Front pages (Contents, Preface, Introduction, List of Figures, Standard Notation, Acknowledgments), i-xvii

DOI: 10.3792/euclid/9781429799881-0

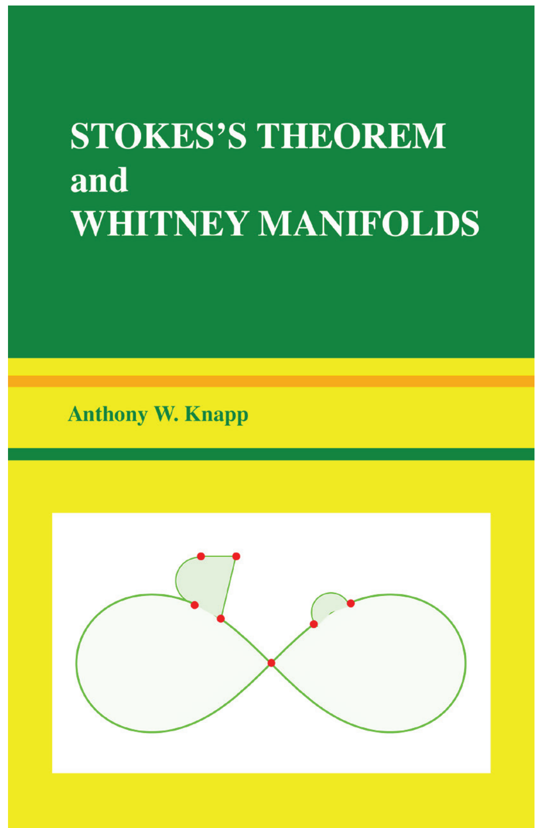
from

Stokes's Theorem and Whitney Manifolds:

A Sequel to
Basic Real Analysis

Anthony W. Knapp

Full Book DOI: 10.3792/euclid/9781429799881
ISBN: 978-1-4297-9988-1



PROJECT euclid

Distributed by Project Euclid.
For copyright information, see the following page.

Anthony W. Knapp
81 Upper Sheep Pasture Road
East Setauket, N.Y. 11733–1729, U.S.A.
Email to: `aknapp@math.stonybrook.edu`
Homepage: `www.math.stonybrook.edu/~aknapp`

Title: Stokes's Theorem and Whitney Manifolds. A Sequel to *Basic Real Analysis*.

Cover: An example of a Whitney domain in two-dimensional space. The green portion is a manifold-with-boundary for which Stokes's Theorem applies routinely. The red dots indicate exceptional points of the boundary where a Whitney condition applies that says Stokes's Theorem extends to the whole domain.

Mathematics Subject Classification (2020): 58–01, 58A05, 58A35, 58C35, 26B20.

Original Digital Edition, not to be sold, no ISBN
©2021 Anthony W. Knapp
Published by the Author

All rights reserved. This file is the original digital edition of the above named book. The text, images, and other data contained in this file, which is in portable document format (PDF), are proprietary to the author, and the author retains all rights, including copyright, in them. The use in this file of trade names, trademarks, service marks, and similar items, even if they are not identified as such, is not to be taken as an expression of opinion whether they are subject to proprietary rights.

The file is made available for limited noncommercial use for purposes of education, scholarship, and research, and for these purposes only, or for fair use as understood in the United States copyright law. Users may freely download this file for their own use and may store it, post it online, and transmit it digitally for purposes of education, scholarship, and research. They may not convert it from PDF to any other format (e.g., EPUB), they may not edit it, and they may not do reverse engineering with it. In transmitting the file to others or posting it online, users must charge no fee, nor may they include the file in any collection of files for which a fee is charged.

Each user of this file is granted permission to make one printed copy of the entire book for personal use toward education, scholarship, and research, and for these purposes only. This personal printed copy must not be sold in whole or part. Any exception to these rules concerning the file and its printing requires written permission from the author.

Except as provided by fair use provisions of the United States copyright law, no extracts or quotations from this file may be used that do not consist of whole pages unless permission has been granted by the author.

The permission granted for use of the whole file and the prohibition against charging fees extend to any partial file that contains only whole pages from this file, except that the copyright notice on this page must be included in any partial file that does not consist exclusively of the front cover page. Such a partial file shall not be included in any derivative work unless permission has been granted by the author.

Anthony W. Knapp

Stokes's Theorem and Whitney Manifolds

A Sequel to *Basic Real Analysis*

Original Digital Edition, 2021

Published by the Author
East Setauket, New York

Anthony W. Knapp
81 Upper Sheep Pasture Road
East Setauket, N.Y. 11733–1729, U.S.A.
Email to: aknapp@math.stonybrook.edu
Homepage: www.math.stonybrook.edu/∼aknapp

Title: Stokes's Theorem and Whitney Manifolds. A Sequel to *Basic Real Analysis*.

Cover: An example of a Whitney domain in two-dimensional space. The green portion is a manifold-with-boundary for which Stokes's Theorem applies routinely. The red dots indicate exceptional points of the boundary where a Whitney condition applies that says Stokes's Theorem extends to the whole domain.

Mathematics Subject Classification (2020): 58–01, 58A05, 58A35, 58C35, 26B20.

Original Digital Edition, not to be sold, no ISBN
©2021 Anthony W. Knapp
Published by the Author

All rights reserved. This file is the original digital edition of the above named book. The text, images, and other data contained in this file, which is in portable document format (PDF), are proprietary to the author, and the author retains all rights, including copyright, in them. The use in this file of trade names, trademarks, service marks, and similar items, even if they are not identified as such, is not to be taken as an expression of opinion whether they are subject to proprietary rights.

The file is made available for limited noncommercial use for purposes of education, scholarship, and research, and for these purposes only, or for fair use as understood in the United States copyright law. Users may freely download this file for their own use and may store it, post it online, and transmit it digitally for purposes of education, scholarship, and research. They may not convert it from PDF to any other format (e.g., EPUB), they may not edit it, and they may not do reverse engineering with it. In transmitting the file to others or posting it online, users must charge no fee, nor may they include the file in any collection of files for which a fee is charged.

Each user of this file is granted permission to make one printed copy of the entire book for personal use toward education, scholarship, and research, and for these purposes only. This personal printed copy must not be sold in whole or part. Any exception to these rules concerning the file and its printing requires written permission from the author.

Except as provided by fair use provisions of the United States copyright law, no extracts or quotations from this file may be used that do not consist of whole pages unless permission has been granted by the author.

The permission granted for use of the whole file and the prohibition against charging fees extend to any partial file that contains only whole pages from this file, except that the copyright notice on this page must be included in any partial file that does not consist exclusively of the front cover page. Such a partial file shall not be included in any derivative work unless permission has been granted by the author.

To Susan

and

To My Children, Sarah and William,

and

To My Grandchildren, Michelle and Scott

CONTENTS

CONTENTS OF *BASIC REAL ANALYSIS*

CONTENTS OF *BASIC ALGEBRA*

PREFACE

This book is a sequel to the author's *Basic Real Analysis*, which systematically developed concepts and tools in real analysis that are vital to every mathematician, whether pure or applied, aspiring or established. The intention was that it and its companion volume, *Advanced Real Analysis*, together would contain what the young mathematician needs to know about real analysis in order to communicate well with colleagues in all branches of mathematics.

The first editions of these books consciously omitted a few topics, the most notable of which were advanced topics in the calculus of several real variables, particularly the integration theorems that relate an integral over a region or surface to an integral over the boundary. These integration theorems go under the general name "Stokes's Theorem" because of the history that will be explained in the Introduction, and they too are tools in real analysis that are vital to every mathematician.

This book aims to treat that topic. Actually the digital second edition of *Basic Real Analysis* dealt with low dimensional aspects of the topic somewhat by addressing arc length, line integrals, and Green's Theorem in the plane in Chapter III. The spirit of the treatment of these matters was the same as the treatment in that book of Riemann integration in one and several variables, careful and thorough, the expectation being that the reader had earlier seen this material presented in a utilitarian fashion. When it comes to surface integrals, however, the method used for addressing arc length breaks down, as was shown toward the end of Section III.13 of *Basic Real Analysis*. Unlike the length of a curve, the area of a surface cannot be defined as the supremeum of some obvious inscribed approximations, and a different approach to the whole subject is needed.

The different approach that we follow is to use material that lies at the beginning of the study of both differentiable topology and algebraic topology. The material in question is the topic of differential forms, including integration of differential forms. The spirit of the treatment is quite different from that of *Basic Real Analysis*, and Chapter I of the present book takes some time to develop differential forms and tools for working with them.

By way of prerequisites, this book relies in part on some real analysis that is treated in Chapters III, V, VI, and X of *Basic Real Analysis*. In addition, it makes use of elementary linear algebra and a certain amount of multilinear algebra that can be found in the author's *Basic Algebra*, Chapter VI, Sections 1–7.

The key theorems that are needed from real analysis are the Inverse and Implicit Function Theorems and the change-of-variables formula for the Lebesgue integral in Euclidean space. The Riemann integral could be used in place of the Lebesgue integral in most circumstances, but at a cost of making certain statements more cumbersome. The key thing that is needed from algebra is some familiarity with the tensor algebra of a real finite dimensional vector space.

A philosophical problem arises in finding the right setting for the integration theorems that are collectively known as Stokes's Theorem and that relate an integral over a region or surface to a integral over the boundary. The integration theorems are most transparent when the sets of integration are rectangular, and we indicate the simple idea in the Introduction. On the other hand, the proofs are most natural when the sets and functions are smooth, as they are for a circle or a ball. Rectangular sets are not smooth. The setting in which the sets and functions are smooth is that of "manifolds-with-boundary," which are defined in Chapter II of the present book. To handle both settings at the same time—rectangular sets and smooth manifolds-with-boundary—the traditional approach is to break the sets of integration into pieces by some kind of triangulation or other cutting of regions into parts. Then one establishes Stokes's Theorem for each piece and adds the results. Pedagogically this approach is unsatisfactory.

A more modern approach is to use "manifolds-with-corners," which are defined and used in the first half of Chapter III. Manifolds-with-corners handle a great many cases without any cutting of regions into pieces, but they are still insufficient to handle all cases of practical interest without additional effort. The second half of Chapter III treats Stokes's Theorem in a still broader context due to Hassler Whitney. Whitney worked with what he called "standard manifolds" but which are more aptly called "Whitney manifolds." Whitney manifolds do indeed handle all cases of practical interest.

Some years ago, aware of the tension between the two standard approaches to Stokes's Theorem via rectangular sets and smooth manifolds, I asked my colleague Blaine Lawson whether one could now finally give a satisfactory exposition of the theorem. At that time he introduced me to manifolds-with-corners and explained to me how one could often use them to avoid the traditional cutting of manifolds into concrete pieces. The resulting situation, although better, was still not satisfactory in my view.

More recently, to help cope with restrictions because of the COVID-19 pandemic, I decided to look at the matter again. Libraries were closed. But during my online reading I encountered Whitney's book *Geometric Integration Theory*, which proves a version of Stokes's Theorem that seems to handle all examples of practical interest without any need at all to cut manifolds into concrete pieces. In response to emailed questions about some passages in Whitney's book, my colleague Chris Bishop introduced me to various notions of dimension and explained to me the relationships among them, pointing to his book *Fractals in Probability and Analysis* written with Yuval Perez for some of the details. I am grateful to both colleagues for sharing information with me.

The problems at the ends of chapters are an important part of the book. Some of them are really theorems, some are significant examples, and a few are just exercises. The reader gets no indication which problems are of which type, nor of which ones are relatively easy. Each problem except perhaps the last one can be solved with tools developed up to that point in the book, plus any additional prerequisites that are noted. Detailed hints appear at the end of the book.

The typesetting was by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX, and the figures were drawn with help from Mathematica.

I invite corrections and other comments from readers. I plan to maintain a list of known corrections on my own Web page.

A. W. KNAPP
November 2020

INTRODUCTION

Stokes's Theorem is a generalization of the Fundamental Theorem of Calculus from one dimension to higher dimensions. In an easy formulation the Fundamental Theorem of Calculus says that

$$\int_a^b F'(x)\,dx = F(b) - F(a)$$

on the closed interval $[a, b]$ if F is a real-valued function with a continuous derivative F'. In thinking how to generalize this theorem while keeping the ideas clear, we shall not be looking for the best possible hypotheses and will be content with assuming in the statement merely that F is smooth (i.e., infinitely differentiable). At any rate the formula relates the integral of the derivative of F over an interval to a linear combination of the values of F at the endpoints.

We encountered two qualitatively similar results in Chapter III of *Basic Real Analysis*, as follows:

(1) One such result was the formula in Proposition 3.47 for the line integral of the gradient of a smooth function over a smooth curve γ in $\mathbb{R}^n$ with domain $[a, b]$, namely

$$\int_\gamma \nabla f \cdot ds = f(\gamma(b)) - f(\gamma(a)).$$

Again the formula relates an integral of a derivative of f over a curve to a linear combination of the values of f at the endpoints of the curve.

(2) The other such result was the formula in Section III.13 concerning Green's Theorem in the plane, namely

$$\iint_U \Big(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\Big)\,dx\,dy = \int_\gamma P\,dx + Q\,dy.$$

Here U is a region in $\mathbb{R}^2$, the curve γ traces out its boundary with the region on the left, and U and γ are assumed to be suitably nice. This formula involves a two-component real-valued function with entries P and Q, and it relates an integral over the region involving first derivatives of P and Q to an integral over the boundary of the values of P and Q.

The first of these results is simply a matter of applying the Fundamental Theorem of Calculus component by component, and it is not mysterious.

Let us consider Green's Theorem in more detail. The idea behind the theorem is clearest for the special case that U is a rectangle with sides parallel to the axes, a case that was considered in Example 1 of Section III.13 of *Basic Real Analysis*. In that case Theorem 3.49 is proved by considering P and Q separately. To handle P, one applies the Fundamental Theorem of Calculus in the x variable and integrates the result in the y variable; to handle Q, one applies the Fundamental Theorem of Calculus in the y variable and integrates the result in the x variable.

Unfortunately the style of proof that works well for a rectangle already runs into technical problems if one tries to prove the theorem for a closed disk in $\mathbb{R}^2$. Example 2 in Section III.13 of *Basic Real Analysis* gives the details. There are two technical problems—(a) the need to impose new parametrizations on a curve and see that its line integrals are unchanged and (b) the need to use Lebesgue integration or some other device to cope with unbounded integrands. Example 3 in Section III.13 shows that for a washer (or annulus) in $\mathbb{R}^2$, further difficulties arise, and the argument uses a decomposition of the region into a number of parts. For a more complicated region, the corresponding decomposition may be expected to be more difficult to describe, and it is not at all apparent how to make a general argument.

Classical treatments of calculus in three variables, or particularly of what is sometimes given the more advanced-sounding name vector analysis, discuss two further theorems of this kind, known respectively as the Divergence Theorem (or the Gauss–Ostrogradsky Theorem) and the Kelvin–Stokes Theorem (or simply Stokes's Theorem).

The Divergence Theorem in $\mathbb{R}^3$ concerns a solid bounded region U in $\mathbb{R}^3$ with a 2 dimensional boundary ∂U. In classical notation it says that

$$\iiint_U \Big(\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}+\frac{\partial R}{\partial z}\Big)\,dx\,dy\,dz = \iint_{\partial U} P\,dy\wedge dz+Q\,dz\wedge dx+R\,dx\wedge dy.$$

Evaluation of a term on the right side involves parametrizing the surface in (x, y, z) space by parameters s and t, and then $dy\wedge dz$, $dz\wedge dx$, and $dx\wedge dy$ are given formally by substituting the product of a two-by-two determinant times $ds\,dt$, specifically

$$dy\wedge dz = \frac{\partial(y,z)}{\partial(s,t)}\,ds\,dt,\quad dz\wedge dx = \frac{\partial(z,x)}{\partial(s,t)}\,ds\,dt,\quad dx\wedge dy = \frac{\partial(x,y)}{\partial(s,t)},$$

and carrying out the double integrations. Some important questions concerning orientations and signs need to be sorted out, but we skip over those for the time being.

In the case that U is a rectangular solid with faces parallel to the axes, the formula can be verified one term at a time by using the Fundamental Theorem of Calculus in the differentiated variable and then integrating in the other two variables, carefully managing the signs that appear from the integrated terms. This computation is the expected generalization of the computation in Example 1 of Section III.13 of *Basic Real Analysis* on Green's Theorem. For more general solids U, one can attempt a similar argument after breaking the original integral into a number of pieces. Once again, it is not at all apparent how to describe such a decomposition of a complicated region, and thus it is not at all apparent how to give a general proof of the Divergence Theorem in this style.

The Kelvin–Stokes Theorem,[1] sometimes known simply as Stokes's Theorem, concerns an oriented 2 dimensional surface S having a 1 dimensional boundary given by a curve γ, the whole manifold plus boundary embedded in $\mathbb{R}^3$. The formula is

$$\begin{aligned}\iint_S \Big(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\Big)\, dy \wedge dz + \Big(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\Big)\, dz \wedge dx + \Big(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\Big)\, dx \wedge dy \\ = \int_\gamma P\, dx + Q\, dy + R\, dz.\end{aligned}$$

When a sketch of proof is given in an elementary text for this theorem in special cases, it often goes by reducing the theorem to Green's Theorem in the plane. When necessary, the surface is cut into pieces and canceling pieces of boundary curve are adjoined.

From an expository point of view, the whole matter is rather unsatisfactory. In 1934 the young French mathematicians André Weil and Henri Cartan had the joint responsibility in Strasbourg for teaching a course on "differential and integral calculus," and they consulted each other frequently. In his autobiography Weil writes of this interaction, saying,[2]

> One point that concerned him [Cartan] was the degree to which we should generalize Stokes' formula in our teaching. This formula is written as follows:
> $$\int_{b(X)} \omega = \int_X d\omega,$$

[1] The theorem was discovered by William Thomson (Lord Kelvin) and communicated to Stokes by letter in 1850.

[2] André Weil, *The Apprenticeship of a Mathematician*, Birkhäuser, Basel, 1992, pp. 99-100.

> where ω is a differential form, $d\omega$ is its derivative, X its domain of integration, and $b(X)$ the boundary of X. There is nothing difficult about this if for example X is the infinitely differentiable image of an oriented sphere and if ω is a form with infinitely differentiable coefficients. Particular cases of this formula appear in classical treatises, but we were not content to make do with these.

Weil goes on to describe how this interaction led a group of young French mathematicians over a period of years to explain systematically much of elementary mathematics in a series of published books going under the title *Eléments de Mathématique* and written with the pseudonym Nicolas Bourbaki.[3]

Ironically although Bourbaki's books eventually developed a wide swath of mathematics rigorously, especially in the 1950s and 1960s, they had not yet treated Stokes's Theorem as of 2018. Possibly the reason was that a suitable framework, conveniently handling all shapes of interest at once, was not developed until well after World War II. Let us elaborate somewhat on the history.

Building on his own work from much earlier and on some work of H. Poincaré and E. Goursat, Elie Cartan[4] had brought a degree of unity to the subject by showing that Green's Theorem, the Divergence Theorem, and the Kelvin–Stokes Theorem were really special cases of the same general theorem. In a course in 1936–1937, whose notes were published as a book in 1945, he showed how to view all three classical theorems as instances of a result about "differential forms" and "exterior differentiation," the unifying formula being the one in the quotation above from Weil's book. Moreover, the theory, which dealt with smooth "manifolds-with-boundary," was not limited to cases in $\mathbb{R}^3$, and the final proof took little more than a couple of pages. The cost of having such a tidy final result for smooth manifolds-with-boundary was that the hard work was transferred into the definitions and verifications necessary to set up the theory. The 1965 book by M. Spivak, *Calculus on Manifolds*, proves Stokes's Theorem just for smooth manifolds-with-boundary,[5] it does so in exactly this way, and it makes the point that the difficulty occurs in setting up the theory. We shall see this cost first hand in the present book in that all of Chapter I and part of Chapter II are devoted to setting up the theory.

In practical applications unfortunately, physicists and engineers need a version of Stokes's Theorem that holds for rectangular sets and other polyhedral sets, as

[3] Although the original six members of the Bourbaki group were all French, mathematicians of other nationalities joined the group later. Members were expected to retire from the group about at age 50.

[4] Father of Henri.

[5] In Spivak's book the manifolds-with-boundary are always embedded in some Euclidean space for the sake of concreteness, but working in such a setting merely adds one unnecessary parameter to the mix and obscures the simplicity of the final formula.

well as for smooth manifolds-with-boundary. This is the matter that concerned H. Cartan in the quotation above. Even as late as the 1950s, rectangular solids and polyhedral sets were best treated directly, essentially by cutting the set into pieces and making an explicit calculation for each piece, while round shapes were best treated as manifolds-with-boundary to which E. Cartan's theory could be applied directly.

In 1961 J. Cerf and A. Douady introduced smooth "manifolds-with-corners," which included solid balls and also rectangular solids. In other words, smooth manifolds-with-corners offered a step toward further unifying the treatment of Stokes's Theorem. The present book will give a proof of Stokes's Theorem for smooth manifolds-with-corners in Sections 1–3 of Chapter III. The argument is really no harder than the argument for smooth manifolds-with-boundary, and one can perhaps regard the setting of manifolds-with-corners as giving a sufficient answer to H. Cartan's question about pedagogy.

It may be a sufficient answer, but it is not completely satisfactory. The corners in the theory of smooth manifolds-with-corners turn out to be of really limited scope. In $\mathbb{R}^3$, for example, when three planes come together at a point, the result is a corner in the sense of the theory, but when four planes come together at a point, the resulting intersection point no longer fits the theory. Thus, for example, the theory applies to a tetrahedron in $\mathbb{R}^3$ but not to a square pyramid.

It turns out that there is a more all-encompassing theory, and it was already known by 1960. Hassler Whitney developed the theory and published it in a book in 1957. The present book concludes Chapter III with the relevant parts of Whitney's theory. Qualitatively Whitney's theory looks at a manifold and boundary and divides the boundary into two sets. One set consists of nice points like those in the E. Cartan theory of smooth manifolds-with-boundary. The other set consists of exceptional points. Whitney's theorem is that if the set of exceptional points is small in a certain precise sense, then everything is fine and the Stokes formula is valid. The theorem handles all smooth manifolds-with-corners. In fact, the theorem appears to handle all situations that might be of interest to physicists and engineers, as well as all those that are of interest to most mathematicians. The proof still takes only a few pages, with its complications concealed in the definitions. One cannot ask for more.

LIST OF FIGURES

STANDARD NOTATION

Item	**Meaning**
$\varnothing$	empty set
$\{x \in E \mid P\}$	the set of x in E such that P holds
E^c	complement of the set E
$E \cup F,\ E \cap F,\ E - F$	union, intersection, difference of sets
$\bigcup_\alpha E_\alpha,\ \bigcap_\alpha E_\alpha$	union, intersection of the sets E_α
$E \subseteq F,\ E \supseteq F$	E is contained in F, E contains F
$E \times F$	product of sets
$(a_1, \ldots, a_n),\ \{a_1, \ldots, a_n\}$	ordered n-tuple, unordered n-tuple
$f : E \to F,\ x \mapsto f(x)$	function, effect of function
$f \circ g,\ f\big\vert_E$	composition of f following g, restriction to E
$f(\,\cdot\,, y)$	the function $x \mapsto f(x, y)$
$f(E),\ f^{-1}(E)$	direct and inverse image of a set
δ_{ij}	Kronecker delta: 1 if $i = j$, 0 if $i \neq j$
c positive, c negative	$c > 0$, $c < 0$
$\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$	integers, rationals, reals, complex numbers
max (and similarly min)	maximum of finite subset of a totally ordered set
$\sum$	sum, possibly with a limit operation
countable	finite or in one-one correspondence with $\mathbb{Z}$
1 or I	identity matrix or function or operator
dim V	dimension of vector space
$\mathbb{R}^n, \mathbb{C}^n$	spaces of column vectors
det A	determinant of A
A^{tr}	transpose of A
$\mathrm{Hom}_{\mathbb{R}}(U, V)$	space of linear functions from U to V
E^o	interior of set E
E^{cl}	closure of set E

ACKNOWLEDGMENTS

The author acknowledges the sources below as the main ones he used in preparing the notes from which this book evolved. The descriptions below have been abbreviated. Full descriptions of the items may be found in the section "Selected References" at the end of the book.

This list is not to be confused with a list of recommended present-day reading for these topics; in some cases newer books deserve attention.

CHAPTER I. Section 1 of Chapter VIII of the author's *Advanced Real Analysis* for Section 1 of the present book. Sections 1–7 of Chapter VI of the author's *Basic Algebra* for Section 2. Spivak's *Calculus on Manifolds*, Warner's *Foundations of Differentiable Manifolds*, and Bott–Tu's *Differential Forms in Algebraic Topology* for Sections 3 and 4. Spivak's *Calculus on Manifolds*, Warner's *Foundations of Differentiable Manifolds* and De Rham's *Variétés différentiables* for Section 5. Ghomi's *Lecture Notes on Differential Geometry* and Lee's *Introduction to Smooth Manifolds* for orientation in Section 6. Chevalley's *Theory of Lie Groups* and Helgason's *Differential Geometry and Symmetric Spaces* for integration in Section 6.

CHAPTER II. Spivak's *Calculus on Manifolds*, Warner's *Foundations of Differentiable Manifolds*, and Bott–Tu's *Differential Forms in Algebraic Topology* for Sections 1 and 2. Lee's *Introduction to Smooth Manifolds* for Section 3. Bott–Tu's *Differential Forms in Algebraic Topology* for Section 4. Spivak's *Calculus on Manifolds* for Section 5.

CHAPTER III. Joyce's paper "On manifolds with corners," Conrad's undated course notes concerning manifolds with corners, and Chapter 1 of Melrose's unfinished book *Differential Analysis on Manifolds* for Sections 1–3. Whitney's *Geometric Integration Theory* and Bishop–Perez's *Fractals in Probability and Analysis* for Sections 4–6.