Chapter 22
3-SPHERES AND HYPERBOLIC 3-SPACES

Let us, then, make a mental picture of our universe: ... as far as possible, a complete unity so that whatever comes into view, say the outer orb of the heavens, shall bring immediately with it the vision, on the one plane, of the sun and of all the stars with earth and sea and all living things as if exhibited upon a transparent globe. Bring this vision actually before your sight, so that there shall be in your mind the gleaming representation of a sphere, a picture holding all the things of the universe ... . Keep this sphere before you, and from it imagine another, a sphere stripped of magnitude and of spatial differences; cast out your inborn sense of Matter, taking care not merely to attenuate it: call on God, maker of the sphere whose image you now hold, and pray Him to enter. And may He come bringing His own Universe ... .

— Plotinus, The Enneads, V.8.9, Burdette, NY: Larson, 1992

In this chapter you will explore hyperbolic 3-space and the 3-dimensional sphere that extrinsically sits in 4-space. But intrinsically, if we zoom in on a point in a 3-sphere or a hyperbolic 3-space, then locally the experience of the space will become indistinguishable from an intrinsic and local experience of Euclidean 3-space. This is also our human experience in our physical universe. We will study these 3-dimensional spaces both because they are possible geometries for our physical universe and in order to see that these geometries are closely related to their 2-dimensional versions.

Try to imagine the possibility of our physical universe being a 3-sphere in 4-space. It is the same kind of imagination a 2-dimensional (2-D) being would need in order to imagine that it was on 2-sphere (ordinary sphere) in 3-space. In Problem **18.6** we thought about how a 2-D bug could determine (intrinsically) that it was on a 2-sphere. Now, we want to explore how the bug could imagine the 2-sphere in 3-space, that is, how could the bug imagine an extrinsic view of the 2-sphere in 3-space. In Problems **22.2** and **22.3** we will use linear algebra to help us talk about and analyze the 3-sphere in 4-space, but this will not solve the problem of imagining the 3-sphere in 4-space.

PROBLEM 22.1 EXPLAIN 2-SPHERE IN 3-SPACE TO A 2-DIMENSIONAL BUG

How would you explain a 2-sphere in 3-space to a 2-D bug living in a (Euclidean) plane or on a 2-sphere so large that it appears flat to the 2-D bug? See Figure 22.1.

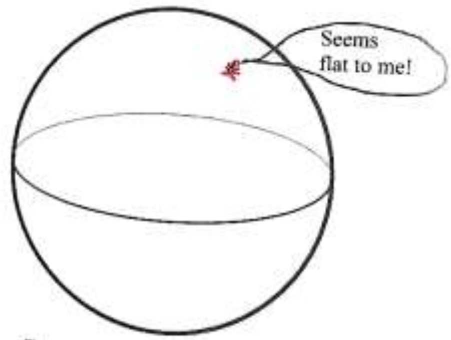


Figure 22.1 2-D bug on a large 2-sphere

SUGGESTIONS

This bug's 2-dimensional experience is very much like the experience of an insect called a water strider that we talked about in Chapter 2. A water strider walks on the surface of a pond and has a very 2-dimensional perception of the universe around it. To the water strider, there is no up or down; its whole universe consists of the surface of the water. Similarly, for the 2-D bug there is no front or back; the entire universe is the 2-dimensional plane.

Living in a 2-D world, the bug can easily understand any notions in 2-space, including plane, angle, distance, perpendicular, circle, and so forth. You can assume the bug is smart and has been in geometry class.

A bug living in a 2-D world cannot directly experience three dimensions, just as we are unable to directly experience four dimensions. Yet, with some help from you, the 2-D bug can begin to imagine three dimensions just as we can imagine four dimensions. One goal of this problem is to try to gain a better understanding of what our experience in our imagination of 4-space might be. Think about what four dimensions might be like, and you may have ideas about the kinds of questions the 2-D bug will have about three dimensions. You may know some answers, as well. The problem is finding a way to talk about them. Be creative!

One important thing to keep in mind is that it is possible to have images of things we cannot see. For example, when we look at a sphere, we can see only roughly half of it, but we can and do have an image of the entire sphere in our minds. We even have an image of the inside of the sphere, but it is impossible to actually see the entire inside or outside of the sphere all at once. Another example — sit in your room, close your eyes, and try to imagine the entire room. It is likely that you will have an image of the entire room, even though you can never see it all at once. Without such images of the whole room it would be difficult to maneuver around the room. The same goes for your image of the whole of the chair you are sitting on or this book you are reading.

Assume that the 2-D bug also has images of things that cannot be seen in their entirety. For example, the 2-D bug may have an image of a circle. Within a 2-dimensional world, the entire circle cannot be seen all at once; the 2-D bug can only see approximately half of the outside of the circle at a time and cannot see the inside at all unless the circle is broken. See Figure 22.2.

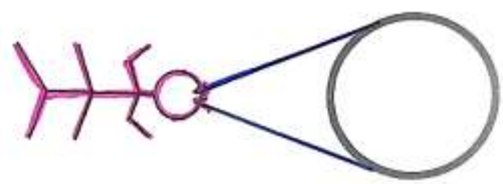


Figure 22.2 The 2-D bug sees a circle

However, from our position in 3-space we can see the entire circle including its inside. Carrying the distinction between what we can see and what we can imagine one step further, the 2-D bug cannot see the entire circle but can imagine in the mind the whole circle including inside and out. Thus, the 2-D bug can only imagine what we, from three dimensions, can directly see. So, the 2-D bug's image of the entire circle is as if it were being viewed from the third dimension. It makes sense, then, that the image of the entire sphere that we have in our minds is a 4-D view of it, as if we were viewing it from the fourth dimension.

When we talk about the fourth dimension here, we are not talking about time, which is often considered the fourth dimension. Here, we are talking about a fourth spatial dimension. A fuller description of our universe would require the addition of a time dimension onto whatever spatial dimensions one is considering.

Try to come up with ways to help the 2-D bug imagine a 2-sphere in 3-space. It may help to think of intersecting planes rotating with respect to each other: How will a 2-D bug in one of the planes experience it? Draw on the bug's experience living in two dimensions as well as some of your own experiences and attempts to imagine four dimensions.

WHAT IS 4-SPACE? VECTOR SPACES AND BASES

We could think of 4-space as $\mathbf{R}^4$:

Let $\mathbf{R}^4$ be the collection of 4-tuples of real numbers (x, y, z, w) with the *distance function (metric)*

$$d\big((a,b,c,d)(e,f,g,h)\big) = \sqrt{(a-e)^2 + (b-f)^2 + (c-g)^2 + (d-h)^2}$$

and

$$\textit{dot product}\ \ (a,b,c,d) \cdot (e,f,g,h) \equiv ae + bf + cg + de.$$

But this would be awkward sometimes because it fixes a given coordinate system; and we will find it geometrically useful to be able to change coordinates (or basis) to fit a particular problem. We will find it more powerful to have a description of space without given coordinates.

So instead of $\mathbf{R}^4$ we will think of 4-space as an (abstract) ***vector space***. A vector space has a point we call the origin **O**. Then designate every point ***P*** by a directed straight line segment **v** from **O** to ***P***. These directed line segments are, of course, what we call ***vectors***. We consider **O** itself also as a vector. If you have studied vector spaces only algebraically, then it may be difficult to see the geometric content. We assume that we can ***add vectors*** by geometrically defining **u** + **v** to be the diagonal of the parallelogram determined by **v** and **u**, as in Figure 22.3. For this definition to make sense we need to assume that we can form the parallelograms. It is also possible to not identify an origin and create what is called an ***affine space***. You can read about geometric affine spaces in [**DG**: Henderson], Appendix A, or [**DG**: Dodson & Poston].

Figure 22.3 Adding vectors

We assume that addition of vectors satisfies, for all **u** and **v** in *V*,

1. **u** + **v** is in *V*, and **O** + **u** = **u**;
2. **u** + **v** = **v** + **u**, and **u** + (**v** + **w**) = (**u** + **v**) + **w**; and
3. there is a vector −**v** in *V* such that **v** + (−**v**) = **O**.

Geometrically, we choose a segment whose length we designate as the unit length 1. With this unit length we can determine the length of any vector **v** — we denote its length by |**v**|. For *r* a real number and **v** a vector, we define geometrically the ***multiplication by scalars*** *r***v** as the vector with length *r*|**v**| and lying on the straight line determined by **v**.
This multiplication by scalars satisfies, for all **u** and **v** in *V* and all real numbers *r* and *s*,

4. *r***u** is in *V*, and 0**u** = **O**, and 1**u** = **u**;
5. (*r* + *s*)**u** = *r***u** + *s***u**, and *r*(**u** + **v**) = *r***u** + *r***v**; and
6. (*rs*)**u** = *r*(*s***u**).

We also assume that we can find the angle θ, $0 \le \theta \le \pi$, between any two vectors and we define the ***Euclidean inner product*** (sometimes called ***the standard inner product***) of two vectors to be:

$$\langle \mathbf{v}, \mathbf{w} \rangle = |\mathbf{v}|\,|\mathbf{w}| \cos\theta, \text{ where } \theta \text{ is the angle between } \mathbf{v} \text{ and } \mathbf{w}.$$

Note that $\langle \mathbf{v}, \mathbf{w} \rangle$ is negative when $\theta > \pi/2$. We can check that this inner product satisfies the following properties, for all **u**, **v**, **w** in *V* and all real numbers *r*,

7. $\langle \mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{w}, \mathbf{v} \rangle$
8. $r\langle \mathbf{v}, \mathbf{w} \rangle = \langle r\mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{v}, r\mathbf{w} \rangle$ for all reals r;
9. $\langle \mathbf{v}+ \mathbf{u}, \mathbf{w} \rangle = \langle \mathbf{v}, \mathbf{w} \rangle + \langle \mathbf{u}, \mathbf{w} \rangle$ and
10. $\langle \mathbf{v}, \mathbf{v} \rangle \geq 0$.

In a more abstract setting, we could simply *define* a ***vector space with inner product*** to be a set with {an origin (**O**), addition of vectors (+), multiplication by scalars, and real-valued function ($\langle , \rangle$)} that satisfies the 10 properties above. Then we define

$$|\mathbf{v}| = \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle} \text{ and } \cos\theta = \frac{\langle \mathbf{v},\mathbf{w} \rangle}{|\mathbf{v}||\mathbf{w}|},\ 0 \leq \theta \leq \pi.$$

Note that $\langle \mathbf{v}, \mathbf{w} \rangle = 0$ implies that **v** and **w** are perpendicular.

A ***subspace*** of a vector space V is a subset $U \subset V$ if $r\mathbf{u} + s\mathbf{v}$ is in U, for every pair of vectors **u**, **v** in U and all real numbers r, s. If $\{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_n\}$ is a finite collection of vectors from V, then we call the ***span of*** $\{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_n\}$, denoted by $\text{sp}\{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_n\}$, the smallest subspace of V containing each of $\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_n$. We say that $\{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_n\}$ are ***linearly independent*** if, for each i, $\mathbf{u}_i$ is not in

$$\text{sp}\{\mathbf{u}_1, \mathbf{u}_2, ...\mathbf{u}_{i-1}, \mathbf{u}_{i+1}, ..., \mathbf{u}_n\}.$$

If $\{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_n\}$ are linearly independent, then we say that

$U = \text{sp}\{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_n\}$ is an n-***dimensional subspace***

and that $\{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_n\}$ is a ***basis*** for U. In particular, we say V is m-***dimensional*** if

$$V = \text{sp}\{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_m\}$$

for some collection $\{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_m\}$ of m linearly independent vectors in V. If, in addition, $\langle \mathbf{u}_i, \mathbf{u}_j \rangle = 0$, $i \neq j$, and $\langle \mathbf{u}_i, \mathbf{u}_j \rangle = 1$ for all $0 \leq i, j \leq m$, then we say that $\{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_m\}$ is an ***orthonormal basis*** for V.

We will need the following result which is usually proved in an abstract linear algebra course:

> **Theorem** (**Gramm-Schmidt Orthogonalization**) *If V is an m-dimensional vector space with n- dimensional subspace U, then V has an orthonormal basis* $\{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_m\}$ *such that* $\{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_n\}$ *is an orthonormal basis for U.*

PROBLEM 22.2 A 3-SPHERE IN 4-SPACE

We will now explore 3-dimensional spheres in 4-space. We shall consider 4-space as a 4-dimensional vector space V^4.

Note that every plane in 3-space has exactly one line perpendicular to it at every point. A line is ***perpendicular*** to a plane if it intersects the plane and is perpendicular to every line in the plane that passes through the intersection point. In V^4 we can similarly

a. *Show that every 2-dimensional subspace (a plane containing the origin* **O***), Π, in V^4 has an* ***orthogonal complement****, $\Pi^\perp$, which is a 2-dimensional subspace (plane) that intersects Π only at* **O** *such that every line through* **O** *in Π is perpendicular to every line through* **O** *in $\Pi^\perp$.*

This was probably proved in your linear algebra course. The easiest proof (we think) is to pick an orthonormal basis for V^4 so that, in the new coordinates, Π is the span of the first two basis vectors. This is possible by the theorem above.

> **DEFINITIONS:** Let a 3-sphere, $\mathbf{S}^3$, be the collection of points in V^4 that are at a fixed distance r from **O**, the origin of V^4 and the ***center*** of the 3-sphere. The number r is called the ***radius*** of the sphere.
>
> We define a ***great circle*** on $\mathbf{S}^3$ to be the intersection of $\mathbf{S}^3$ with 2-dimensional subspace (a plane through the center **O**) in V^4.
>
> We define a ***great 2-sphere*** on $\mathbf{S}^3$ to be the intersection of $\mathbf{S}^3$ with any 3-dimensional subspace of V^4 (that passes through **O**).

b. *Show that every great 2-sphere in the 3-sphere has reflection-in-itself symmetry.*

Choose an orthonormal basis for V^4 so that the great 2-sphere is in the 3-subspace spanned by the first three basis elements.

c. *Show that every great circle has the symmetries in $\mathbf{S}^3$ of rotation through any angle and reflection through any great 2-sphere perpendicular to the great circle. Because these are principle symmetries of a straight line in 3-space, it makes sense to call these great circles* ***geodesics in $\mathbf{S}^3$****.*

Choose an orthonormal basis for V^4 so that the great circle is in the plane spanned by the first two basis elements.

d. *If two great circles in $\mathbf{S}^3$ intersect, then they lie in the same great 2-sphere.*

SUGGESTIONS FOR PROBLEM 22.2

Thinking in four dimensions may be a foreign concept to you, but believe it or not, it is possible to visualize a 4-dimensional space. Remember, the fourth dimension here is not time, but a fourth spatial dimension. We know that any two intersecting lines that are linearly independent (that do not coincide) determine a 2-dimensional plane. If we then add another line that is not in this plane, the three lines span a 3-space. When lines such as these are used as coordinate axes for a coordinate system, then they are typically taken to be orthogonal — each line is perpendicular to the others. Now to get 4-space, imagine a fourth line that is perpendicular to each of these original three. This creates the fourth dimension that we are considering.

Although we cannot experience all four dimensions at once, we can easily imagine any three at a time, and we can easily draw a picture of any two. This is the secret to looking at four dimensions. These 3- or 2-dimensional subspaces look exactly the same as any other

3-space or plane that you have seen before. This holds true for any subspace of V^4 — because, in any orthonormal basis, the basis vectors are orthogonal, any set of three of these will look the same and will determine (span) a subspace geometrically identical to our familiar 3-space, and any set of two basis elements will look like any other and will determine (span) a 2-dimensional plane. (You may find helpful Jeff Week's app http://geometrygames.org/Draw4D/index.html.)

For all of these problems, you should not be looking at projections of the 3-sphere into a plane or a 3-space, but rather looking at the part of the 3-sphere that lies in a subspace. For example, because the 3-sphere is defined as the set of points a distance r from the origin **O** in V^4, if you take any 3-dimensional subspace (through **O**) of V^4, then the part of the 3-sphere that lies in this 3-dimensional subspace is the set of points a distance r from its center **O** in the 3-subspace. So, any 3-dimensional subspace of V^4 intersects the 3-sphere in a 2-sphere, which you know all about by now, and you can easily visualize.

It is generally best, for all of the problems here, to draw pictures of various planes (2-dimensional subspaces) through the 3-sphere because they are easy to draw on a piece of paper. Remember, only include in your picture those geometric objects that lie in the plane you are drawing: a great circle that lies in this plane would be drawn as a circle, while another great circle that passed through this plane would intersect this plane only in two points. See Figure 22.4.

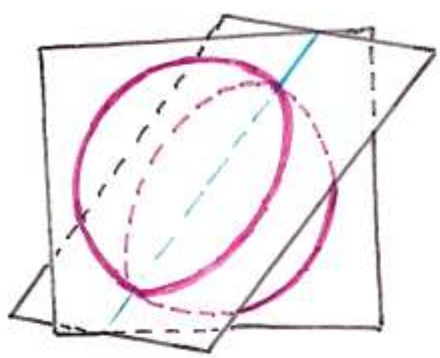


Figure 22.4 Intersecting great circles

For this particular problem, you are looking at the 3-sphere extrinsically. A good way to proceed is to draw several planes as outlined above and try to get an idea of how the planes relate to one another when combined into a 4-dimensional space. Once you have an understanding of how the different planes interact in four dimensions, it is fairly easy to show how the great circles of a 3-sphere behave.

PROBLEM 22.3 HYPERBOLIC 3-SPACE, UPPER HALF-SPACE

As mentioned previously, there is no smooth isometric embedding of a hyperbolic plane into 3-space and, thus, no analytic isometric description. In the same way there is no isometric analytic description of a hyperbolic 3-space in 4-space. Instead we will describe hyperbolic 3-space in terms of the upper-half-space model that is analogous to the upper-half-plane model for the hyperbolic plane, which was described in Chapter 17.

DEFINITION: Let $\mathbf{R}^{3+} = \{(x, y, z) \text{ in } \mathbf{R}^3 \mid z > 0\}$ and call it the ***upper half-space***.

In Chapter 17 we started with the annular hyperbolic plane and then defined a coordinate map $\mathbf{z}: \mathbf{R}^{2+} \to \mathbf{H}^2$. Now we do not have an isometric model $\mathbf{H}^3$ but, instead, we have to start with the upper half-space and use **z** to define $\mathbf{H}^3$. Recall that $\mathbf{z}: \mathbf{R}^{2+} \to \mathbf{H}^2$ has distortion ρ / b at the point (a,b) in $\mathbf{R}^{2+}$, where ρ is the radius of the annuli. As we saw in Chapter 17, we can study the geometry of the hyperbolic plane $\mathbf{H}^2$ by considering it to be the upper half-plane with angles as they are in $\mathbf{R}^{2+}$ and distances distorted in $\mathbf{R}^{2+}$ by ρ / b at the point (a, b). So now we use this idea to define hyperbolic 3-space $\mathbf{H}^3$.

DEFINITIONS: Define the ***upper-half-space model*** of hyperbolic space $\mathbf{H}^3$ to be the upper half-space $\mathbf{R}^{3+}$ with angles as they are in $\mathbf{R}^{3+}$ and with distances distorted by ρ / c at the point (a, b, c). We call ρ the ***radius*** of $\mathbf{H}^3$.

We define a ***great semicircle*** in $\mathbf{H}^3$ to be the intersection of $\mathbf{H}^3$ with any circle that is in a plane perpendicular to the boundary of $\mathbf{R}^{3+}$ and whose center is in the boundary of $\mathbf{R}^{3+}$ or the intersection of $\mathbf{R}^{3+}$ with any line perpendicular to the boundary of $\mathbf{R}^{3+}$. The ***boundary*** of $\mathbf{R}^{3+}$ are those points in $\mathbf{R}^3$ with $z = 0$.

We define a ***great hemisphere*** in $\mathbf{R}^{3+}$ to be the intersection of $\mathbf{R}^{3+}$ with a sphere whose center is on the boundary of $\mathbf{R}^{3+}$ in $\mathbf{R}^3$ or the intersection of $\mathbf{R}^{3+}$ with any plane that is perpendicular to the boundary of $\mathbf{R}^{3+}$ in $\mathbf{R}^3$. See Figure 22.5.

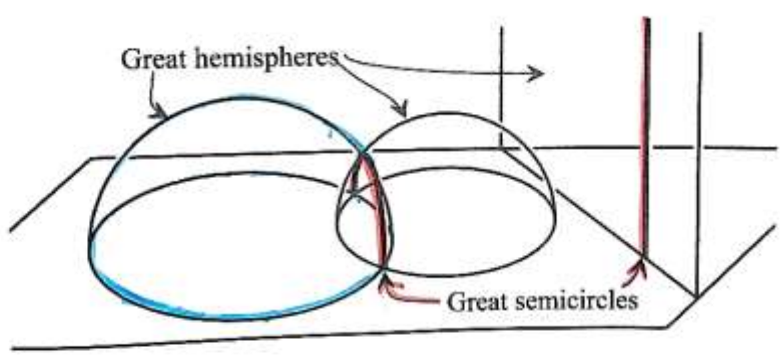


Figure 22.5 Upper-half-space model of $\mathbf{H}^3$

a. *Show that inversion through a great hemisphere in $\mathbf{R}^{3+}$ has distortion 1 in $\mathbf{H}^3$ and, thus, is an isometry in $\mathbf{H}^3$ and can be called a (hyperbolic) reflection through the great hemisphere.*

Look back at Problem **17.3**. Note that any inversion in a sphere when restricted to a plane containing the center of the sphere is an inversion of the plane in the circle formed by the intersection of the plane and the sphere.

b. *Show that, given a great semicircle* [*or great hemisphere*], *there is a hyperbolic reflection* (inversion through a great hemisphere) *that takes the great semicircle* [*hemisphere*] *to a vertical half-line* [*half-plane*] *in the upper half-space.*

Look back at Problems **16.2b** and **17.3b**. Note that any inversion in a sphere when restricted to a plane containing the center of the sphere is an inversion of the plane in the circle formed by the intersection of the plane and the sphere.

Any vertical half-plane is precisely an upper-half-plane model of $\mathbf{H}^2$. Thus, we conclude that each great hemisphere in $\mathbf{H}^3$ has the geometry of $\mathbf{H}^2$.

c. *Show that every great semicircle* Γ *has the symmetries in* $\mathbf{H}^3$ *of reflection through any great hemisphere perpendicular to* Γ *and rotation about* Γ *through any angle. Because these are principal symmetries of a straight line in 3-space it makes sense to call these great semicircles* ***geodesics in*** $\boldsymbol{H}^3$.

For the reflection, look in the half-plane containing Γ and in this plane use the arguments of Problem **16.2a**. For the rotation, look at great hemispheres containing Γ and restrict your attention to their intersections with the vertical half-plane that passes through the center of Γ and is perpendicular to Γ; then refer to Problem **5.4a**.

d. *If two great semicircles in* $\mathbf{H}^3$ *intersect, then they lie in the same great hemisphere.*

Use part **b** to assume that one of the great semicircles is a vertical half-line.

PROBLEM 22.4 DISJOINT EQUIDISTANT GREAT CIRCLES

a. *Show that there are two great circles in* $\mathbf{S}^3$ *such that every point on one is a distance of one-fourth of a great circle away from every point on the other and vice versa.*

Is there anything analogous to this in $\mathbf{H}^3$ *or in ordinary 3-space? Why?*

SUGGESTIONS

This problem is especially interesting because there is no equivalent theorem on the 2-sphere; we know that on the 2-sphere, all great circles intersect, so they can't be everywhere equidistant. The closest analogy on the 2-sphere is that a pole is everywhere equidistant from the equator. When we go up to the next dimension, this pole "expands" to a great circle such that every point on this great circle is everywhere equidistant from the equator. While this may seem mind-boggling, there are ways of seeing what is happening.

An important difference created by adding the fourth dimension lies in the orthogonal complement to a plane. In 3-space, the orthogonal complement of a plane is a line that passes through a given point. This means that for any given point on the plane (the origin is always a convenient point), there is exactly one line that is perpendicular to the plane at that point. Now what happens when you add the fourth dimension? In 4-space, the orthogonal complement to a plane is a plane. This means that every line in one plane is perpendicular to every line in the other plane. To understand how this is possible, think about how it works in 3-space and refer to Figure 22.6, where we are depicting $\mathbf{R}^4$ the 4-space with *x*, *y*, *z*, *w* coordinates. Now look at the *xy*-plane and the *zw*-plane. What do you notice? Why is every line through the center in one of these planes perpendicular to every line through the center in the other?

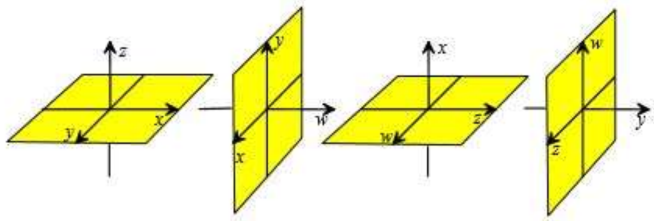


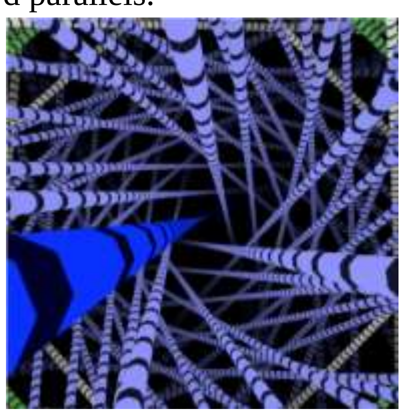
Figure 22.6 Orthogonal planes

Knowing this, look at the two great circles in terms of the planes in which they lie, and look at the relationships between these two planes, that is, where and how they intersect. Also, try to understand how great circles can be everywhere equidistant.

If we rotate along a great circle on a 2-sphere, all points of the sphere will move except for the two opposite poles of the great circle. If you rotate along a great circle on a 3-sphere, then the whole 3-sphere will move except for those points that are a quarter great circle away from the rotating great circle. Therefore, if you rotate along one of the two great circles you found above, the other great circle will be left fixed. But now rotate the 3-sphere simultaneously along both great circles at the same speed. Now every point is moved and is moved along a great circle!

b. *Write an equation for this rotation* (in *x, y, z, w* coordinates*) and check that each point of the 3-sphere is moved at the same speed along some great circle. Show that all of the great circles obtained by this rotation are equidistant from each other* (in the sense that the perpendicular distance from every point on one great circle to another of the great circles is a constant).

These great circles are traditionally called ***Clifford parallels***, named after William Clifford (1845–1879, English). See [**DG**: Thurston], pp 103–04, and [**DG**: Penrose] for readable discussions of Clifford parallels.



A computer generated view of the Clifford parallels in S 3 [Orbifold construction of the modes of the Poincaré dodecahedral space - Scientific Figure on ResearchGate. Available from: https://www.researchgate.net/figure/A-computer-generated-view-of-the-Clifford-parallels-in-S-3_fig1_231086397 [accessed 6 Feb, 2020]]

Symmetries of...	Reflection through...	Reflection through...	Half-turn about...	Rotation about...	Translation along...
line $l \subset \mathbf{E}^2$	l	line $\perp l$	point in l	NA	l
great circle $g \subset \mathbf{S}^2$	g	great circle $\perp g$	point/pair in g	poles of g	g
great semicircle $g \subset \mathbf{H}^2$	g	great semicircle $\perp g$	point/pair in g	NA	g
line $l \subset \mathbf{E}^3$	plane $\supset l$	plane $\perp l$	line $\perp l$ intersecting l	l	l
great circle $g \subset \mathbf{S}^3$	great sphere $\supset g$	great sphere $\perp g$	great circle $\perp g$ intersecting g	g	g
great semicircle $g \subset \mathbf{H}^3$	great hemisphere $\supset g$	great hemisphere $\perp g$	great semicircle $\perp g$ intersecting g	g	g
plane $P \subset \mathbf{E}^3$	P	plane $\perp P$	line in P	line $\perp P$	line $\subset P$
great sphere $G \subset \mathbf{S}^3$	G	great sphere $\perp G$	great circle in G	great circle $\perp G$	great circle $\subset G$
great hemisphere $G \subset \mathbf{S}^3$	G	great hemisphere $\perp G$	great semicircle in G	great semicircle $\perp G$	great semicircle $\subset G$

Figure 22.7 Symmetries in Euclidean, spherical, and hyperbolic spaces

PROBLEM 22.5 HYPERBOLIC AND SPHERICAL SYMMETRIES

We are now ready to see that the symmetries of great circles and great 2-spheres in a 3-sphere [and great semicircles and great hemispheres in a hyperbolic 3-space] are the same as the symmetries of straight lines and (flat) planes in 3-space. If g is a great circle in the 3-sphere, then let $g^\perp$ denote the great circle (from Problem **22.4**) every point of which is $\pi/2$ from every point of g.

a. *Check the entries in the table* (Figure 22.7), *which gives a summary of various symmetries of lines, great circles, and great semicircles and of (flat) planes, great 2-spheres, and great hemispheres.*

DEFINITION: A surface in a 3-sphere or in a hyperbolic 3-space is called ***totally geodesic*** if, for any pair of points on the surface, there is a geodesic (with respect to $\mathbf{S}^3$ or $\mathbf{H}^3$) that joins the two points and lies entirely in the surface.

b. *Show that a great 2-sphere in* $\mathbf{S}^3$ *(with radius r) is a totally geodesic surface and is itself a 2- sphere of the same radius r.*

c. *Show that a great hemisphere is a totally geodesic surface in* $\mathbf{H}^3$ *(with radius r) and is isometric to a hyperbolic plane with the same radius r.*

In the upper-half-space model there is a hyperbolic reflection that takes every great hemisphere to a plane perpendicular to the boundary. (See Problem **22.3**.)

PROBLEM 22.6 TRIANGLES IN 3-DIMENSIONAL SPACES

Show that if A, B, C are three points in $\mathbf{S}^3$ [*or in* $\mathbf{H}^3$] *that do not all lie on the same geodesic, then there is a unique great 2-sphere* [*hemisphere*], $\mathbf{G}^2$, *containing A, B, C. Thus, we can define ΔABC as the (small) triangle in* $\mathbf{G}^2$ *with vertices A, B, C. With this definition,* ***triangles in*** $\mathbf{S}^3$ **[*or in*** $\mathbf{H}^3$**] *have all the properties that we have been studying of small triangles on a sphere*** **[*or triangles in a hyperbolic plane*].**

SUGGESTIONS

Think back to the suggestions in Problems **22.2** and **22.3** — they will help you here, as well. Take two of the points, *A* and *B*, and show that they lie on a unique plane through the center, **O**, of the 3-sphere [or a unique plane perpendicular to the boundary of $\mathbf{R}^{3+}$]. Then show that there is a unique (shortest) geodesic in this plane. See Figure 22.8.

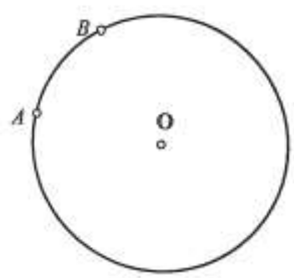


Figure 22.8 Great circle through A and B

Think of *A*, *B*, and *C* as defining three intersecting great circles [or semicircles]. On a 3-sphere, look at the planes in which these great circles lie and where the two planes lie in relation to one another. In hyperbolic 3-space, use a hyperbolic reflection to send one of the great semicircles to a vertical line.

Be sure that you show that the great 2-sphere (hemisphere) containing *A, B, C* is unique.