the open set U .

with respect to Lebesgue measure. Notationally we just drop the signs \wedge . This integral raises some convergence questions, but we can avoid them either by assuming that ω has compact support in U or by working with the linear functional

$$f \mapsto \int_U f(x_1, \dots, x_m) F(x_1, \dots, x_m) dx_1 \cdots dx_m$$

defined for f in $C_{\text{com}}(U)$.

When $U = \mathbb{R}^m$, what happens to this definition of integration of m forms on \mathbb{R}^m if the variables are written in a different order? For example, suppose that the positions of x_1 and x_2 are interchanged. The coefficient F of ω is unchanged, but the alternating tensor becomes $dx_2 \wedge dx_1 \wedge dx_3 \wedge \cdots \wedge dx_m$, which is the negative of $dx_1 \wedge dx_2 \wedge dx_3 \wedge \cdots \wedge dx_m$. Meanwhile the Lebesgue integral is unchanged if we replace $dx_1 dx_2 dx_3 \cdots dx_m$ by $dx_2 dx_1 dx_3 \cdots dx_m$. So we are off by a minus sign. The answer to this seeming contradiction is that orientation is playing a role in the definition of integration of an m form, a role that does not show up in the notation.²²

Consider now any oriented smooth manifold M in the sense defined earlier in this section. The theorem below defines a notion of integration of top-degree differential forms that generalizes the one in open subsets of \mathbb{R}^m . After proving the theorem, we shall relate its statement to the Riesz Representation Theorem.²³

Theorem 1.29. If ω is a smooth m form on the *oriented* smooth manifold M of dimension $m \geq 0$, then there exists a unique linear functional $f \mapsto \int_M f \omega$ on the space $C_{\text{com}}(M)$ of continuous functions of compact support on M with the property that whenever (M_α, α) is a *positive* compatible chart with local coordinates $\alpha = (x_1, \dots, x_m)$ and f is a member of $C_{\text{com}}(M)$ supported in M_α , then the value of the linear functional on any f that is compactly supported in M_α is

$$\int_M f \omega = \int_{\alpha(M_\alpha)} (f \circ \alpha^{-1})(x_1, \dots, x_m) F_\alpha(x_1, \dots, x_m) dx_1 \cdots dx_m, \quad (*)$$

where α is given in local coordinates by (x_1, \dots, x_m) and the local expression for ω in the local coordinates of $\alpha(M_\alpha)$ is

$$(\alpha^{-1})^* \omega = F_\alpha(x_1, \dots, x_m) dx_1 \wedge \cdots \wedge dx_m \quad (**)$$

with $F_\alpha : \alpha(M_\alpha) \rightarrow \mathbb{R}$ smooth.²⁴ The integral on the right side of $(*)$ is understood to be an ordinary Lebesgue integral with respect to Lebesgue measure.

²²We shall use notation like $\int_M f \omega$ in this text, but notation like $\int_{M,o} f \omega$ that indicates an orientation o along with M and the integrand $f \omega$, might serve as a better reminder that the orientation affects the value.

²³Theorem 11.1 of *Basic Real Analysis*.

²⁴The notation $(\alpha_i^{-1})^* \omega$ is no mystery. It refers to the pullback of ω under α^{-1} , i.e., the “push forward” of ω from its domain M_α to the open set $\alpha(M_\alpha)$ in Euclidean space. In other words, it is indeed the “local expression for ω in the local coordinates.”

REMARKS.

(1) In other words the expression $\int_M f\omega$ is being defined by the right side of (*). The content of the theorem is that the definition does not depend on the choice of local coordinates.

(2) Theorem 1.29 remains true if “manifold” in the statement is replaced by “manifold-with-boundary,” which is a notion to be defined in Chapter II, or by “manifold-with-corners” or “Whitney manifold,” which are notions to be defined in Chapter III. The proof requires no change other than an updating of the reference to the existence of a partition of unity.

(3) Once again: In the definition of $\int_M f\omega$, the notation “ M ” includes both M and its orientation. If the orientation is changed, then the value of the integral may change. The orientation enters the statement of the theorem in the requirement that (M_α, α) be a positive compatible chart.

(4) For our purposes the main role of having f present in the formula is to relate integration of differential m forms to Lebesgue integration in measure theory. We shall have more to say about this point after the end of the proof of the theorem. In the applications of this theorem after this section in this book, all the m forms that are involved in integration will have compact support within the set of integration, and then inclusion of f in the formula becomes a frill. Accordingly we shall tend to drop f in applications of this formula after this section.

(5) With f dropped, the formula of the theorem can be written briefly as

$$\int_M \omega = \int_{M_\alpha} \omega = \int_{\alpha(M_\alpha)} (\alpha^{-1})^* \omega$$

if ω is compactly supported in M_α . Orientations are implicit throughout the three members of this equation, the orientation on the right side being the standard orientation on Euclidean space.

PROOF. Let us first dispose of the case $m = 0$. Then ω is a 0 form, which is a real-valued function on the points of the discrete space. The integral $\int_M f\omega$ is to be interpreted as the sum over the points of the product of the value of f by the value of ω times the value of the orientation at the point, namely ± 1 . This factor ± 1 is what by convention $F_\alpha(x_1, \dots, x_m) dx_1 \wedge \dots \wedge dx_m$ reduces to when $m = 0$.

For the remainder of the proof, assume that $m > 0$. Whenever f is compactly supported in M_α , then $f \circ \alpha^{-1}$ is compactly supported in $\alpha(M_\alpha)$ and the right side of (*) is well defined. Thus let us define

$$\int_{M_\alpha} f\omega = \int_{\alpha(M_\alpha)} (f \circ \alpha^{-1})(x_1, \dots, x_m) F_\alpha(x_1, \dots, x_m) dx_1 \cdots dx_m.$$

This definition satisfies a certain consistency condition. To see this, suppose that f is compactly supported in an intersection $M_\alpha \cap M_\beta$. Then by our definition we

have also

$$\int_{M_\beta} f \omega = \int_{\beta(M_\beta)} (f \circ \beta^{-1})(y_1, \dots, y_m) F_\beta(y_1, \dots, y_m) dy_1 \cdots dy_m. \quad (\dagger)$$

To see that the right sides of (*) and (†) are equal, we use the change of variables formula for multiple integrals.²⁵ The change of variables $y = \beta \circ \alpha^{-1}(x)$ in (*) expresses y_1, \dots, y_m as functions of x_1, \dots, x_m , and (†) therefore is

$$\begin{aligned} &= \int_{\alpha(M_\alpha \cap M_\beta)} f \circ \beta^{-1} \circ \beta \circ \alpha^{-1}(x_1, \dots, x_m) \\ &\quad \times F_\beta \circ \beta \circ \alpha^{-1}(x_1, \dots, x_m) \left| \det \left\{ \frac{\partial y_i}{\partial x_j} \right\}_{i,j=1,\dots,m} \right| dx_1 \cdots dx_m. \end{aligned}$$

The right side here will be equal to the right side of (*) if it is shown that

$$F_\alpha \stackrel{?}{=} (F_\beta \circ \beta \circ \alpha^{-1}) \left| \det \left\{ \frac{\partial y_i}{\partial x_j} \right\}_{i,j=1,\dots,m} \right|. \quad (\dagger\dagger)$$

Now

$$\begin{aligned} F_\alpha dx_1 \wedge \cdots \wedge dx_m &= (\alpha^{-1})^* \omega && \text{from (**)} \\ &= (\beta \circ \alpha^{-1})^* (\beta^{-1})^* \omega \\ &= (\beta \circ \alpha^{-1})^* (F_\beta dy_1 \wedge \cdots \wedge dy_m) && \text{from (**)} \\ &= (F_\beta \circ \beta \circ \alpha^{-1}) \det \left\{ \frac{\partial y_i}{\partial x_j} \right\}_{i,j=1,\dots,m} dx_1 \wedge \cdots \wedge dx_m \\ &&& \text{by Proposition 1.17.} \end{aligned}$$

Thus

$$F_\alpha = (F_\beta \circ \beta \circ \alpha^{-1}) \det \left\{ \frac{\partial y_i}{\partial x_j} \right\}_{i,j=1,\dots,m}. \quad (\ddagger)$$

Since $\det \left\{ \frac{\partial y_i}{\partial x_j} \right\}_{i,j=1,\dots,m}$ is everywhere positive, equality in (††) follows from (‡).

Therefore

$$\int_{M_\alpha} f \omega = \int_{M_\beta} f \omega$$

whenever f is compactly supported in $M_\alpha \cap M_\beta$.

²⁵Theorem 6.32 of *Basic Real Analysis*.

For future reference later in this section and also for use in the next chapter, we rewrite (‡) in terms of coordinates as

$$F_\alpha(y_1, \dots, y_m) = F_\beta(x_1, \dots, x_m) \det \left\{ \frac{\partial y_i}{\partial x_j} \right\}_{i,j=1,\dots,m}. \quad (\ddagger\ddagger)$$

To define $\int_M f \omega$ for general f in $C_{\text{com}}(M)$, we select finitely many open coordinate neighborhoods M_{α_i} that together cover the support of f , and we use Theorem 1.25 to form a smooth partition of unity $\{\psi_{\alpha_i}\}$ subordinate to the finite open cover $\{M_{\alpha_i}\}$ of the support of f . Then we can define

$$\int_M f \omega = \sum_i \int_{M_{\alpha_i}} (\psi_{\alpha_i} f) \omega. \quad (\S)$$

Let us see that this definition is unchanged if the smooth partition of unity is changed. Indeed, suppose that $\{M_{\beta_j}\}$ is a second finite open cover of the support of f . Let $\{\phi_{\beta_j}\}$ be a smooth partition of unity subordinate to the finite open cover $\{M_{\beta_j}\}$ of the support of f . Linearity of the Lebesgue integral allows us to write the right side of (§) as

$$= \sum_i \sum_j \int_{M_{\alpha_i}} (f \psi_{\alpha_i} \phi_{\beta_j}) \omega. \quad (\S\S)$$

If $f \psi_{\alpha_i} \phi_{\beta_j}$ is not identically 0, it is supported in M_{α_i} and also in M_{β_j} . The fact that (*) equals (†), which we proved above, means that we get the same result for $\int_M f \psi_{\alpha_i} \phi_{\beta_j}$ whether we treat f as a function supported in M_{α_i} or we treat it as a function supported in M_{β_j} , i.e.,

$$\int_{M_{\alpha_i}} (f \psi_{\alpha_i} \phi_{\beta_j}) \omega = \int_{M_{\beta_j}} (f \psi_{\alpha_i} \phi_{\beta_j}) \omega.$$

Thus (§§) is

$$= \sum_j \sum_i \int_{M_{\beta_j}} (f \psi_{\alpha_i} \phi_{\beta_j}) \omega = \sum_j \int_{M_{\beta_j}} (f \phi_{\beta_j}) \omega,$$

and this is the value of $\int_M f \omega$ we get by using the partition of unity $\{\phi_{\beta_j}\}$. \square

When ω is fixed, it is apparent from (§) and the integral formula for $\int_{M_{\alpha_i}} (\psi_{\alpha_i} f) \omega$ that the map $f \mapsto \int_M f \omega$ is a linear functional on $C_{\text{com}}(M)$. In dimension $m \geq 1$, we say that the m form ω is **everywhere positive** relative to the given atlas if each local expression (**) has $F_\alpha(x_1, \dots, x_m)$ everywhere positive on $\alpha(U_\alpha)$. In dimension 0, a 0 form ω is interpreted as everywhere positive if the pointwise product of ω and the orientation is everywhere positive.

When ω is everywhere positive, the linear functional $f \mapsto \int_M f \omega$ is positive in the sense that $f \geq 0$ implies $\int_M f \omega \geq 0$. By the Riesz Representation Theorem,²⁶ there exists a unique (regular²⁷) Borel measure $d\mu_\omega$ on M such that

$$\int_M f \omega = \int_M f(x) d\mu_\omega(x)$$

for all $f \in C_{\text{com}}(M)$. The next two propositions tell how to create and recognize everywhere positive m forms ω .

Proposition 1.30. If an m dimensional manifold M with $m \geq 1$ admits a nowhere-vanishing m form ω , then M can be oriented so that ω is everywhere positive. Conversely if M is oriented, then M admits a nowhere-vanishing m form ω .

REMARKS. This proposition will allow us to classify the possible ways of orienting a smooth manifold M of dimension $m \geq 1$. An **orientation** of M is an equivalence class of nowhere-vanishing m forms on M , two such being equivalent if each is an everywhere positive function times the other. Indeed, the constructions in the proof below show that any nowhere-vanishing m form yields an atlas of compatible charts exhibiting M as oriented, that equivalent such forms lead to the same atlas, and that inequivalent such forms lead to distinct atlases. If a given orientation of M comes from a nowhere-vanishing m form ω_0 , then the orientation that corresponds to $-\omega_0$ is called the **opposite orientation** to the given one. In Theorem 1.29, changing matters so that the oriented manifold M has the opposite orientation has the effect of multiplying $\int_M f \omega$ by -1 .

PROOF. Suppose that M admits a nowhere-vanishing m form ω . Let $\{(M_\alpha, \alpha)\}$ be any atlas for M . The components of each M_α are open and cover M_α , and there is no loss of generality in assuming that each M_α is connected. For each M_α , let F_α be the function in (**) of Theorem 1.29 in the local expression for ω in $\alpha(M_\alpha)$. Specifically

$$(\alpha^{-1})^* \omega = F_\alpha(x_1, \dots, x_m) dx_1 \wedge \cdots \wedge dx_m$$

with $F_\alpha : \alpha(M_\alpha) \rightarrow \mathbb{R}$ smooth. Since ω is nowhere vanishing and M_α is connected, F_α has constant sign on $\alpha(M_\alpha)$. If the sign is positive, we retain (M_α, α) in the atlas. If the sign is negative, we redefine²⁸ α by following it with the map $(x_1, x_2, \dots, x_m) \mapsto (-x_1, x_2, \dots, x_m)$, and then the redefined F_α is everywhere positive; in this case we instead include the redefined (M_α, α) in

²⁶Theorem 11.1 of *Basic Real Analysis*.

²⁷On any separable locally compact Hausdorff space, and in particular on any smooth manifold, all Borel measures are regular.

²⁸This redefinition is possible since $m \geq 1$.

the atlas. In this way we can arrange that all F_α are everywhere positive on their domains. Referring to (§§) in the proof of Theorem 1.29, we see that each function $\det \left(\frac{\partial y_i}{\partial x_j} \right)$ is positive on its domain. Hence M is oriented. Since the F_α are all everywhere positive, ω is everywhere positive relative to this orientation.

Conversely suppose that M is oriented. Let $\{(M_\alpha, \alpha)\}$ be an atlas that exhibits M as oriented. For each α , define a smooth differential m form ω_α on M_α by

$$\omega_\alpha = \alpha^*(dx_1 \wedge \cdots \wedge dx_m).$$

If the intersection $M_\alpha \cap M_\beta$ is nonempty, then points in the intersection have also

$$\begin{aligned} \omega_\beta &= \beta^*(dy_1 \wedge \cdots \wedge dy_m) \\ &= (\beta \circ \alpha^{-1} \circ \alpha)^*(dy_1 \wedge \cdots \wedge dy_m) \\ &= \alpha^*(\beta \circ \alpha^{-1})^*(dy_1 \wedge \cdots \wedge dy_m) && \text{by Proposition 1.18f} \\ &= \alpha^*(\beta \circ \alpha^{-1}) \det \left\{ \frac{\partial y_i}{\partial x_j} \right\} dx_1 \wedge \cdots \wedge dx_m && \text{by Proposition 1.17} \\ &= \det \left\{ \frac{\partial y_i}{\partial x_j} \right\} \alpha^*(dx_1 \wedge \cdots \wedge dx_m) \\ &= \det \left\{ \frac{\partial y_i}{\partial x_j} \right\} \omega_\alpha. \end{aligned}$$

In other words,

$$\omega_\beta(p) = \lambda_{\beta\alpha}(p)\omega_\alpha(p) \quad (*)$$

for all points $p \in M_\alpha \cap M_\beta$ and some everywhere-positive function $\lambda_{\beta\alpha}$.

Let K be a compact subset of M to be specified. The various open sets M_α of charts cover K , we let $\{\alpha_1, \dots, \alpha_k\}$ be a finite subcover, and we use Theorem 1.25 to choose a smooth partition of unity $\{\psi_{\alpha_i}, 1 \leq i \leq k\}$ of M subordinate to the finite open cover $\{M_{\alpha_i}, 1 \leq i \leq k\}$ of K . Let $\omega = \sum_{i=1}^k \psi_{\alpha_i} \omega_{\alpha_i}$. The m form ω_{α_i} is nowhere-vanishing on M_{α_i} , being the pullback to M_{α_i} from $\alpha_i(M_{\alpha_i})$ of a nowhere-vanishing differential form on $\alpha_i(M_{\alpha_i})$. We can extend its domain to all of M by setting it equal to 0 off M_{α_i} , and then the product $\psi_{\alpha_i} \omega_{\alpha_i}$ is a smooth m form on M . Hence the sum ω is smooth on M .

Consider any point p in K . Since $\sum \psi_{\alpha_i} = 1$ and each $\psi_{\alpha_i} \geq 0$, some $\psi_{\alpha_i}(p)$ is nonzero. Then also $\psi_{\alpha_i} \omega_{\alpha_i}(p)$ is nonzero. If any other index j has $\psi_{\alpha_j}(p) \neq 0$, then (*) shows that $\psi_{\alpha_j} \omega_{\alpha_j}(p)$ is a positive multiple of $\psi_{\alpha_i} \omega_{\alpha_i}(p)$. Then it follows that $\omega(p)$ is not zero. In other words, the m form ω is nowhere vanishing on K .

If M were compact, we would be done at this point. In the general case we begin with the following observation: if we had specified in advance an open set U containing K , we could have arranged that ω vanishes at all points outside U simply by multiplying ω by a smooth function that is 1 on K and is 0 off U ; Lemma 1.27 provides such a function. That being so, let $E_0 = \emptyset \subset E_1 \subset E_2 \subset \cdots$ be an exhausting sequence of compact subsets of M . Each set E_j is contained in the interior E_{j+1}° of the next member of the sequence. For each $j \geq 0$, repeat the above procedure for the compact set $E_{j+1} - E_j^\circ$ in place of K , obtaining a smooth differential form ω_j , and arrange that the form ω_j vanishes off E_{j+2}° . The form ω_j is nowhere vanishing on $E_{j+1} - E_j^\circ$, and the coefficients of the forms all have the same sign at all points where any of them is nonzero. Each point p has some j for which p is in $E_{j+1} - E_j^\circ$, and then a neighborhood of p lies in $E_{j+2}^\circ - E_j^\circ$. The points of that neighborhood are all outside $E_{k+1} - E_k^\circ$ for $k \geq j+2$ and $k+1 \leq j$. Thus that neighborhood meets at most the three sets $E_{j+2} - E_{j+1}^\circ$, $E_{j+1} - E_j^\circ$, and $E_j - E_{j-1}^\circ$ in the sequence. Consequently $\omega = \sum_{j=0}^{\infty} \omega_j$ is a well defined smooth m form. The form ω is nonvanishing at least at all points of $\bigcup_{j=0}^{\infty} (E_{j+1} - E_j^\circ)$; in other words, ω is nowhere vanishing. \square

Proposition 1.31. If a connected manifold M is oriented and if ω is a nowhere-vanishing smooth m form on M , then either ω is everywhere positive or $-\omega$ is everywhere positive.

REMARKS. The proposition says that the problem of finding nowhere-vanishing forms of the top degree m can be solved one connected component at a time: the manifold M is orientable if and only if each connected component is orientable, a connected component is orientable if and only if it has two equivalence classes of nowhere-vanishing m forms rather than just one, and nonvanishing m forms can be assembled for M one component at a time in arbitrary fashion.

PROOF. At each point p of M , all the functions F_α representing ω locally by means of a positive compatible chart as in (***) of the statement of Theorem 1.29 have $F_\alpha(\alpha(p))$ nonzero of the same sign because of (†††), the nowhere-vanishing of ω , and the fact that M is oriented. Let S be the subset of M where this common sign is positive. Possibly replacing ω by $-\omega$, we may assume that S is nonempty. We show that S is open and closed. Let p be in S and let (M_{α_0}, α_0) be a positive compatible chart about p . Then $F_{\alpha_0}(\alpha_0(p)) > 0$ since p is in S , and hence $F_\alpha(\alpha(q))$ is positive at all points q in the neighborhood M_{α_0} of p for the one value α_0 of α . Since the sign is the same for the α 's of all positive compatible charts, $F_\alpha(\alpha(q)) > 0$ for all α such that q is in $M_{\alpha_0} \cap M_\alpha$. Hence S is open. Let $\{p_n\}$ be a sequence in S converging to p in M , and let (M_{α_0}, α_0) be a positive compatible chart about p . Then $F_{\alpha_0}(\alpha_0(p_n)) > 0$ for large n , and hence $F_{\alpha_0}(\alpha_0(p)) \geq 0$ by continuity. Since ω is nowhere vanishing, $F_{\alpha_0}(\alpha_0(p)) > 0$.

Since the sign is the same for all positive compatible charts, $F_\alpha(\alpha(p))$ is > 0 for all α . Therefore p is in S , and S is closed. Since M is connected and S is nonempty open closed, $S = M$. \square

Propositions 1.30 and 1.31 together give us a better understanding of the notion of positive chart that was defined just before the four examples in this section. If M is connected and orientable, then there are exactly two possibilities for a nowhere-vanishing form of top degree m up to equivalence, and these are negatives of each other. If we fix the orientation, say in terms of the m form ω , then the positive compatible charts (M_α, α) are exactly the charts for which $(\alpha^{-1})^*\omega$ is a positive function times $dx_1 \wedge \cdots \wedge dx_m$. The set of such positive charts is an atlas.

Let us now examine the effect of mappings on orientation. Because orientation is determined by a differential m form η , we can check the effect of a mapping Φ by examining the pullback $\Phi^*\eta$. The situation is clearest in the case of a diffeomorphism.

Let M and N be oriented smooth manifolds of dimension m , and let $\Phi : M \rightarrow N$ be a diffeomorphism. If η is a nowhere-vanishing m form on N , then $\Phi^*\eta$ will be an m form on M , and Proposition 1.17 shows that it is nowhere vanishing. In fact, we can argue locally, writing η in local coordinates as the wedge product of m nowhere-vanishing 1 forms. Then Proposition 1.17 gives a local expression for $\Phi^*\eta$ as the wedge product of nowhere-vanishing 1 forms on M . Consequently the globally defined m form $\Phi^*\eta$ is nowhere vanishing.

We say that Φ is **orientation preserving** if whenever the nowhere-vanishing m form η is everywhere positive, then the nowhere-vanishing m form $\Phi^*\eta$ is everywhere positive. Similarly Φ is **orientation reversing** if whenever the nowhere-vanishing m form η is everywhere positive, then the nowhere-vanishing m form $\Phi^*\eta$ is everywhere negative. If Φ is orientation preserving, then for every positive chart (M_α, α) in the atlas for M , the chart $(\Phi(M_\alpha), \alpha \circ \Phi^{-1})$ is positive relative to the atlas for N . Consequently the atlas of compatible charts for N can be taken to be $\{(\Phi(M_\alpha), \alpha \circ \Phi^{-1})\}$. Then the change of variables formula for multiple integrals may be expressed using pullbacks as in the following proposition.

Proposition 1.32. Let M and N be oriented manifolds of dimension m , and let $\Phi : M \rightarrow N$ be an orientation-preserving diffeomorphism. If ω is any smooth m form on N , then

$$\int_N f\omega = \int_M (f \circ \Phi)\Phi^*\omega$$

for every $f \in C_{\text{com}}(N)$.

PROOF. Let the atlas for M be $\{(M_\alpha, \alpha)\}$, and take the atlas for N to be $\{(\Phi(M_\alpha), \alpha \circ \Phi^{-1})\}$. It is enough to prove the result for f compactly supported in a particular $\Phi(M_\alpha)$. For such f , Theorem 1.29 gives

$$\int_N f \omega = \int_{\alpha \circ \Phi^{-1}(\Phi(M_\alpha))} f \circ \Phi \circ \alpha^{-1}(x_1, \dots, x_m) F_\alpha(x_1, \dots, x_m) dx_1 \cdots dx_m, \quad (*)$$

where F_α is the function with

$$((\alpha \circ \Phi^{-1})^{-1})^* \omega = F_\alpha(x_1, \dots, x_m) dx_1 \wedge \cdots \wedge dx_m. \quad (**)$$

The function $f \circ \Phi$ is compactly supported in M_α , and Theorem 1.29 gives also

$$\int_M (f \circ \Phi) \Phi^* \omega = \int_{\alpha(M_\alpha)} f \circ \Phi \circ \alpha^{-1}(x_1, \dots, x_m) F_\alpha(x_1, \dots, x_m) dx_1 \cdots dx_m \quad (\dagger)$$

since

$$(\alpha^{-1})^* \Phi^* \omega = ((\alpha \circ \Phi^{-1})^{-1})^* \omega = F_\alpha(x_1, \dots, x_m) dx_1 \wedge \cdots \wedge dx_m$$

by (**). The right sides of (*) and (\dagger) are equal, and hence so are the left sides. \square

The above discussion of diffeomorphisms and pullbacks extends to “immersions” between two smooth manifolds of the same dimension. If M and N are smooth manifolds, then an **immersion** Φ of M into N is a smooth function, not necessarily one-one, of M into N such that the derivative $D\Phi(p)$ is one-one from $T_p(M)$ into $T_{\Phi(p)}(N)$ for each point p in M . In this case when M and N have the same dimension, then the same argument as above shows for each nowhere-vanishing m form η on N that $\Phi^* \eta$ is a nowhere-vanishing m form on M . The next proposition is a consequence.

Proposition 1.33. The Möbius band of Example 3 in this section is not orientable.

PROOF. We assume that the Möbius band M has already been shown to be a manifold; this step is carried out in Problem 29 at the end of the chapter. To address orientability, we consider M as defined directly in terms of the parameters (s, t) in Example 3, rather than as a parametrically defined subset of \mathbb{R}^3 . In the setup of the example, the subset $\mathbb{R} \times (-1, 1)$ gets mapped onto M in such a way that (s, t) maps to the same point of M as $(s + 2\pi, -t)$, and hence also to the same point as $(s + 4\pi, t)$. We carry out this process in two stages. In the first stage we pass from the manifold $\mathbb{R} \times (-1, 1)$ to the manifold $S^1 \times (-1, 1)$ by taking the remainder modulo 4π in the s variable. Representatives of members of the image are the pairs (s, t) with $0 \leq s < 4\pi$ and $-1 < t < 1$. In the second stage

we identify any pair (s, t) with the pair $(s + 2\pi, -t)$. This carries $S^1 \times (-1, 1)$ onto M and is a smooth 2-to-1 mapping that we call Φ ; it is an immersion.

If we write h for the function that interchanges each pair (s, t) in $S^1 \times (-1, 1)$ with its mate $(s + 2\pi, -t)$, with $s + 2\pi$ understood to be adjusted by 4π if necessary so that it lies in $[0, 4\pi)$, then h is a diffeomorphism of $S^1 \times (-1, 1)$ onto itself that satisfies $h^*(ds \wedge dt) = -ds \wedge dt$. In other words, h is orientation reversing. Moreover, we have $\Phi = \Phi \circ h$. Arguing by contradiction, suppose that M is orientable. Then Proposition 1.30 supplies a nowhere-vanishing differential 2 form η on it. Passing to pullbacks from the equation $\Phi = \Phi \circ h$, we obtain

$$\Phi^*\eta = (\Phi h)^*\eta = h^*\Phi^*\eta.$$

The 2 form $\Phi^*\eta$, being nowhere vanishing, has to equal $F ds \wedge dt$ for some nowhere-vanishing function F on $S^1 \times (-1, 1)$. Then we are led to

$$F ds \wedge dt = h^*(F ds \wedge dt) = h^*(F)h^*(ds \wedge dt) = -(F \circ h) ds \wedge dt,$$

which is a contradiction since F has constant sign. \square

7. Problems

1. Show that if $K_1 \subset K_2 \subset K_3 \subset \dots$ is an exhausting sequence for a smooth manifold M and if C is a compact subset of M , then there is some j such that $C \subseteq K_j$.
2. The circle $S^1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$ was defined as a smooth manifold of dimension 1 in Section 1 by means of two charts (C_1, φ_1) and (C_2, φ_2) , where

$$\begin{aligned} C_1 &= S^1 - \{(0, +1)\} \quad \text{and} \quad \varphi_1(x, y) = \left(\frac{x}{1-y}\right), \\ C_2 &= S^1 - \{(0, -1)\} \quad \text{and} \quad \varphi_2(x, y) = \left(\frac{x}{1+y}\right). \end{aligned}$$

In Example 2 in Section 6, it was defined by means of two charts

$$\begin{aligned} M_1 &= S^1 - \{(-1, 0)\} \quad \text{and} \quad \psi_1(\cos t, \sin t) = t \quad \text{for} \quad -\pi < t < \pi, \\ M_2 &= S^1 - \{(+1, 0)\} \quad \text{and} \quad \psi_2(\cos t, \sin t) = t \quad \text{for} \quad 0 < t < 2\pi. \end{aligned}$$

What steps need to be carried out to show that these smooth manifolds are the same? Carry out one such step.

3. Set-theoretically, the real n dimensional projective space $M = \mathbb{R}P^n$ can be defined as the result of identifying each member x of the unit sphere S^n in \mathbb{R}^{n+1} with its antipodal point $-x$. Let $[x] \in \mathbb{R}P^n$ denote the class of $x \in S^n$. Do the following:
- (a) Show that $d([x], [y]) = \min\{|x - y|, |x + y|\}$ is well defined and makes $\mathbb{R}P^n$ into metric space such that the function $x \mapsto [x]$ is continuous and carries open sets to open sets.
- (b) For each j with $1 \leq j \leq n + 1$, define

$$\alpha_j[(x_1, \dots, x_{n+1})] = \left(\frac{x_1}{x_j}, \dots, \frac{x_{j-1}}{x_j}, \frac{x_{j+1}}{x_j}, \dots, \frac{x_{n+1}}{x_j} \right)$$

on the domain $M_{\alpha_j} = \{(x_1, \dots, x_{n+1}) \mid x_j \neq 0\}$. Show that the system $\{(M_{\alpha_j}, \alpha_j) \mid 1 \leq j \leq n + 1\}$ is an atlas for $\mathbb{R}P^n$ and that the function $x \mapsto [x]$ from S^n to $\mathbb{R}P^n$ is smooth.

4. Prove that if p and q are two points in a connected smooth manifold, then there exists a diffeomorphism of the manifold mapping p to q .
5. The **product** of two manifolds M and N with respective atlases $\{(M_\alpha, \alpha)\}$ and $\{(N_\beta, \beta)\}$ is the set $M \times N$ with an atlas consisting of all charts $(M_\alpha \times N_\beta, \alpha \times \beta)$.
- (a) Show that $M \times N$ is a smooth manifold and that the projections $M \times N \rightarrow M$ and $M \times N \rightarrow N$ are smooth.
- (b) Show that if p is in M and q is in N , then the maps $i_p : N \rightarrow M \times N$ and $j_q : M \rightarrow M \times N$ given by $i_p(n) = (p, n)$ and $j_q(m) = (m, q)$ are smooth immersions.
6. Prove in \mathbb{R}^3 that if f is real-valued and F is vector-valued, then $\operatorname{div} \operatorname{curl} F = 0$ and that $\operatorname{curl} \operatorname{grad} f = 0$.
7. Prove by induction on the dimension that if ω is a smooth differential 1 form on \mathbb{R}^n with $d\omega = 0$, then $\omega = df$ for some smooth real-valued function f defined on \mathbb{R}^n .
8. The proof in Problem 7 has to depend on special properties of \mathbb{R}^n as the domain of ω because of the following example: Let ω be the 1 form on $\mathbb{R}^2 - \{(0, 0)\}$ defined by

$$\omega = \frac{-y}{x^2 + y^2} dx + \frac{x}{x^2 + y^2} dy.$$

Define a function θ on $\mathbb{R}^2 - \{(x, 0) \mid x \geq 0\}$ by

$$\theta(x, y) = \begin{cases} \arctan y/x & \text{if } x > 0 \text{ and } y > 0 \\ \pi + \arctan y/x & \text{if } x < 0 \\ 2\pi + \arctan y/x & \text{if } x > 0 \text{ and } y < 0 \\ \pi/2 & \text{if } x = 0 \text{ and } y > 0 \\ 3\pi/2 & \text{if } x = 0 \text{ and } y < 0, \end{cases}$$

where \arctan is the inverse function on \mathbb{R} to \tan on $(-\pi/2, \pi/2)$.

- (a) Verify that $d\omega = 0$ on the domain of ω .
 - (b) Verify that if f is smooth on the domain of θ and if $\omega = df$ there, then f and θ have respective first partial derivatives equal on the domain of θ .
 - (c) Observe that a function f as in (b) has to be $f = \theta + \text{constant}$ on the domain of θ and cannot extend continuously to $\mathbb{R}^2 - \{(0, 0)\}$. Conclude that the equation $df = \omega$ has no smooth solution f on $\mathbb{R}^2 - \{(0, 0)\}$.
9. If E and F are disjoint compact subsets of a smooth manifold M , prove that there exist functions $f \geq 0$ and $g \geq 0$ in $C_{\text{com}}^\infty(M)$ such that f is identically 1 on E and identically 0 on F and such that g is identically 0 on E and identically 1 on F .
10. Let U be a nonempty connected open set in \mathbb{R}^m . Call a smooth k form ω on U **elementary** if it can be written as

$$\omega = d\varphi_1 \wedge \cdots \wedge d\varphi_k$$

for some set of k functions in $C^\infty(U)$.

- (a) Prove that in this case, $\omega = d\eta$ for some smooth $k - 1$ form η .
- (b) Prove that any k form ω on U that can be written as

$$\omega = f_1(x_1) f_2(x_2) \cdots f_k(x_k) dx_1 \wedge \cdots \wedge dx_k$$

is elementary.

11. Let $\Phi : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be defined by $\Phi(r, s, t) = (r + s + t, rs + st + rt, rst)$. Compute $\Phi^*(dx \wedge dy)$ in terms of $dr \wedge ds$, $dr \wedge dt$, and $ds \wedge dt$.

Problems 12–18 introduce the notion of “contraction” (also called “interior multiplication”) of a smooth differential form by a smooth vector field and use it to analyze the orientability of spheres. Let M be a smooth manifold and let X and X_1, \dots, X_{k-1} be smooth vector fields on M . If ω is a smooth differential k form on M , i.e., a member of $\Omega^k(M)$, then the **contraction** $c_X(\omega)$ of ω by X is defined pointwise on M by

$$c_X(\omega)_p((X_1)_p, \dots, (X_{k-1})_p) = k \omega_p((X)_p, (X_1)_p, \dots, (X_{k-1})_p).$$

12. Under the hypotheses above, expand $c_X(\omega)$ and the smooth vector fields within each chart by using the methods of Sections 1 and 3, and conclude that $c_X(\omega)$ is smooth and therefore c_X carries $\Omega^k(M)$ into $\Omega^{k-1}(M)$. Check also that $c_X(\omega)$ is $C^\infty(M)$ linear in the X variable.
13. (a) Prove for $k \geq 1$ and for all $\omega_1, \dots, \omega_k$ in $\Omega^1(M)$ that

$$c_X(\omega_1 \wedge \dots \wedge \omega_k) = \sum_{i=1}^k (-1)^{i-1} \omega_i(X) (\omega_1 \wedge \dots \wedge \widehat{\omega}_i \wedge \dots \wedge \omega_k).$$

- (b) Deduce as a consequence that

$$c_X(\omega \wedge \eta) = c_X(\omega) \wedge \eta + (-1)^k \omega \wedge c_X(\eta)$$

if ω is in $\Omega^k(M)$ and η is in $\Omega^l(M)$.

14. Show that if $i : S \rightarrow M$ a one-one smooth immersion between manifolds and if ω is in $\Omega^k(M)$, then the member $i^*(\omega)$ of $\Omega^k(S)$ can be regarded as the restriction of ω to S . (This problem will be applied shortly to the immersion $i : S^n \rightarrow \mathbb{R}^{n+1}$.)
15. Show that if a connected smooth manifold M has an atlas with just two charts and the charts have connected intersection, then M is orientable. Deduce that the unit sphere S^n in \mathbb{R}^{n+1} is orientable if $n \geq 2$. A separate argument is needed to see that S^1 is orientable. The next three problems will produce an explicit nowhere-vanishing smooth n form on S^n , which has to exist by Proposition 1.30.
16. Let $i : S^n \rightarrow \mathbb{R}^{n+1}$ be the inclusion mapping, which is a one-one smooth immersion. For any point $p = (x_1, \dots, x_{n+1})$ in S^n and its image $i(p)$ in \mathbb{R}^{n+1} , check in two ways that \mathbb{R}^{n+1} is the direct sum of the tangent space to S^n at p and the 1 dimensional space $\mathbb{R}p$:
- (a) First check via inner products and linear algebra, using the naive geometric interpretation of the tangent space as being geometrically tangent to the sphere at p .
- (b) Second check via the definitions in this chapter of notions related to “tangent space.” Specifically let $r = (r_1, \dots, r_{n+1})$ be any member of \mathbb{R}^{n+1} such that the dot product $p \cdot r$ equals 0. Define a smooth curve γ_r in S^n for $|t| < \epsilon$ with $\epsilon > 0$ sufficiently small by

$$\gamma_r(t) = \frac{p + tr}{|p + tr|}.$$

Observe that $f \mapsto \frac{d}{dt} f(\gamma_r(t))|_{t=0}$ defines a derivation of the space of germs of smooth functions at p on S^n and therefore is a member X_r of $T_p(S^n)$. Show that the mapping $r \mapsto X_r$ is linear in r and is one-one, hence is onto $T_p(S^n)$. By dimensionality, conclude that $T_p(\mathbb{R}^{n+1}) = i(T_p(S^n)) \oplus \mathbb{R}X$, where $X = \{X_p\}$ is the vector field with $X_p = \sum_{j=1}^{n+1} x_j \frac{d}{dx_j}$ in $T_p(\mathbb{R}^{n+1})$.

17. With i , p , and X as in the previous problem, let $\omega = dx_1 \wedge \cdots \wedge dx_{n+1}$ on \mathbb{R}^{n+1} . Define a smooth n form η on S^n by $\eta = i^*(c_X(\omega))$. Using the results of Problems 14 and 16, prove that the n form η on S^n is nowhere vanishing.
18. With $M = \mathbb{R}^{n+1}$ and $\omega = dx_1 \wedge \cdots \wedge dx_{n+1}$, compute $c_X(\omega)$ for $X = \sum_{j=1}^{n+1} x_j(\partial/\partial x_j)$, showing that

$$c_X(\omega) = \sum_{j=1}^{n+1} (-1)^{j-1} x_j dx_1 \wedge \cdots \wedge \widehat{dx_j} \wedge \cdots \wedge dx_{n+1}.$$

(The differential form η of the previous problem involves also an application of i^* . Problem 14 observes that this application is just a matter of restricting domains, and it is customary not to incorporate it into the explicit notation.)

Problems 19–23 treat in a more general setting the orientation question that Proposition 1.33 settled for the Möbius band. Let M be a connected smooth manifold of dimension m , and let h be a diffeomorphism of M onto itself such that $h^2 = 1$ and such that $h(x) = x$ for no x .

19. (a) For x and y in M , define $x \sim y$ if $x = y$ or $y = h(x)$. Show that \sim is an equivalence relation.
- (b) Write $[x]$ for the equivalence class of x , and let N denote the set of equivalence classes. If $d(x, y)$ is a metric for M such that $d(h(x), h(y)) = d(x, y)$, prove that the formula $d_0([x], [y]) = \min\{d(x, y), d(x, h(y))\}$ defines d_0 as a metric on N in such a way that the function $x \mapsto [x]$ of M onto N is continuous and open.
- (c) Show how to define charts that make the metric space N into a smooth manifold of dimension m for which that the quotient map $h(x) = [x]$ of M onto N is smooth and is an immersion.
20. Guided by the proof of Proposition 1.33, prove that if M is oriented and h is orientation reversing, then N is not orientable.
21. Using the charts constructed in Problem 19c, prove that if M is oriented and h is orientation preserving, then N is orientable.
22. The real projective space $\mathbb{R}P^n$ is defined in Problem 3 and also arises from Problem 19 when M is taken to be the sphere S^n and h is taken to be the antipodal map $h(x) = -x$.
- (a) Show that the smooth structures defined on $\mathbb{R}P^n$ by means of Problems 3 and 19c are the same.
- (b) The sphere S^n in \mathbb{R}^{n+1} is orientable for $n \geq 1$ by Problem 15, and Problems 17–18 exhibited a nowhere-vanishing n form on it. Show that the antipodal map of $h : S^n \rightarrow S^n$ is orientation reversing if n is even and is orientation preserving if n is odd.

23. Conclude from Problems 20–22 that $\mathbb{R}P^n$ is orientable if n is odd and ≥ 1 and that it is not orientable if n is even and ≥ 2 .

Problems 24–30 concern graphs, smooth immersions, “submanifolds,” and “embeddings.” A **submanifold** of a smooth manifold M is a subset S that has a smooth manifold structure of its own for which the inclusion $i : S \rightarrow M$ is a one-one immersion. A submanifold S of the manifold M is said to be **embedded** if the inclusion is a homeomorphism of S onto its image in M , i.e., if the manifold topology for S coincides with the subspace topology.

24. Let U be a nonempty open set in \mathbb{R}^n , and let $f : U \rightarrow \mathbb{R}^k$ be a continuous function, not necessarily smooth. The **graph** of f , written $\text{Graph}(f)$, is the subset of \mathbb{R}^{n+k} of all points $(x, f(x))$ for x in U . Make $\text{Graph}(f)$ into a smooth manifold with an atlas having just one chart, defined as (U, α) with $\alpha(x) = (x, f(x))$.
- (a) Verify that the mapping of U onto $\text{Graph}(f)$ given by $\alpha(x) = (x, f(x))$ is a diffeomorphism of U onto $\text{Graph}(f)$.
- (b) Let $I : \text{Graph}(f) \rightarrow U \times \mathbb{R}^k$ be the inclusion mapping, and let $p : U \times \mathbb{R}^k \rightarrow \mathbb{R}^k$ be the projection to the second coordinate. Then the composition of the maps

$$U \xrightarrow{\alpha} \text{Graph}(f) \xrightarrow{I} U \times \mathbb{R}^k \xrightarrow{p} \mathbb{R}^k$$

is $x \mapsto f(x)$, which need not be smooth. What is going on?

25. Let U be a nonempty open set in \mathbb{R}^n , and let $f : U \rightarrow \mathbb{R}$ be a smooth function. Define $\varphi : U \times \mathbb{R}^k \rightarrow U \times \mathbb{R}^k$ by $\varphi(x, y) = (x, y - f(x))$.
- (a) Verify that φ is a diffeomorphism.
- (b) Observe that $\varphi(\text{Graph}(f)) = \{(u, v) \in U \times \mathbb{R}^k \mid v = 0\}$. In other words, $\text{Graph}(f)$ is exhibited as the level set for level $v = 0$ in \mathbb{R}^k of the smooth function φ .
26. Let $\gamma : (-\pi/2 \rightarrow 3\pi/2) \rightarrow \mathbb{R}^2$ be the function given by $\gamma(t) = (\sin 2t, \cos t)$. Its image looks something like the numeral 8 and is pictured in Figure 1.3. Show that γ is a one-one immersion, that its image is compact, and that it is not a smooth embedding.

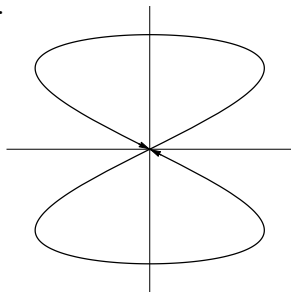


FIGURE 1.3. Numeral 8 from a one-one smooth immersion.

27. View S^1 as the set of elements in \mathbb{C} of the form $e^{i\theta}$ for θ in \mathbb{R} , define the 2 dimensional torus T^2 to be the product $S^1 \times S^1$, and fix an irrational real number c . This problem observes that $\gamma(t) = (e^{2\pi it}, e^{2\pi ict})$ is a one-one immersion from \mathbb{R} into T^2 but is not a smooth embedding. Its image is therefore a submanifold of T^2 but not an embedded submanifold.
- Check that indeed $\gamma(t)$ is one-one and is an immersion.
 - Show for each $\epsilon > 0$ that some nonzero integer k has $|\gamma(k) - \gamma(0)| < \epsilon$.
 - Deduce that $\gamma(0)$ is a limit point of $\gamma(\mathbb{Z})$, and conclude that γ is not a homeomorphism with its image and therefore cannot be a smooth embedding.
28. This problem gives a mechanism for defining a manifold parametrically, i.e., as the image of a vector-valued function of several variables.
- Let F be the smooth function from an interval of \mathbb{R} into \mathbb{R}^2 given by $F(t) = \begin{pmatrix} x \\ y \end{pmatrix}$. Suppose that $x'(t_0) \neq 0$. Prove that the set of points $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$ for t near t_0 is the embedded graph of a smooth function and is therefore an embedded submanifold in \mathbb{R}^2 of dimension 1.
 - Let U be a nonempty open subset of \mathbb{R}^n , let $F : U \rightarrow \mathbb{R}^k$ be a smooth function with $n < k$, and let $J(x)$ be the n -by- k Jacobian matrix of F at $x \in U$ with entries $\partial F_i / \partial x_j$. Suppose for each $x \in U$ that the rank of the matrix $J(x)$ is n , i.e., that $J(x)$ has n linearly independent columns. Use the Inverse Function Theorem to show for each x_0 in U that the set of points $F(x)$ in \mathbb{R}^k for x near x_0 is an embedded submanifold in \mathbb{R}^k of dimension n .
29. This problem constructs the Möbius band of Example 3 in Section 6 as a smooth 2 dimensional manifold in \mathbb{R}^3 . (See Figure 1.2 for a picture.)
- Example 3 of Section 6 explicitly defines three functions x, y, z as functions of the pair (s, t) for $-\infty < s < \infty$ and $-1 < t < 1$. Show that the Jacobian matrix of the function $(s, t) \mapsto (x, y, z)$ has rank two at every point (s, t) , i.e., that the columns of the Jacobian matrix are linearly independent for each pair (s, t) .
 - For fixed t , the functions x, y, z are periodic functions of period 4π in the variable s . Explain why this means that the function (x, y, z) of (s, t) descends to a smooth function into \mathbb{R}^3 with domain $M = \mathbb{R}/4\pi\mathbb{Z} \times (-1, 1)$.
 - Conclude that the image of the smooth function in (b) is a smooth manifold of dimension 2.
30. This problem gives a mechanism for defining a manifold implicitly, i.e., as the 0 locus of a vector-valued function of several variables.

- (a) The unit circle in \mathbb{R}^2 is the set where $x^2 + y^2 = 1$. Define $F(x, y) = x^2 + y^2 - 1$, so that the circle is the set where $F(x, y) = 0$. The Jacobian matrix of F is

$$J(x, y) = \begin{pmatrix} \frac{\partial F}{\partial x} & \frac{\partial F}{\partial y} \end{pmatrix} = (2x \quad 2y).$$

Explain how the Implicit Function Theorem implies that near any point (x_0, y_0) on the circle for which $\frac{\partial F}{\partial x}(x_0, y_0) \neq 0$, the intersection of the circle with a suitable neighborhood of (x_0, y_0) is the graph of a smooth function $x = x(y)$. Why is this graph a smooth manifold?

- (b) Repeat (a) for the unit sphere S^n in \mathbb{R}^{n+1} .
- (c) More generally let U be a nonempty open subset of \mathbb{R}^n , let $F : U \rightarrow \mathbb{R}^n$ be a smooth function with $n > k$, and let $J(x)$ be the n -by- k Jacobian matrix of F at $x \in U$, with entries $\partial F_i / \partial x_j$. Suppose for each $x \in U$ that the rank of $J(x)$ is k , i.e., that $J(x)$ has k linearly independent columns. Use the Implicit Function Theorem to show that the subset of points $x \in U$ with $F(x) = 0$ is a smooth manifold of dimension $n - k$.