Chapter 1

What Is Straight?

Straight is that of which the middle is in front of both extremities.
— Plato, *Parmenides,* 137 E [**AT**: Plato]

A *straight line* is a line that lies symmetrically with the points on itself.
— Euclid, *Elements*, Definition 4 [Appendix A]

To draw a circle, we generally do not start with a model of a circle; instead we use a fundamental property of circles that the points on a circle are a fixed distance from a center as in Euclid's definition of a circle (see Appendix A, Definition 15). Based on this definition we use a compass to draw a circle. What about drawing a straight line: Is there a tool (serving the role of a compass) that will draw a straight line? One could say we can use a straightedge for constructing a straight line. Well, how do you know that your straightedge is straight? How can you check that something is straight? What does "straight" mean? Think about it — this is part of Problem **1.1** below.

History: How Can We Draw a Straight Line?

You can try to use Euclid's definition above. If you fold a piece of paper the crease will be straight — the edges of the paper needn't be straight. This utilizes mirror symmetry to produce the straight line.

Figure 1.0.5 Folding a straight line

Traditional cabinet makers also use symmetry to determine straightness — they put two boards face to face, simultaneously plane the edges until they look straight, and then turn one board over so the planed edges are touching as in Figure 1.1. If the edges are not straight, there will be gaps between the boards where one can see light shining through.

Figure 1.1 Cabinet maker's method for checking straightness

In Problem **1.1** we will explore more deeply symmetries of a straight line.

When grinding an extremely accurate flat mirror, the following technique is sometimes used: Take three approximately flat pieces of glass and put pumice between the first and second pieces and grind them together. Then do the same for the second and the third pieces and then for the third and first pieces. Repeat many times and all three pieces of glass will become very accurately flat. Do you see in Figure 1.2. why this works? What does this have to do with straightness?

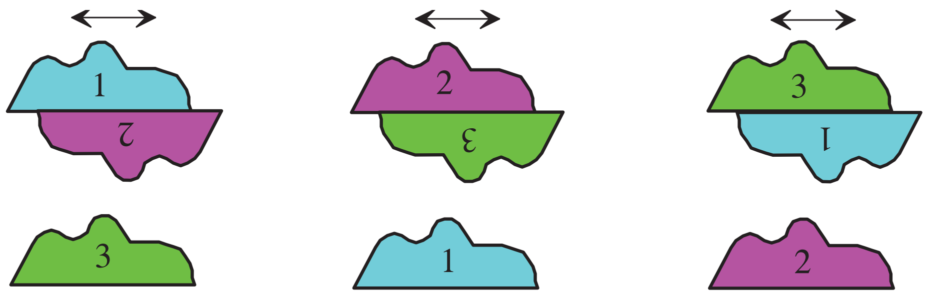


Figure 1.2 Grinding flat mirrors

We can also use the usual high school definition, “A straight line is the shortest distance between two points.” That is why a straight line can be produced by stretching a string. Plato’s description of straight can be experienced when ‘sighting along’ a row of trees, as in Figure 1.2a.

Figure 1.2.5a Sighting along straight line

This use of symmetry, stretching, and folding can also be extended to other surfaces, as we will see in Chapters 2, 4, and 5. Sometimes we can get confused reading in the literature that straight line is an “undefined” term or that straight lines on the sphere are “defined to be arcs of great circles.” We find that putting “straight” in the context of the four historical strands helps clarify this: “Symmetry” comes mostly from the Art/Pattern Strand, “undefined term” comes from the axiomatics in the Building Structures Strand, and “shortest distance” comes mostly from the Navigation/Stargazing Strand.

Still the question of whether there is a mechanism analogous to a compass that will draw an accurate straight line is left unanswered. We can find answers to this question in the history of mechanics, which leads us into the Motion/Machines Strand and to another meaning of “straight”. We focus on a particular kind of mechanism called a *linkages* (rigid bars constrained to be near a plane and joined at their ends by rivets or pivots).

The simplest linkage is a fixed compass which can be considered as a single rigid bar which is constrained to pivot on a fixed point at one end while the other end traces a circle. The goal is to find a linkage that will connect this circular motion with a straight line motion.

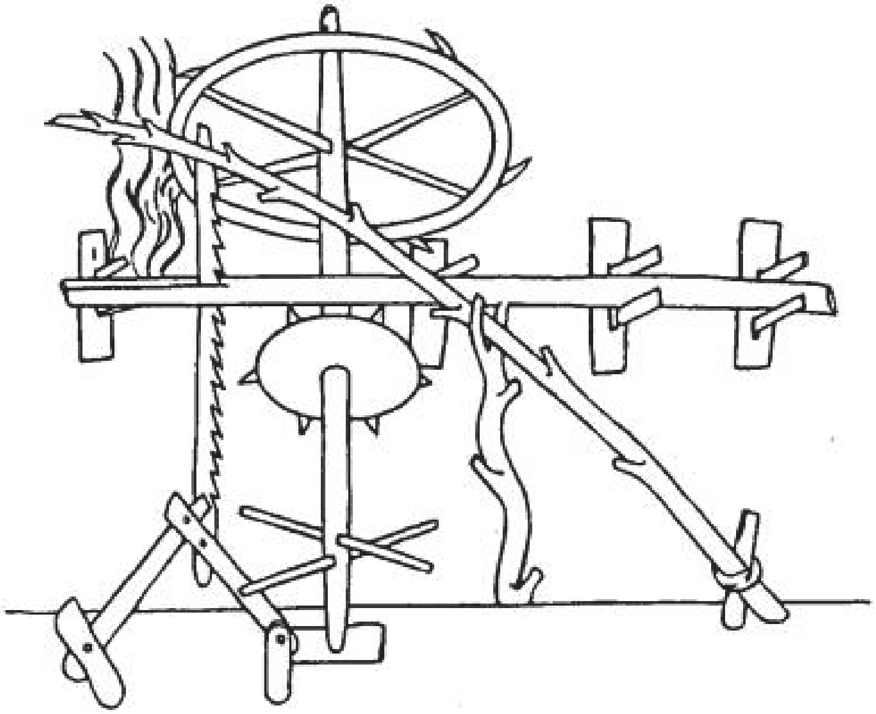


Figure 1.3 Up-and-down sawmill of the 13th century

Turning circular motion into straight line motion has been a practical engineering problem since at least the Middle Ages. As we can see in some 13th century drawings of a sawmill in Figure 1.3, linkages (rigid bars constrained to be near a plane and joined at their ends by rivets) were in use by that time and probably were originated much earlier. Georgius Agricola's (1494–1555) geological writings [**ME**: Agricola] reflect firsthand observations not just of rocks and minerals, but of every aspect of mining technology and practice of the time. In the pictures of his work one can see link work that was widely used for converting the continuous rotation of a water wheel into a reciprocating motion suited to piston pumps. In 1588, Agostino Ramelli published his book [**ME**: Ramelli] on machines where linkages were widely used. Both of these books are readily available; see the Bibliography.

In the late 18th century people started turning to steam engines for power. James Watt (1736–1819), a highly gifted designer of machines, worked on improving the efficiency and power of steam engines. In a steam engine, the steam pressure pushes a piston down a straight cylinder. Watt's problem was how to turn this linear motion into the circular motion of a wheel (such as on steam locomotives). It took Watt several years to design the straight-line linkage that would change straight-line motion to circular one. Later, Watt told his son,

> Though I am not over anxious after fame, yet I am more proud of the parallel motion than of any other mechanical invention I have ever made. (quoted in [**ME**: Ferguson 1962], p 197)

"Parallel motion" is a name Watt used for his linkage, which was included in an extensive patent of 1784. Watt's linkage was a good solution to the practical engineering problem. In Figure 1.4, Watt's linkage is the parallelogram with associated links in the upper left corner.

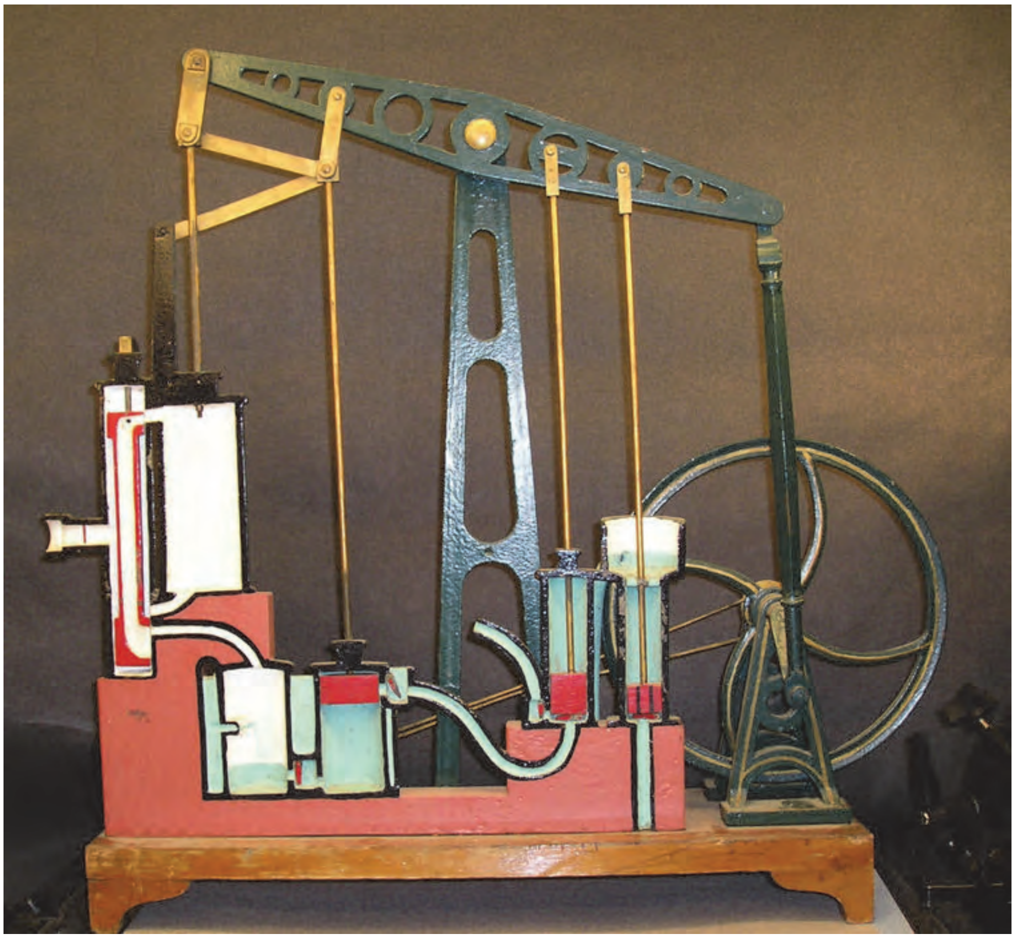


Figure 1.4 A steam engine with Watt's "parallel motion" linkage

However, this solution did not satisfy mathematicians, who knew that Watt's linkage can draw only an approximate straight line. Mathematicians continued to look for a planar straight-line linkage. A linkage that would draw an exact straight-line was not found until 1864–1871, when a French army officer, Charles Nicolas Peaucellier (1832–1913), and a Russian graduate student, Lipmann I. Lipkin (1851–1875), independently developed a linkage that draws an exact straight line. See Figure 1.5. There is not much known about Lipkin. Some sources mentioned that he was born in Lithuania and was a graduate student of Chebyshev in Saint Petersburg but died before completing his doctoral dissertation. (For more discussion of this discovery see also Philip Davis's delightful little book *The Thread* [**EM**: Davis], Chapter IV.)

Figure 1.5 Peaucellier/Lipkin linkage for drawing a straight line

The linkage in Figure 1.5 works because, as we will show in Problem **16.3**, the point Q will only move along an arc of a circle of radius $(s^2 - d^2)f/(g^2 - f^2)$. This allows one to draw an arc of a large circle without using its center. When the lengths g and f are equal, then Q draws the arc of a circle with infinite radius. (See [**EG**: Hilbert], pp. 272–273, for another discussion of this linkage.) So in the Motion/Machines Strand we find another notion of straight line – as a circle of infinite radius. (See the text near Figure 11.4 for discussion of circles of infinite radius).

Problem 1.1 When Do You Call a Line Straight?

In the spirit of the approach to geometry discussed in the Preface, we begin with a question that encourages you to explore deeply the concept of straightness. We ask you to build a notion of straightness from your experiences rather than accept a certain number of assumptions about straightness. Though it is difficult to formalize, straightness is a natural human concept.

a. *How can you check in a practical way if something is straight? How do you construct something straight — lay out fence posts in a straight line, or draw a straight line? Do this without assuming that you have a ruler, for then we will ask, "How can you check that the ruler is straight?"*

At first, look for examples of straightness in your experiences. Go out and actually try walking along a straight line and then along a curved path; try drawing a straight line and then checking that a line already drawn is straight. As you look for properties of straight lines that distinguish them from non-straight lines, you will probably remember the following statement (which is often taken as a definition in high school geometry): *A straight line is the shortest distance between two points.* But can you ever measure the lengths of all the paths between two points? How do you find the shortest path? If the shortest path between two points is in fact a straight line, then is the converse true? Is a straight line between two points always the shortest path? We will return to these questions in later chapters.

A powerful approach to this problem is to think about lines in terms of symmetry. This will become increasingly important as we go on to other surfaces (spheres, cones, cylinders, and so forth). Two of the symmetries of lines are as follows:

- *Reflection symmetry in the line*, also called *bilateral symmetry* — reflecting (or mirroring) an object over the line.

Figure 1.6 Reflection symmetry of a straight line

- *Half-turn symmetry* — rotating 180° about any point on the line.

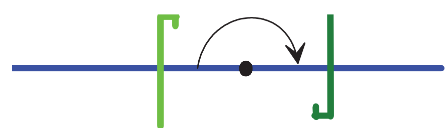


Figure 1.7 Half-turn symmetry of a straight line

Although we are focusing on a symmetry *of the line* in each of these examples, notice that the symmetry is not a property of the line by itself but includes the line and the space around the line. The symmetries preserve the local environment of the line. Notice how in reflection and half-turn symmetry the line and its local environment are both part of the symmetry action and how the relationship between them is integral to the action. In fact, reflection in the line does not move the line at all but exhibits a way in which the spaces on the two sides of the line are the same.

Definitions. An ***isometry*** is a transformation that preserves distances and angle measures. A ***symmetry of a figure*** is an isometry of a region of space that takes the figure (or the portion of it in the region) onto itself. You will show in Problem **11.3** that every isometry of the plane is either a translation, a rotation, a reflection, or a composition of them.

b. *What symmetries does a straight line have*?

Try to think of other symmetries of a line as well (there are quite a few). Some symmetries hold only for straight lines, while some work for other curves as well. Try to determine which symmetries are specific to straight lines and why. Also think of practical applications of these symmetries for constructing a straight line or for determining if a line is straight.

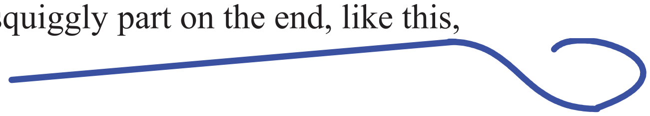
c. *What is in common among the different notions of straightness? Can you write a definition of "straight line"*?

Look for things that you call "straight." Where do you see straight lines? Why do you say they are straight? Look for both physical lines and nonphysical uses of the word "straight". What symmetries does a straight line have? How do they fit with the examples that you have found and those mentioned above? Can we use any of the symmetries of a line to define straightness? The intersection of two (flat) planes is a straight line — why does this work? Does it help us understand "straightness"?

Imagine (or actually try!) walking while pulling a long thread with a small stone attached. When will the stone follow along your path? Why? This property is used to pick up a fallen water skier. The boat travels by the skier along a straight line and thus the tow rope follows the path of the boat. Then the boat turns in an arc in front of the skier. Because the boat is no longer following a straight path, the tow rope moves in toward the fallen skier. What is happening?

Another idea to keep in mind is that straightness must be thought of as a local property. Part of a line can be straight even though the whole line may not be. For example, if we agree that this line is straight,

and then we add a squiggly part on the end, like this,



would we now say that the original part of the line is not straight, even though it hasn't changed, only been added to? Also note that we are not making any distinction here between "line" and "line segment." The more generic term "line" generally works well to refer to any and all lines and line segments, both straight and non-straight.

You are likely to bring up many ideas of straightness. It is necessary to think about what is common among all of these straight phenomena.

Think about and formulate some answers for these questions before you read any further. Do not take anything for granted unless you see why it is true. No answers are predetermined. You may come up with something that we have never imagined. Consequently, it is important that you persist in following your own ideas.

THE SYMMETRIES OF A LINE

Reflection-in-the-line symmetry: It is most useful to think of reflection as a "mirror" action with the line as an axis rather than as a "flip-over" action that involves an action in 3-space. In this way we can extend the notion of reflection symmetry to a sphere (the flip-over action is not possible on a sphere). Notice that this symmetry cannot be used as a definition for straightness because we use straightness to define reflection symmetry. This same comment applies to most of the other symmetries discussed below.

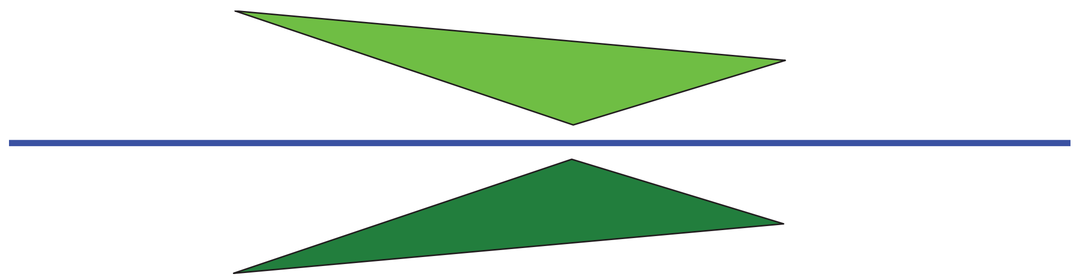


Figure 1.8 Reflection-in-the-line symmetry

- *Practical application*: We can produce a straight line by folding a piece of paper because this action forces symmetry along the crease. Above we showed a carpenter's example.

In Figures 1.8–1.14, the light green triangle is the image of the dark green triangle under the action of the symmetry on the space around the line.

Reflection-perpendicular-to-the-line symmetry: A reflection through *any* axis perpendicular to the line will take the line onto itself. Note that circles also have this symmetry about any diameter. See Figure 1.9.

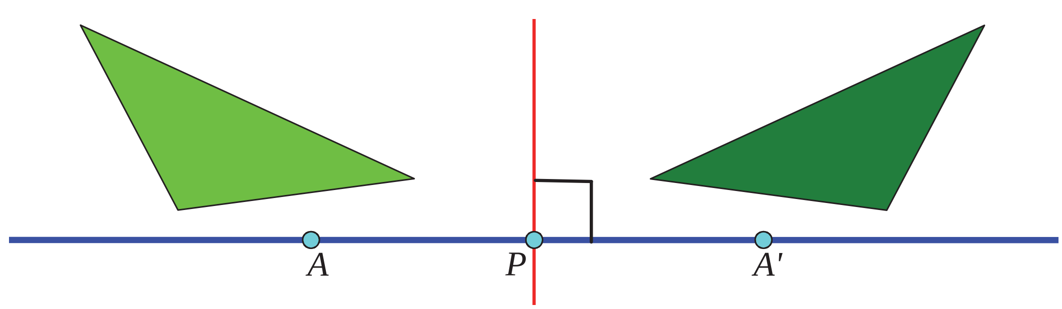


Figure 1.9 Reflection-perpendicular-to-the-line symmetry

- *Practical applications*: You can tell if a straight segment is perpendicular to a mirror by seeing if it looks straight with its reflection. Also, a straight line can be folded onto itself.

Half-turn symmetry: A rotation through half of a full revolution about any point P on the line takes the part of the line before P onto the part of the line after P and vice versa. Note that some non-straight lines, such as the letter Z, also have half-turn symmetry — but not about *every* point. See Figure 1.10.

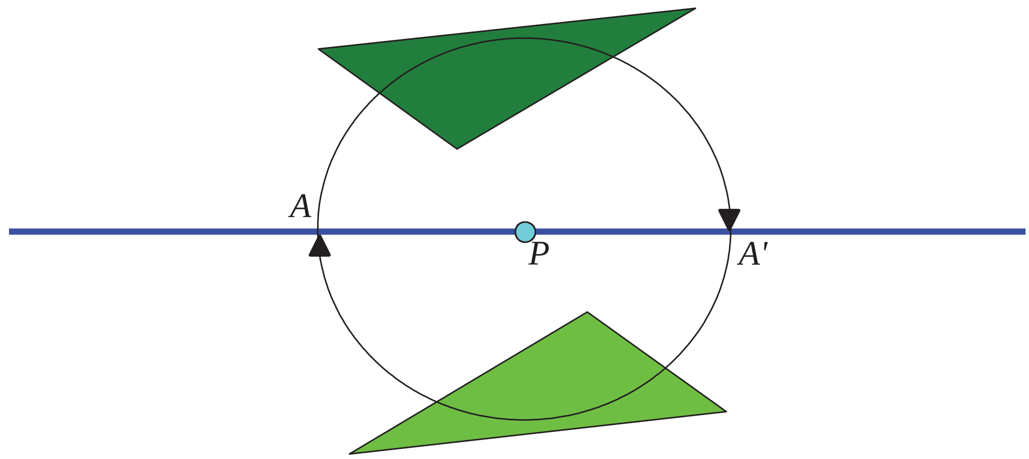


Figure 1.10 Half-turn symmetry

- *Practical applications*: Half-turn symmetry exists for the slot on a screw and the tip of the screwdriver (unless you are using Phillips-head screws and screwdrivers, which also have quarter-turn symmetry) and thus we can more easily put the tip of the screwdriver into the slot. Also, this symmetry is involved when a door (in a straight wall) opens up flat against the wall.

Rigid-motion-along-itself symmetry: For straight lines in the plane, we call this *translation symmetry*. Any portion of a straight line may be moved along the line without leaving the line. This property of being able to move rigidly along itself is not unique to straight lines; circles (rotation symmetry) and circular helixes (screw symmetry) have this property as well. See Figure 1.11.

- *Practical applications*: Slide joints such as in trombones, drawers, nuts and bolts, and so forth, all utilize this symmetry.

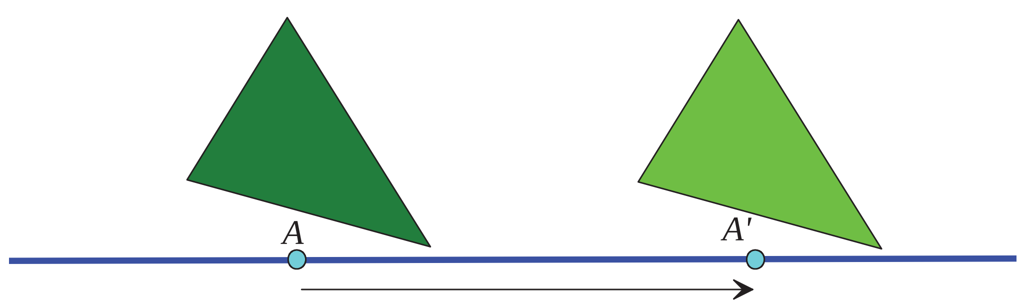


Figure 1.11 Rigid-motion-along-itself symmetry

3-dimensional-rotation symmetry: In a 3-dimensional space, rotate the line around itself through *any* angle using itself as an axis.

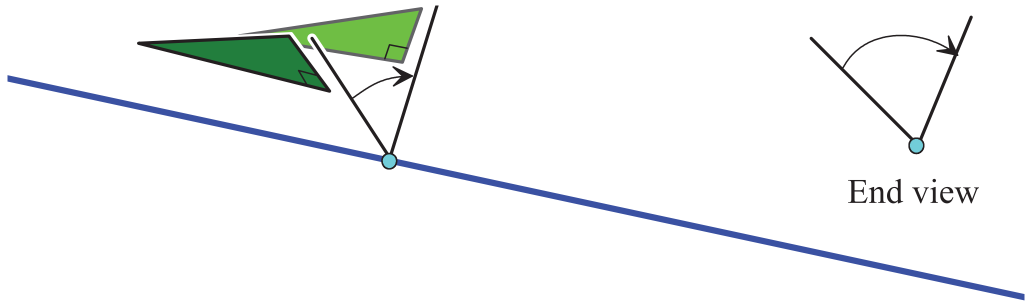


Figure 1.12 3-dimensional-rotation symmetry

- *Practical applications*: This symmetry can be used to check the straightness of any long thin object such as a stick by twirling the stick with itself as the axis. If the stick does not appear to wobble, then it is straight. This is used for pool cues, axles, hinge pins, and so forth.

Central symmetry or point symmetry: Central symmetry through the point P sends any point A to the point on the line determined by A and P at the same distance from P but on the opposite side from P as in Figure 1.13. In two dimensions central symmetry does not differ from half-turn symmetry in its end result, but they do differ in the ways we imagine them and construct them.

- In 3-space, central symmetry produces a result different than any single rotation or reflection (though we can check that it does give the same result as the composition of three reflections through mutually perpendicular planes). To experience central symmetry in 3-space, hold your hands in front of you with the palms facing each other and your left thumb up and your right thumb down. Your two hands now have approximate central symmetry about a point midway between the center of the palms; and this symmetry cannot be reproduced by any single reflection or rotation.

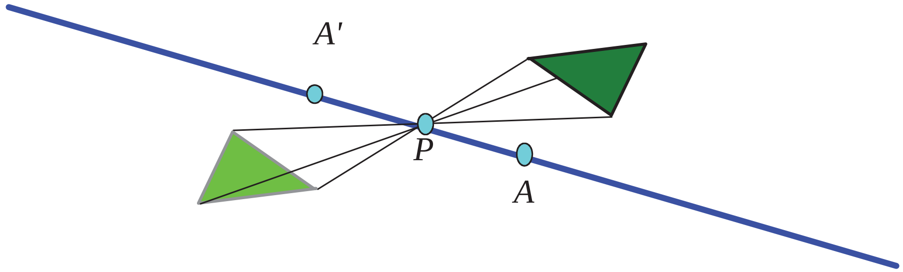


Figure 1.13 Central symmetry

Similarity or self-similarity "quasi-symmetry": Any segment of a straight line (and its environs) is similar to (that is, can be magnified or shrunk to become the same as) any other segment. See Figure 1.14. This is not a symmetry because it does not preserve distances but it could be called a "quasi-symmetry" because it *does* preserve the measure of angles.

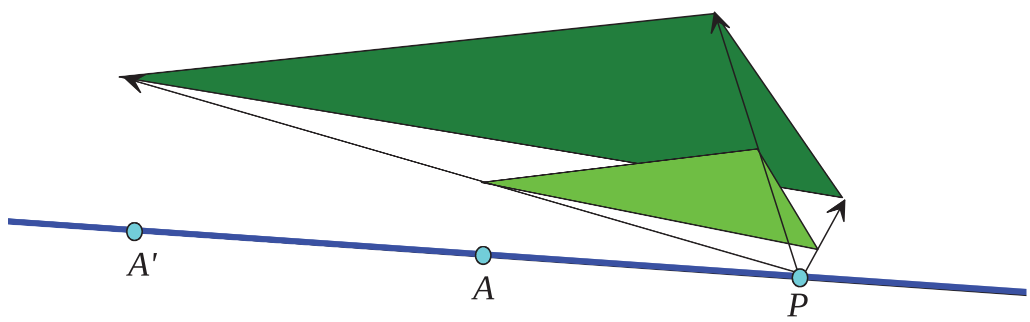


Figure 1.14 Similarity "quasi-symmetry"

- Logarithmic spiral is a self-similar curve for which at any point the angle between tangent and radius is the same. It is common in nature as a natural growth curve of plants and seashells. See examples in Figure 1.15.

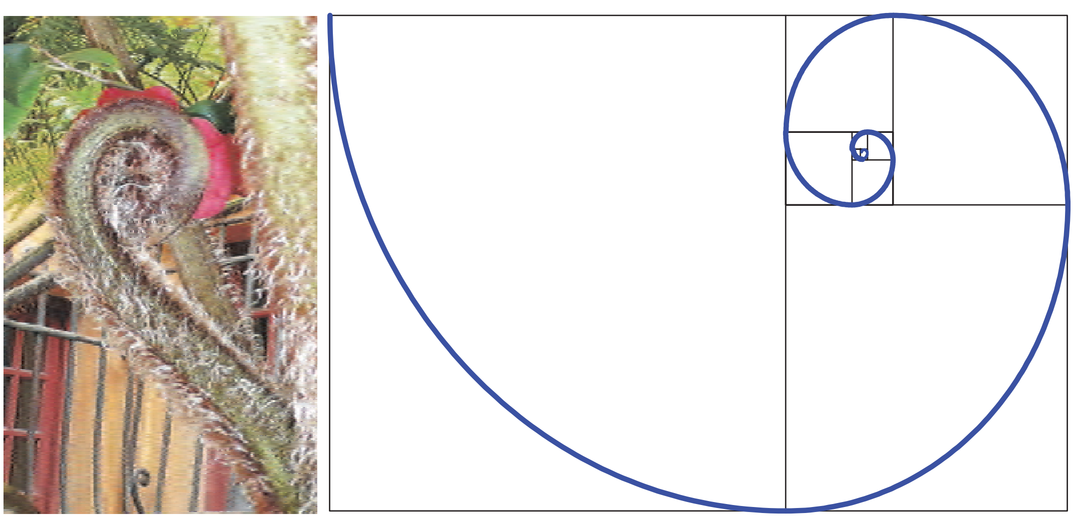


Figure 1.15 Fern growth spiral and logarithmic spiral

Clearly, other objects besides lines have some of the symmetries mentioned here. It is important for you to construct your own examples and attempt to find an object that has all of the symmetries but is not a line. This will help you to experience that straightness and the seven symmetries discussed here are intimately related. You should come to the conclusion that while other curves and figures have some of these symmetries, only straight lines have all of them.

LOCAL (AND INFINITESIMAL) STRAIGHTNESS

Previously, we saw how a straight line has reflection-in-the-line symmetry and half-turn symmetry: One side of the line is the same as the other. But, as pointed out above, straightness is a local property: Whether a segment of a line is straight depends only on what is near the segment and does not depend on anything happening away from the line. Thus each of the symmetries must be thought of (and experienced) as applying only locally. This will become particularly important later when we investigate straightness on the cone and cylinder. (See the discussions in Chapter 4.) For now, it can be experienced in the following way:

> *When a piece of paper is folded not in the center, the crease is still straight even though the two sides of the crease on the paper are not the same.* (See Figure 1.0a.)
>
> *What is the role of the sides when we check for straightness using reflection symmetry*? *Think about what is important near the crease in order to have reflection symmetry*.

When we talk about straightness as a local property, you may bring out a notion of scale. For example, if you see only a small portion of a very large circle, it will be indistinguishable from a straight line. This can be experienced easily on many of the modern graphing programs for computers. Also, you can experience zooming by experimenting with the camera on a smart phone. If a curve is smooth (or differentiable), then if we "zoom in" on any point of the curve, eventually the curve will be indistinguishable from a straight line segment. See Figure 1.17.

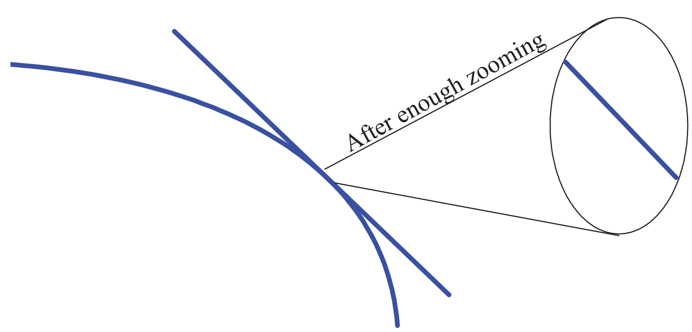


Figure 1.17 Infinitesimally straight

We sometimes use the terminology, *infinitesimally straight*, in place of the more standard terminology, *differentiable.** We say that a curve is *infinitesimally straight* at a point p if there is a straight line l such that if we zoom in enough on p, the line and the curve become indistinguishable. When the curve is parametrized by arc length this is equivalent to the curve having a well-defined velocity vector at each point.

In contrast, we can say that a curve is *locally straight at a point* if that point has a neighborhood that is straight. In the physical world the usual use of both *smooth* and *locally straight* is dependent on the scale at which they are viewed. For example, we may look at an arch made out of wood — at a distance it appears as a smooth curve; then as we move in closer we see that the curve is made by many short straight pieces of finished (planed) boards, but when we are close enough to touch it, we see that its surface is made up of smooth waves or ripples, and under a microscope we see the non-smoothness of numerous twisting fibers. See Figure 1.18.

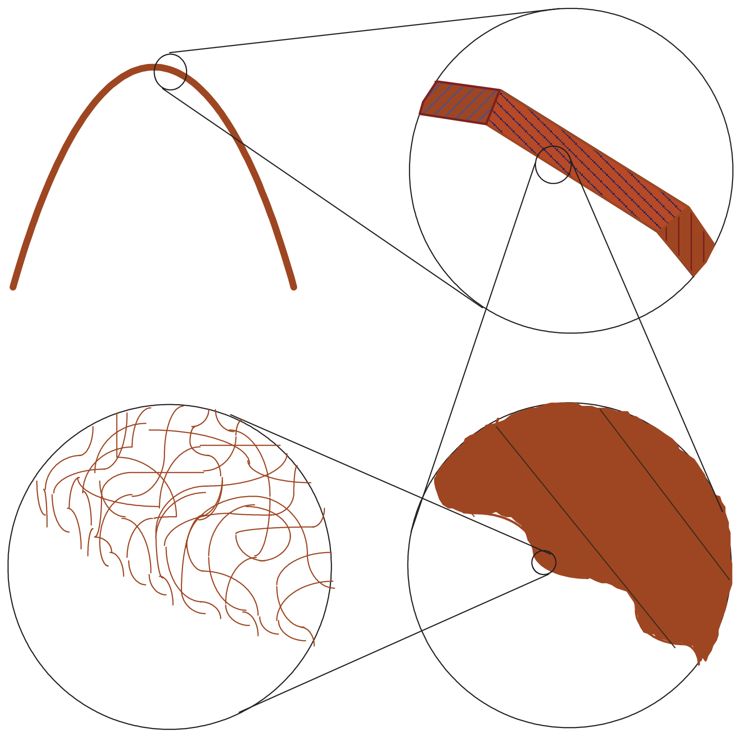


Figure 1.18 Straightness and smoothness depend on the scale

Endnotes:

* This is equivalent to the usual definitions of being differentiable at p. For example, if $t(x) = f(p) + f'(p)(x - p)$ is the equation of the line tangent to the curve $(x,f(x))$ at the point $(p,f(p))$, then, given $\varepsilon > 0$ (the distance of indistinguishability), there is a $\delta > 0$ (the radius of the zoom window) such that, for $|x - p| < \delta$ (for x within the zoom window), $|f(x) - t(x)| < \varepsilon$ [$f(x)$ is indistinguishable from $t(x)$]. This last inequality may look more familiar in the form

$$f(x) - t(x) = f(x) - f(p) - f'(p)(x - p) = \{ [f(x) - f(p)]/(x - p) - f'(p) \}(x - p) < \varepsilon.$$

In general, the value of δ might depend on p as well as on ε. Often the term *smooth* is used to mean continuously differentiable, which the interested reader can check is equivalent (on closed finite intervals) to, for each $\varepsilon > 0$, there being one $\delta > 0$ that works for all p