Chapter 21

MECHANISMS

The mathematical investigations referred to bring the whole apparatus of a great science to the examination of the properties of a given mechanism, and have accumulated in this direction rich material, of enduring and increasing value. What is left unexamined is however the other, immensely deeper part of the problem, the question: How did the mechanism, or the elements of which it is composed, originate? What laws govern its building up? — F. Reuleaux, Kinematics (1876), p. 3

In this chapter we will study mechanisms, which for our purposes we define as collections of rigid bodies with moveable connections having the purpose of transforming motion. We have already studied two mechanisms: an angle-trisecting mechanism in Problem **15.4d** and the Peaucellier-Lipkin straight-line mechanism in Problem **16.3**. A machine can be considered as a combination of mechanisms connected together in a way to do useful work. In this chapter we will use the Law of Cosines and the Law of Sines for the plane and sphere in Problems **21.1** and **21.2**. Otherwise the material demands only basic understanding of plane and spheres.

INTERACTIONS OF MECHANISMS WITH MATHEMATICS

We mentioned in Chapter 0 that one of the strands in the history of geometry is the Motion/Machines Strand; and we showed in Chapter 1 how this strand led to mechanism for producing straight line motion. It is known that the Greeks, in particular Aristotle, studied the so-called simple machines: the wheel, lever, pulleys, and inclined plane. He also described gear wheel drive in windlasses and pointed out that the direction of rotation

is reversed when one gear wheel drives another gear wheel. Archimedes made devices to multiply force or torque many times and studied spirals and helices for mechanical purposes. Archimedes screw idea is still used.

Archimede's screw still in use

We know Hero's formula for the area of a triangle but in his time, he was better known as an engineer. Most of his inventions are known from the writings of the Roman engineer Vitruvius. However, there was little application of Greek natural science to engineering in antiquity. In fact, engineering contributed far more to science than science did to engineering until the latter half of the 19th century. For more discussion of this history, see [**ME**: Kirby], p. 43.

One of the simplest mechanisms used in human activities are linkages. Perhaps an idea of using linkages came into somebody's mind because a linkage resembles a human arm. We can find linkages in old drawings of various machines in 13th century. See Figure 1.3.

Leonardo da Vinci drawing of machine from *Codex Madridi* (1493)

Leonardo da Vinci's *Codex Madridi* (1493) contained a collection of machine elements, *Elementi Macchinali*, and he invented a lathe for turning parts with elliptical cross section, using a four-bar linkage (see Problem **21.1**). Georgius Agricola (1494–1555) is considered a founder of geology as a discipline but he gave descriptions of machines used in mining,

and that is where we can find pictures of linkages used in these machines. Gigantic linkages, principally for mine pumping operations, connected water wheels at the riverbank to pumps high up on the hillside. Such linkages consisted mostly of what we call four-bar linkages; see Problem **21.1**.
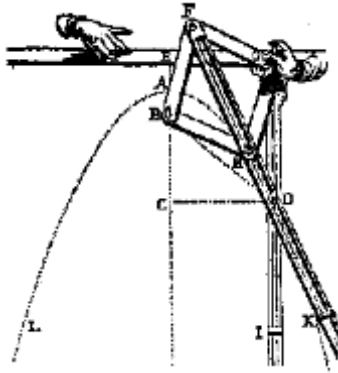

Figure 21.1 Mechanism for drawing parabola (1657)

Mathematicians got interested in linkages first for geometric drawing purposes. We know about some such devices from ancient Greek mathematics. (For example, see Problem **15.4** about devices for trisecting an angle.) When Rene Descartes published his *Geometry* (1637) he did not create a curve by plotting points from an equation. There were always first given geometrical methods for drawing each curve with some apparatus, and often these apparatuses were linkages. See, for example, in figure above the mechanism for drawing a parabola that appear in the works of Franz von Schooten (1615–1660) that were a popularization of Descartes' work. Isaac Newton developed mechanisms for the generation of algebraic curves of the third degree. This tradition of seeing curves as the result of geometric actions can be found also in works of Roberval, Pascal, and Leibniz. Mechanical devices for drawing curves played a fundamental role in creating new symbolic languages (for example, calculus) and establishing their viability. The tangents, areas, and arc length associated with many curves were known before any algebraic equations were written. Critical experiments using curves allowed for the coordination of algebraic representations with independently established results from geometry. For more detailed discussion of the ideas in the last two paragraphs, see [**HI**: Dennis], Chapter 2.

 Linkages are closely related with kinematics or geometry of motion. First it was the random growth of machines and mechanisms under the pressure of necessity. Much later, algebraic speculations on the generation of curves were applied to physical problems.

Two great figures appeared in 18th century, Leonard Euler (1707–1783) and James Watt (1736–1819). Although their lives overlap there was no known contact between them. But both were involved with the "geometry of motion." Watt, instrument maker and engineer, was concerned with designing mechanisms that produce desired motions. Watt's search for a mechanism to relate circular motion with straight-line motion is discussed in the historical introduction to Chapter 1. Euler's theoretical results were unnoticed for a century by the

engineers and mathematicians who were devising linkages to compete or supersede Watt's mechanism.
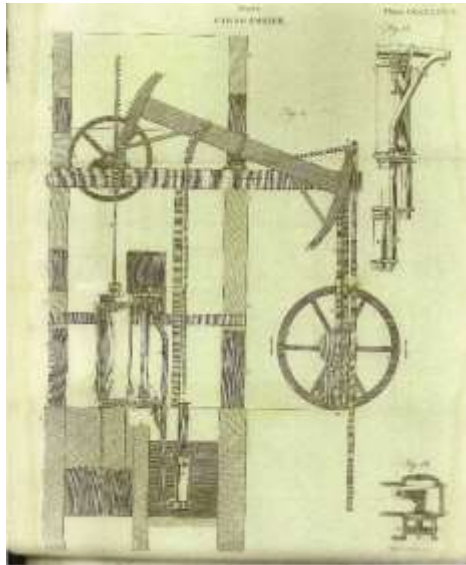
Watt's steam engine

The fundamental idea of the geometric analysis of motion (kinematics) stems from Euler, who wrote in 1775,

> The investigation of the motion of a rigid body may be conveniently separated into two parts, the one geometrical, the other mechanical. In the first part, the transference of the body from a given position to any other position must be investigated without respect to the causes of motion, and must be represented by analytical formulae, which will define the position of each point of the body. This investigation will therefore be referable solely to geometry ... . [Euler, Novi commentarii Academiae Petrop., vol. XX, 1775. Translation in Willis, Principles of Mechanism, 2nd ed., p. viii, 1870.]

These two parts are sometimes called kinematics (geometry of motion) and kinetics (the mechanics of motion). Here we can see beginnings of the separation of the general problem of dynamics into kinematics and kinetics.

Franz Reuleaux (1829–1905) divided the study of machines into several categories and one of them was study of the geometry of motion ([**ME**: Reuleaux], pp. 36–40). In the proliferation of machines at the height of the Industrial Revolution, Reuleaux was systematically analyzing and classifying new mechanisms based on the way they constrained motion. He hoped to achieve a logical order in engineering. The result would be a library of mechanisms that could be combined to create new machines. He laid the foundation for a systematic study of machines by determining the basic building blocks and developing a system for classifying known mechanism types. Reuleaux created at Berlin a collection of over 800 models of mechanisms and authorized a German company, Gustav Voigt, *Mechanische Werkstatt*, in Berlin, to manufacture these models so that technical schools could use them for teaching engineers about machines.

Figure 21.2 Reuleaux models (photo Prof. F. Moon)

In 1882, Cornell University acquired 266 of such models, and now the remaining 219 models is the largest collection of Reuleuax kinematic mechanisms in the world. See examples in Figure 21.2. In 2003 a team of Cornell mathematicians, engineers, and librarians released a digital Reuleaux kinematic model Web site [**ME**: KMODDL;

   http://wayback.archive-it.org/2566/20180418122029/http://kmoddl.library.cornell.edu/]

The Web site contains photos, mathematical descriptions, historical descriptions, moving virtual reality images, simulations, learning modules (for middle school through undergraduate), and downloadable files for 3D printing. In this chapter we will look at the mathematics related to a few of these mechanisms in order to show geometry in Machine/Motion Strand.

There will be discussions of more recent history later in the chapter.

PROBLEM 21.1 FOUR-BAR LINKAGES

A four-bar linkage is a mechanism that lies in a plane (or spherical surface) and consists of four bars connected by joints that allow rotation only in the plane (or sphere) of the mechanism. See Figure 21.3.
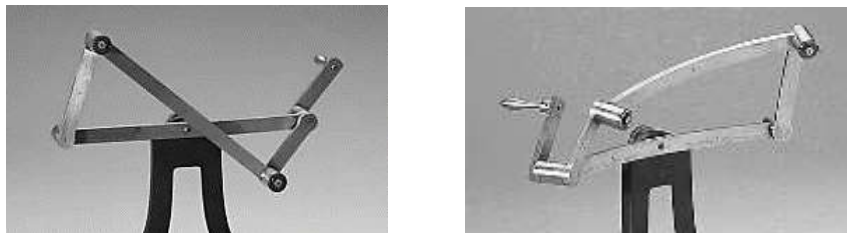

Figure 21.3   Reuleaux four-bar linkages: planar and spherical

In normal practice one of the links is fixed so that it does not move. In Figure 21.4 we assume that the link OC is fixed and investigate the possibilities of motion for the other three links. We call the link *OA* the **input crank** and link *CB* the **output crank**. Similarly, we call the angle θ the input angle and angle ϕ the **output angle**.
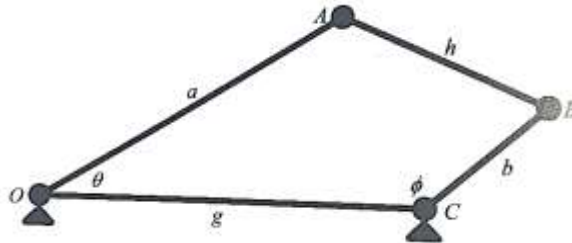
Figure 21.4 Four-bar linkage

a. (Plane) *The input crank will be able to swing opposite C (where the input angle is $\pi = 180$ degrees) only if $a + g < b + h$. If $a + g > b + h$, there will be a maximum input angle, θ_{max}, satisfying*

$$\cos \theta_{max} = [\,(g^2 + a^2) - (h+b)^2\,]\,/2ag$$

What happens if $a + g = b + h$?

You may find it helpful to experiment with four-bar linkages that you make out of strips of cardboard; and/or to play with some online simulations. Apply the Law of Cosines (Problem **20.2**).

We can now do the same analysis on the sphere, using the spherical Law of Cosines (**20.2**) and the special absolute value $|l|_S$ (see the section Triangle Inequality just before Problem **6.3**). On the plane, $|l|S = |l| =$ length of l. On the sphere, $|l|_S =$ (shortest) distance between the endpoints of l.

b. (Sphere) *The input crank will be able to swing opposite C (where the input angle is $\pi = 180$ degrees) only if $|a + g|_S < b + h$. If $|a + g|_S > b + h$, there will be a maximum input angle, θ_{max}, satisfying,*

$$\cos \theta_{max} = [\,\cos(h+b) - \cos a \cos g\,]/\sin a \sin g,$$

where the edge lengths are measured by the radian measure of the angle they subtend at the center of the sphere. What happens if $|a + g|_S = b + h$?

We can now finish this problem working with both the plane and the sphere at the same time.

c. (Plane or Sphere) *Similarly, whenever $|b - h| > |g - a|$ there will be a minimum input angle, θ_{min}, satisfying*

$$\cos \theta_{min} = [\,g^2 + a^2 - |b-h|^2\,]\,/2ag \text{ (plane)},$$
$$\cos \theta_{min} = [\,\cos|b-h| - \cos a \cos g\,]/\sin a \sin g \text{ (sphere)}.$$

The minimum angle is actualized when either B-A-C is straight or A-C-B is straight, as in Figure 21.5.
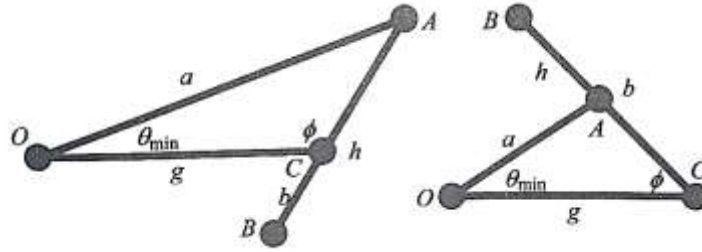
Figure 21.5 Minimum input angles

Thus, we have four types of input cranks:

1. A ***crank*** if the link *OA* can freely rotate completely around *O*. In this case,
$$b + h > |a + g|_S \text{ and } |a - g| > |b - h|.$$
2. A ***0-rocker*** if there is a maximum input angle but the link *OA* can rotate freely past $\theta = 0$. Then
$$b + h < |a + g|_S \text{ and } |a - g| > |b - h|.$$
3. A ***π-rocker*** if there is a minimum input angle but the link *OA* can rotate freely past $\theta = \pi$. Then
$$b + h > |a + g|_S \text{ and } |a - g| < |b - h|.$$
4. A ***rocker*** if there is a maximum input angle and minimum input angle. In this case,
$$b + h < |a + g|_S \text{ and } |a - g| < |b - h|.$$

The analysis of the output crank is exactly symmetric to above with the lengths *a* and *b* interchanged. In particular,

d. *If* $|b + g|_S > a + h$, *then there is a maximum output angle* ϕ_{max} *that satisfies*
$$\cos \phi_{max} = [(g^2 + b^2) - (h+a)^2]/2ag \text{ (plane)}$$
$$\cos \phi_{max} = [\cos(h+a) - \cos b \cos g]/\sin b \sin g \text{ (sphere)}.$$

If $|a - h| > |g - b|$, *there is a minimum output angle* ϕ_{min} *min that satisfies*
$$\cos \phi_{min} = [g^2 + b^2 - |a-h|^2]/2bg \text{ (plane)},$$
$$\cos \phi_{min} = [\cos|a-h| - \cos b \cos g]/\sin b \sin g \text{ (sphere)}.$$

Thus, we have four types of output cranks:
1. A ***crank*** if the link *CB* can freely rotate completely around *C*. In this case,
$$a + h > |b + g|_S \text{ and } |b - g| > |a - h|.$$
2. A ***0-rocker*** if there is a maximum output angle but link *CB* can rotate freely past $\phi = 0$. Then
$$a + h < |b + g|_S \text{ and } |b - g| > |a - h|.$$
3. A ***π-rocker*** if there is a minimum output angle but the link *CB* can rotate freely past $\phi = \pi$. Then
$$a + h > |b + g|_S \text{ and } |b - g| < |a - h|.$$

4. A *rocker* if there is a maximum output angle and minimum output angle. In this case,
$$a + h < |b + g|_S \text{ and } |b - g| < |a - h|.$$

e. *Putting these together we get eight types of four-bar linkages.*

Make a model of each type using cardboard strips or explore them with GeoGebra.

1. A *double crank* in which the input and output links are cranks.
$$b + h > |a + g|_S,$$
$$|a - g| > |b - h|, a + h > |b + g|_S, \text{ and } |b - g| > |a - h|.$$

2. A *crank-rocker* if the input link is a crank and the output link is a rocker.
$$b + h > |a + g|_S,$$
$$|a - g| > |b - h|, a + h < |b + g|_S, \text{ and } |b - g| < |a - h|.$$

3. A *rocker-crank* if the input link is a rocker and the output link is a crank.
$$b + h < |a + g|_S,$$
$$|a - g| < |b - h|, a + h > |b + g|_S, \text{ and } |b - g| > |a - h|.$$

4. A *rocker-rocker* if both the input and the output link are rockers.
$$b + h < |a + g|_S,$$
$$|a - g| < |b - h|, a + h < |b + g|_S, \text{ and } |b - g| < |a - h|.$$

5. A *00 double rocker* if there are maximum input and output angles and no minimums, thus both cranks move across the fixed OC.
$$b + h < |a + g|_S,$$
$$|a - g| > |b - h|, a + h < |b + g|_S, \text{ and } |b - g| > |a - h|.$$

6. A *0π double rocker* if the input angle has a maximum and no minimum but the output angle has a minimum but no maximum.
$$b + h < |a + g|_S,$$
$$|a - g| > |b - h|, a + h > |b + g|_S, \text{ and } |b - g| < |a - h|.$$

7. A *π0 double rocker* if the input angle has a minimum and no maximum but the output angle has a maximum but no minimum.
$$b + h > |a + g|_S,$$
$$|a - g| < |b - h|, a + h < |b + g|_S, \text{ and } |b - g| > |a - h|.$$

8. A *ππ double rocker* if the input and output angles both have minimums but no maximums and move freely on the ends of OC.
$$b + h > |a + g|_S,$$
$$|a - g| < |b - h|, a + h > |b + g|_S, \text{ and } |b - g| < |a - h|.$$

Check that the other eight combinations (a 0 or π rocker combined with a crank or rocker) are not possible. This can be done either analytically (using the inequalities) or geometrically (by noting symmetries).

All the other four-bar linkages are the cases when one or more of the inequalities become equalities, in each of these cases the linkage can be folded. That is, the linkage has a configuration in which all the links line up with OC. Some four-bar linkages can be folded

in more than one way; for example, the linkage with $a = h = b = g$ can be folded in three different ways (*Try it!*). (Examples in GeoGebra created by Steve Phelps https://www.geogebra.org/m/xmAST89t)

PROBLEM 21.2 UNIVERSAL JOINT

Almost all vehicles with an engine in front that drives the rear wheels have a drive shaft that transmits the power from the engine to the rear axle. It is important that the drive shaft be able to bend as the vehicle goes over bumps. The usual way to accomplish this "bending" is to put in the drive shaft a ***universal joint*** (also known as Hooke's joint or Cardan's joint). See Figure 21.6.

In 1676, Robert Hooke (1635–1703) published a paper on an optical instrument that could be used to study the sun safely. In order to track the sun across the sky, the device featured a control handle fitted with a new type of joint that allowed twisting motion in one shaft to be passed on to another, no matter how the two shafts were orientated. Hooke gave this the name "universal joint." This joint was earlier suggested by Leonardo da Vinci and also is attributed to Girolamo Cardano. Therefore, on the European continent it got name "Cardan's joint," but in Britain the name of "Hooke's joint" was used.

Figure 21.6   Universal joint

a. *The universal joint can be considered to be a spherical four-bar linkage with* $a = b = h = \pi/2$*. The fixed (grounded) link g is the angle between the input and output shafts that can be adjusted in the range* $\pi/2 < g \leq \pi$*.*

The links a, h, and g are not actually links; however, the constraints of the mechanism operate as if they were spherical links. See Figure 21.7.

b. *For what lengths of the fixed link (angles between the input and output shafts) is the universal joint a double crank* (see **21.1e**)?

This is of utmost importance in its automotive use because as an automobile goes over bumps the angle *g* between the shafts will change.

We now look at the relationship between the rotation of the input shaft to the rotation of out shaft. One of the problems with the universal joint is that, though one rotation of the input shaft results in one rotation of the output shaft, the rotations are not in sync during the revolution.

c. *Check that Figure 21.7 is correct. In particular,*

i. *A is the pole for the great circle (dashed in the figure) passing through O and B; and the great circle arcs h and a intersect this great circle at right angles.*

ii. *Likewise, B is the pole for the great circle passing through A and C; and the great circle arcs h and b also intersect this great circle at right angles.*

iii. *The arc h must bisect the lune determined by these two dashed great circles. The angle of this lune at P must be $\pi/2$ as marked.*

iv. *If we use α to label the angle at A and β to label the angle at B, as in the figure, then radian measure of the arc OB is α, and the radian measure of the arc AC is β.*
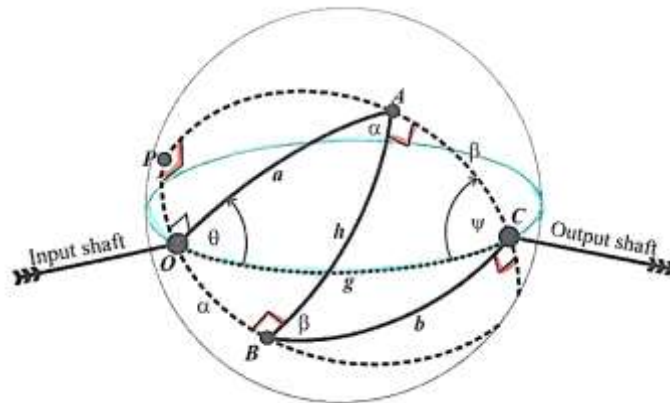

Figure 21.7 Universal joint as a spherical mechanism

The angle θ measures the rotation of the input shaft and the angle ψ measures the rotation of the output shaft. Note that, when $\theta = 0$ then $\psi = 0$, also. As θ changes in the positive counterclockwise direction, ψ will be changing in the negative clockwise direction; thus, when θ is positive, ψ will be negative and we will need to use $-\psi$ when denoting the angle in the triangle OCA.

d. *The input and output angles satisfy $\tan(-\psi) = \tan\theta/(-\cos g)$. In practice, g is near π and $-\cos g$ is positive. Note that when $g = \pi$, $\tan(-\psi) = \tan\theta$ and thus the two shafts turn in unison.*

Hint: Apply the Spherical Pythagorean Theorem (Theorem **20.2a**) to the right triangle *OPC*, and the Law of Sines (Problem **20.3b**) to triangles *OCA* and *OCB*.

PROBLEM 21.3   REULEAUX TRIANGLE AND CONSTANT WIDTH CURVES

What is this triangle?   If an enormously heavy object must be moved from one spot to another, it may not be practical to move it on wheels. Instead the object is placed on a flat platform that in turn rests on cylindrical rollers. As the platform is pushed forward, the rollers left behind are picked up and put down in front. An object moved this way over flat horizontal surface does not bob up and down as it rolls along. The reason is that cylindrical rollers have a circular cross section, and a circle is closed curve with *constant width*. What does this mean? If a closed convex curve is placed between two parallel lines and the lines are moved together until they touch the curve, the distance between the parallel lines is the curve's width in one direction. Because a circle has the same width in all directions, it can be rotated between two parallel lines without altering the distance between the lines.

Figure 21.8 Mechanisms utilizing a Reuleaux triangle: Reauleaux model and rotary (Wankel) engine

Is the circle the only curve with constant width? Actually, there are infinitely many such curves. The simplest noncircular such curve is named the Reuleaux triangle. Mathematicians knew it earlier (some authors refer to Leonard Euler in 18th century), but Reuleaux was the first to demonstrate and use its constant width properties. In Figure 21.8 there is a Reuleaux model using the Reuleaux triangle and an image of inside of a Wankel engine (similar to that used in some Mazda automobiles) showing the rotor in the shape of a Reuleaux triangle.

A Reuleaux triangle can be constructed starting with an equilateral triangle of side *s* and then replacing each side by a circular arc with the other two original sides as radii. See Figure 21.9.
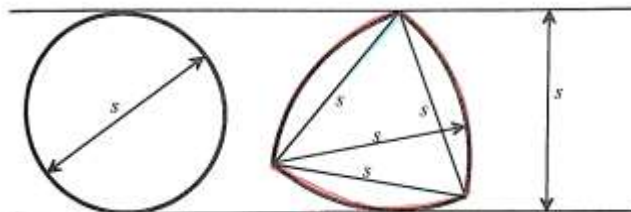

Figure 21.9 Circle and Reuleaux triangle of same width

a. *Why will the Reuleaux triangle make a convenient roller but not a convenient wheel?*
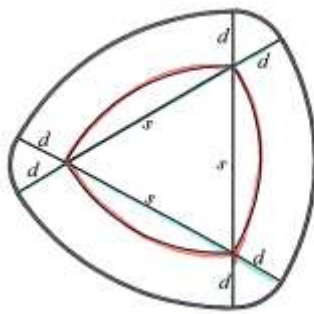

Figure 21.10 Smoothed Reuleaux triangle

The Reuleaux triangle has corners, but if you want to smooth out the corners you can extend a Reuleaux triangle a uniform distance *d* on every side as in Figure 21.10. Then you can

b. *Show that the resulting curve has constant width s + 2d.*

Figure 21.11 Constant width coins

Other symmetrical curves of constant width result if you start with a regular pentagon (or any regular polygon with an odd number of sides) and follow similar procedures. See Figure 21.11 for examples of British coins that are the shape of a constant width curve based on the heptagon. *What advantages would these British coins have due to their shape?*

But here is one surprising method of constructing curves with constant width: Draw as many straight lines as you please such that each line intersects all the others. See Figure 21.12. On one of the lines start with a point sufficiently far away from the intersections. Now draw an arc from this point to an adjacent line, with the compass point at the intersection of the two lines. Then, starting from the end of this arc, draw another arc connecting to the next line with the compass point at the intersection of these two lines. Proceed in this manner from one line to the next, as indicated in Figure 21.12.
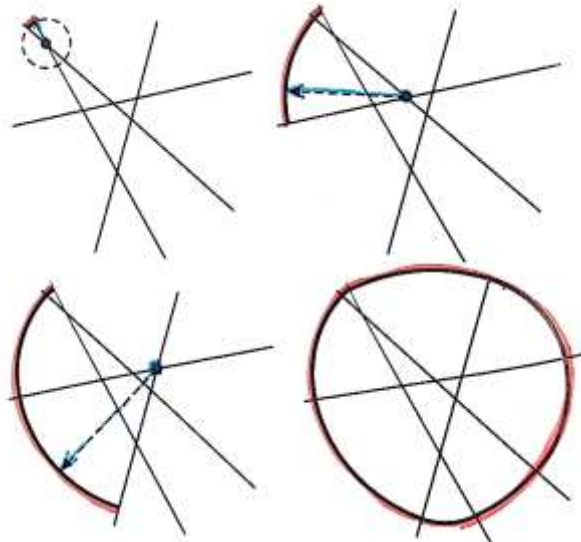
Figure 21.12 Creating irregular curves of constant width

If you do it carefully, the curve will close and will have a constant width. (You can try to prove it! It is not difficult at all.) The curves drawn in this way may have arcs of as many different circles as you wish. The example in Figure 21.12 shows steps in drawing such curves, but you will really enjoy making your own. After you have done that, you can make several more copies of it and check that your wheels really roll!

    c. *Prove that the procedure in the last paragraph produces a curve of constant width as long as all the intersection points are inside the curve. Can you specify how far out you have to start in order for this to happen?*

The Reuleaux triangle has been used to make a drill bit that will drill a (almost) square hole. See Figure 21.13.
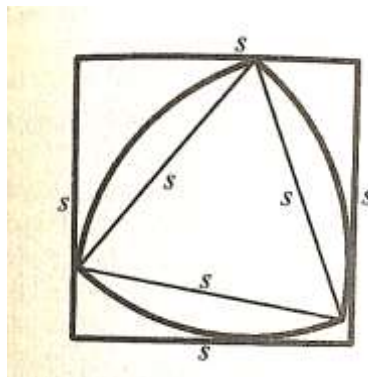

Figure 21.13 Reuleaux triangle in a square

    d. *A Reuleaux triangle of width s can be turned completely around within a square of side s in such a way that, at each time of the motion, the Reuleaux triangle will be tangent simultaneously to all four sides of the square. Describe the small spaces in the corners of the square that the Reuleaux triangle will not reach.*

The interested reader may find more information and references about Reuleaux triangles online [http://pi.math.cornell.edu/~dtaimina/Reuleaux/Reuleaux.htm]. Another good source is "Mechanical Circle-Squaring" by Barry Cox and Stan Wagon, *The College Mathematics Journal*, vol. 40, No.4, Sept.2009, p. 238-247.

To sharpen your interest, we list some further properties of Reuleaux triangles:

1. The inscribed and circumscribed circles of an arbitrary curve of constant width h are concentric and the sum of their radii is equal to h.
2. Among curves with constant width h, the circle bounds the region of greatest area and the Reuleaux triangle bounds the region of least area.

3. Any curve of constant width h has perimeter equal πh.

4. The corners of a Reuleaux triangle are the sharpest possible on a curve with constant width.

5. The circle is the only curve of constant width with central symmetry.

6. For every point on a curve of constant width there exists another point with the distance between the two equaling the width of the curve, and the line joining these two points is perpendicular to the support lines at both points. (Support lines are lines that touch the curve and the curve lies totally on one side of the line.)

7. There is at least one supporting line through every point of a curve of constant width.

8. If a circle has three (or more) points in common with a curve of constant breadth h, then the length of the radius of the circle is at most h.

INVOLUTES

Reuleaux used the geometric idea of involutes in the design of several mechanisms including a pump (the middle image in Figure 21.2) and gear design (Figure 21.15). Before we discuss these, we must describe the geometry of the involute. Focus on one of the four arms in the pump in Figure 21.14.
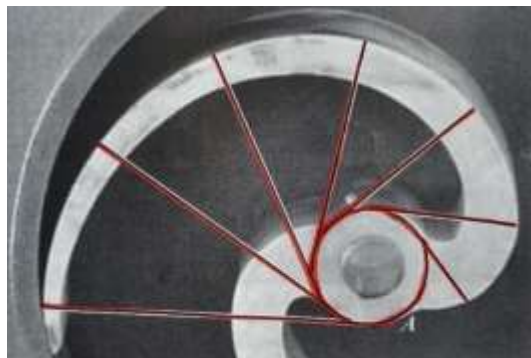

Figure 21.14 Involute arm

Imagine that the outlined circle in picture is a spool that has a black thread with white edging wrapped around it in such a way that when fully wound the end of the thread is at the point *A*. Now imagine unwinding the thread, keeping the spool fixed, and keeping the thread pulled taut. The end of the thread traces the outer edge of the spiral arm. This curve is called the *involute of a circle* (in this case, the outlined circle).

Instead of keeping the spool fixed and unwinding the thread, we could rotate the spool and pull the thread taut in the same direction. It is this latter view that we will use in analyzing the spiral pump.

Examine the picture of the spiral pump in Figure 21.15. Now imagine that there is thread rolled around the left spool and then pulled taut and wrapped around the right spool as indicated in the picture. Place a small circle on the thread at the place that two arms touch each other. Now, instead of unwrapping the thread, we will turn the two spools at the same rate, always keeping the thread taut. Since the spiral arms are in the shape of an involute, the small circle will follow the outer edge of both spiral arms. Thus, as the spools rotate, the spiral arms will stay in contact.
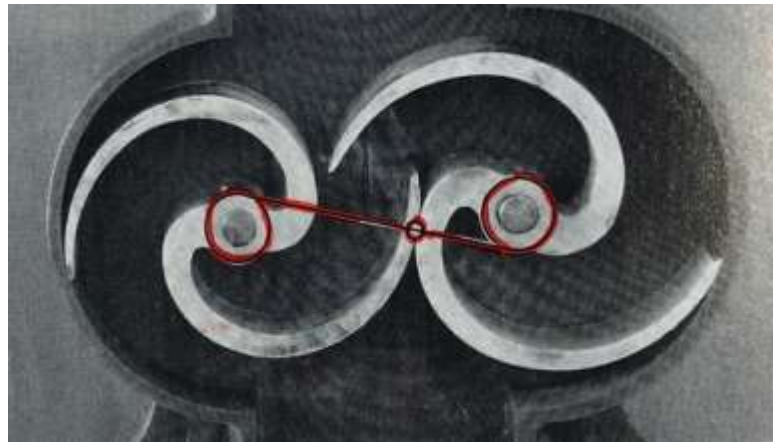

Figure 21.15 Spiral pump with thread and small circle

The two rotors in the spiral pump can be thought of as gears with two teeth. The same discussion above illustrates why it is advantageous for gear teeth, in general, to be in the shape of an involute curve so that the gear teeth stay in contact throughout uniform turning of the gears. If the axes of two engaging gears are parallel, then the involute is a planar involute, as described above. However, if the axes are not parallel, then the gears can be considered as on a sphere whose center is at the intersection of the two axes. An involute of a circle (on plane or sphere) can be described either as unrolling a taut string from the circle or as rolling of a straight line along the circle.

Figure 21.16 Spherical involute curve

Reuleaux designed several models to illustrate spherical involutes by rolling a straight line (great circle) on a small circle. See Figure 21.16. You can watch a tutorial about modelling involute gears https://www.youtube.com/watch?v=DqBOva04lcE

LINKAGES INTERACT WITH MATHEMATICS

After Descartes and others used linkages to draw curves, it was natural for mathematicians to ask the question of what curves could be drawn by linkages. In [**ME**: Kempe 1876; https://archive.org/details/howtodrawstraigh00kemprich/page/n6], A. B. Kempe gave a proof that any algebraic curve may be described by a linkage.

David's copy of Todhunter's *How to Draw a Straight Line*

The idea for Kempe's proof, as discussed in [**ME**: Artobolevski], is as follows:

Consider the algebraic curve $f(x, y) = 0$, which can be expressed in the form $\Sigma\, A_{mn}\, x^m\, y^n = 0$, where the coefficients A_{mn} are constant. Thus, the generation of the curve reduces to a series of mathematical operations. Kempe's idea was that each of these mathematical operations can be fulfilled by individual linkages, which can then be linked together into a kinematic chain of linkages. Linkages needed for this are as follows:

a. Linkage for translating a point along a given straight line (for example, the Peaucellier-Lipkin linkage in Problem **16.3**);

b. Linkage for projecting a given point onto a given line;

c. Linkage that cuts off on one axis a segment equal to a given segment on the other axis;

d. Linkage that determines a straight line that passes through a given point and is parallel to a given line;

e. Linkage that, given two segments *r* and *s* on one line and one segment *t* on another line, will obtain a second segment *u* on the second line such that *r/s = t/u* (Multiplier);

f. Linkage for the addition of two given segments (Adder).

For more details on this proof, see [**ME**: Artobolevski; https://archive.org/details/ArtobolevskyMechanismsInModernEngineeringDesignVol1/page/n13] For a different proof with more explicit pictures of the constructions of linkages and their combinations, see [**ME**: Yates, https://archive.org/details/YatesHandbookCurves1947], Section 11.

In a different direction, there was for many years an open question that appeared in robotics, topology, discrete geometry, and pattern recognition:

> *Given a linear* **chain** *of links* (each one connected to the next to form a polygonal path without self-intersections) *or a* **cycle** *of links* (a linear chain with the first and last links joined), *then is it possible to find a motion of the chain or cycle during which there continue to be no self-intersections and, at the end of the motion, the chain forms a straight line and the cycle forms a convex polygon.* [See Figure 21.17.]
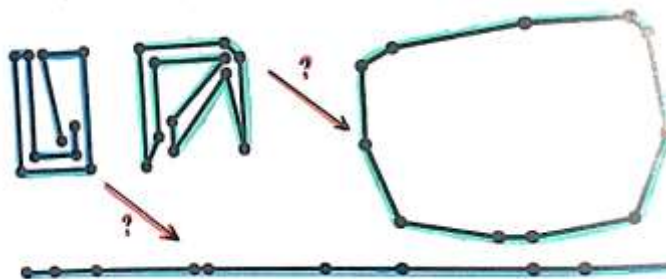

Figure 21.17   Straightening and convexifying linkages

In 2002, Robert Connelly, Erik D. Demaine, and Günter Rote published a paper, "Straightening Polygonal Arcs and Convexifying Polygonal Cycles" [**ME**: Connelly; https://erikdemaine.org/papers/LinkageTR/paper.pdf], in which they solved this problem positively and, in addition, proved that their motion is piecewise differentiable, does not decrease the distance between any pair of vertices, and preserves any symmetry present in the initial configuration.