Chapter 17

PROJECTIONS (MODELS) OF HYPERBOLIC PLANES

In recent times the mathematical public has begun to occupy itself with some new concepts which seem to be destined, in the case they prevail, to profoundly change the entire order of classical geometry. — E. Beltrami (1868), when he developed the projective disk model (Problem **17.5**)

In this chapter we will study projections of a hyperbolic plane onto the plane and use these "models" to prove some results about the geometry of hyperbolic planes. In the case of hyperbolic planes, it is customary to call these "***models***" instead of "projections" because it was thought that there were no surfaces that were hyperbolic planes. As in the case of spherical projections, any projections (models) of the hyperbolic plane must distort some geometric properties; and with models it is more difficult to gain the intrinsic and intuitive experiences that are possible with the hyperbolic surfaces discussed in Chapter 5. Nevertheless, these models do give the most analytically accurate picture of hyperbolic planes and allow for more accurate and precise constructions and proofs. We take as our starting point the geodesic rectangular coordinates presented in Problem **5.2**. In order to connect these coordinates to the study of the models, we will need the results on circles from Chapter 14 and an analytic sophistication that is not necessary in other chapters in this book. However, no technical results from analysis are needed. The reader may bypass most of the analytic technicalities (which occur in Problems **17.1** and **17.2**) if the reader is willing to assume the results of Problem **17.2**, which make the connections between an annular hyperbolic plane and the upper-half-plane model and prove which curves in the upper half-plane correspond to geodesics in the annular hyperbolic plane. The basic properties of geodesics and constructions in the upper- half-plane model (and therefore in annular hyperbolic planes) are investigated in Problem **17.3**. We continue our work on the area of triangles by investigating in Problem **17.4** ideal and 2/3-ideal triangles. Other popular models of hyperbolic planes are contained in Problem **17.5** (Poincaré disk model) and Problem **17.6** (projective disk model).

DISTORTION OF COORDINATE SYSTEMS

The reader should review the description of the annular hyperbolic plane in Chapter 5 and the discussion in Problem **5.2**. There we defined geodesic rectangular coordinates on the annular hyperbolic plane as the map $\boldsymbol{x}$: $\boldsymbol{R}^2 \to H^2$ defined as indicated in Figure 17.1.

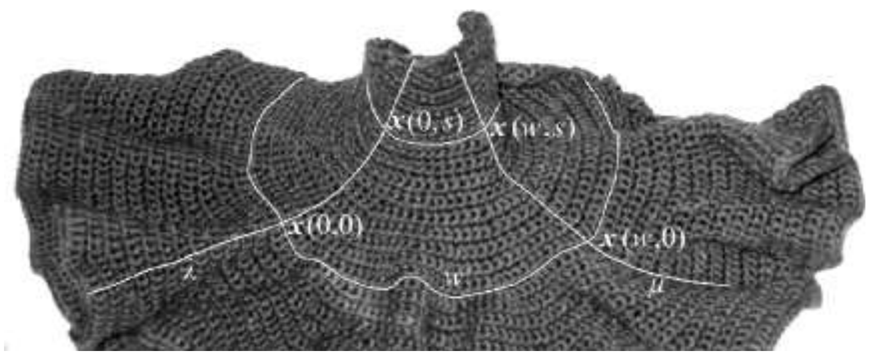


Figure 17.1 Geodesic rectangular coordinates on annular hyperbolic plane

In Problem **5.2** we showed that the coordinate map $\boldsymbol{x}$ is one-to-one and onto from the whole of $\boldsymbol{R}^2$ onto the whole of the annular hyperbolic plane. Horizontal lines map onto the annular strips and vertical lines map onto radial geodesics. Then we showed the following:

> **5.2b.** *Let λ and μ be two radial geodesics on a hyperbolic plane with radius ρ. If the distance between λ and μ along the base curve is w, then the distance between them at a distance s from the base curve is w* $\exp(-s/\rho)$.

Thus, the coordinate chart $\boldsymbol{x}$ preserves (does not distort) distances along the (vertical) second coordinate curves but at $\boldsymbol{x}(a, b)$ the distances along the first coordinate curve are distorted by the factor of $\exp(-b/\rho)$ when compared to the distances in $\boldsymbol{R}^2$. To be more precise,

DEFINITION. Let $\boldsymbol{y}$: $A \to B$ be a map, and let $t \longmapsto \lambda(t)$ be a curve in A. Then the ***distortion*** of $\boldsymbol{y}$ along λ at the point $p = \lambda(0)$ is defined as

$$\lim_{t\to 0} \frac{\text{the arc length along } y(\lambda) \text{from } y(\lambda(t) \text{to } y(\lambda(0))}{\text{the arc length along } \lambda \text{ from } \lambda(t) \text{to } \lambda(0)}$$

In the case of the above coordinate curves, λ is the path in $\boldsymbol{R}^2$, $t \longmapsto (a+t,b)$ or $t \longmapsto (a,b+t)$, and the distortions of $\boldsymbol{x}$ along the coordinate curves are

$$\lim_{t\to 0}\frac{\text{the arc length from } \boldsymbol{x}(a+t,b) \text{ to } \boldsymbol{x}(a,b)}{|(a+t,b)-(a,b)|} = \frac{t\exp\left(-\frac{b}{\rho}\right)}{t} = \exp\left(-\frac{b}{\rho}\right)$$

and

$$\lim_{t\to 0}\frac{\text{the arc length from } \boldsymbol{x}(a,b+t) \text{to } \boldsymbol{x}(a,b)}{|(a,b+t)-(a,b)|} = \frac{t}{t} = 1.$$

We seek a change of coordinates that will distort distances equally in both coordinate directions. The reason for seeking this change (as we will see below) is that if distances are distorted the same in both coordinate directions, then the chart will preserve angles. (Remember, we call such a chart ***conformal***.)

We cannot hope to have no distortion in both coordinate directions (if there were no distortion, then the chart would be an isometry), so we try to make the distortion in the second coordinate direction the same as the distortion in the first coordinate direction. After a little experimentation we find that the desired change is $\boldsymbol{z}(x,y) = \boldsymbol{x}(x, \rho\ln(y/\rho))$, with the domain of $\boldsymbol{z}$ being the upper half-plane

$\boldsymbol{R}^{2+} \equiv \{ (x,y) \in \boldsymbol{R}^2 \mid y > 0 \}$, where $\boldsymbol{x}$ is the geodesic rectangular coordinates defined above. This is usually called the ***upper-half-plane model*** of the hyperbolic plane. The upper-half-plane model is a convenient way to study the hyperbolic plane — think of it as a map of the hyperbolic plane in the same way that we use planar maps of the spherical surface of the earth.

PROBLEM 17.1 A CONFORMAL COORDINATE SYSTEM

Show that the distortion of $\boldsymbol{z}$ *along both coordinate curves*
$x \to \boldsymbol{z}(x, b)$ *and* $y \to \boldsymbol{z}(a, y)$ *at the point* $\boldsymbol{z}(a, b)$ *is* ρ/b.

It may be best to first try this for $\rho = 1$. For the first coordinate direction, use the result of Problem **5.2b**. For the second coordinate direction, use the fact that the second coordinate curves in geodesic rectangular coordinates are parametrized by arc length. Use first-semester calculus where necessary.

Lemma 17.1. *If the distortion of* $\boldsymbol{z}$ *at the point* $p = (a, b)$ *is the same* [say $\Delta(p)$] *along each coordinate curve, then at* (a, b) *the distortion of* $\boldsymbol{z}$ *has the same value along any other curve* $\lambda(t) = \boldsymbol{z}(x(t), y(t))$ *that passes through* p; *and* $\boldsymbol{z}$ *preserves angles at* p (*that is,* $\boldsymbol{z}$ *is conformal*).

Proof. Suppose that $\lambda(0) = (x(0), y(0)) = (a, b) = p$. Assuming that the annular hyperbolic plane can be locally isometrically (that is, preserving distances and angles) embedded in 3-space (see Problem **5.3**), the distortion of $\boldsymbol{z}$ along λ at p is

$$\lim_{t\to 0}\frac{\text{the arc length along } z(\lambda)\text{from } z(\lambda(t) to\ z(\lambda(0))}{\text{the arc length along } \lambda \text{ from } \lambda(t)\text{to } \lambda(0)}$$
$$= \lim_{t\to 0}\frac{\frac{1}{|t|}\text{the arc length along } z(\lambda)\text{from } z(\lambda(t)\text{to } z(\lambda(0))}{\frac{1}{|t|}\text{the arc length along } \lambda \text{ from } \lambda(t) to\ \lambda(0)}$$
$$= \frac{\text{speed of } z(\lambda(t) at\ t=0}{\text{speed of } \lambda(t) t=0} = \frac{\left|\frac{d}{dt}z(\lambda(t))\right|}{\left|\frac{d}{dt}\lambda(t)\right|}\ at\ t=0 = \frac{|(z\circ\lambda)'(0)|}{|\lambda'(0)|}$$

Therefore, along the first coordinate curve $t \longmapsto (a+t,b)$ the distortion is

$$\frac{\left|\frac{d}{dt}z(a+t,b)\right|}{\left|\frac{d}{dt}(a+t,b)\right|}\ at\ t=0 = \text{the norm of the partial derivative, } |z_1(p)|.$$

Similarly, the distortion along the second coordinate curve is $|\boldsymbol{z}_2(p)|$. The velocity vector of the curve $\boldsymbol{z}(\lambda(t)) = \boldsymbol{z}(x(t),y(t))$ at p is

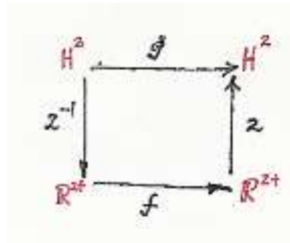
$$\frac{d}{dt}z(x(t),y(t))_{t=0} = z_1(p)\frac{d}{dt}\ x(t)_{t=0} + z_2(p)\frac{d}{dt}\ y(t)_{t=0}$$

Thus, the velocity vector, $(z\circ\lambda)'$ at $t = 0$ is a linear combination of the partial derivative vectors, $\boldsymbol{z}_1(p)$ and $\boldsymbol{z}_2(p)$ — note that these vectors are orthogonal. Therefore, the velocity vectors of curves through $p = \lambda(0)$ all lie in the same plane called the tangent plane at $\boldsymbol{z}(p)$. Also note that the velocity vector,
$(z\circ\lambda)'$, depends only on the velocity vector, $\lambda'(0)$, not on the curve λ. Thus, $\boldsymbol{z}$ induces a linear map (called the differential $d\boldsymbol{z}$) that takes vectors at $p = \lambda(0)$ to vectors in the tangent plane at $\boldsymbol{z}(p)$. This differential is a similarity that multiples all length by $\Delta(p)$ and thus preserves angles. The distortion of $\boldsymbol{z}$ along λ is also $\Delta(p)$.

Definition. In the above situation we call $\Delta p)$ the ***distortion of the map z at the point p*** and denote it **dist(*z*)(*p*)**.

Problem 17.2 Upper Half-Plane Is Model of Annular Hyperbolic Plane

We were able to prove in Problem **5.1** that there are reflections about the radial geodesics but only assumed (based on our physical experience with physical models) the existence of other geodesics and reflections through them. To assist us in looking at transformations of the annular hyperbolic space (with radius ρ), we use the upper-half-plane model. If f is a transformation taking the upper half plane $\boldsymbol{R}^{2+}$ to itself, then we have



We see that $g = z \circ f \circ z^{-1}$ is a transformation from the annular hyperbolic plane to itself. We call g the transformation of H^2 that ***corresponds*** to f.

We will call f an ***isometry of the upper-half-plane model*** if the corresponding g is an isometry of the annular hyperbolic plane. To show that g is an isometry, you must show that the transformation $g = z \circ f \circ z^{-1}$ preserves distances. Remember that distance along a curve is equal to the integral of the speed along the curve. Thus, it is enough to check that the distortion of g at each point is equal to 1. Before we do this, we must first show that

a. *The distortion of an inversion i_C with respect to a circle Γ at a point P, which is a distance s from the center C of Γ, is equal to r^2/s^2, where r is the radius of the circle.* See Figure 17.2.

Hint: Because the inversion is conformal, the distortion is the same in all directions. Thus, check the distortion along the ray from C through P. The distance along this ray of an arbitrary point can be parametrized by $t \longmapsto ts$. Use the definition of distortion given in Problem **17.1**.

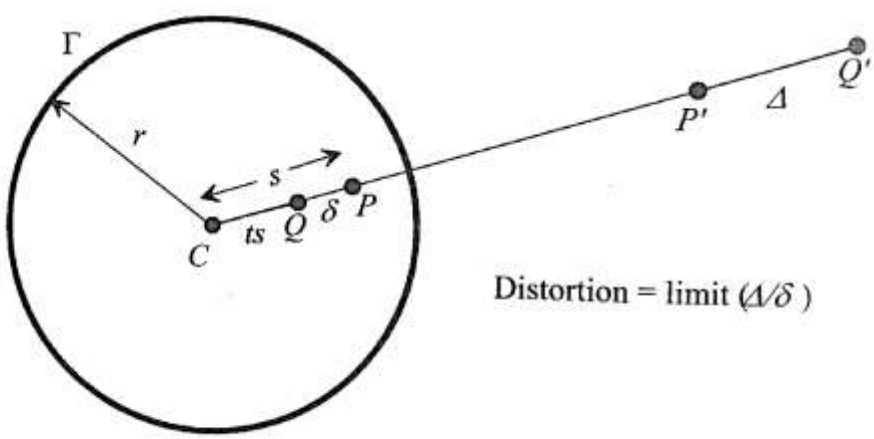


Figure 17.2 Distortion of an inversion

b. *Let f be the inversion in a circle whose center is on the x-axis. Show that f takes $\mathbf{R}^{2+}$ to itself and that $g = z \circ f \circ z^{-1}$ has distortion 1 at every point and is thus an isometry.*

OUTLINE OF A PROOF:

1. Note that each of the maps $\boldsymbol{z}$, $\boldsymbol{z}^{-1}$, f is conformal and have at each point a distortion that is the same for all curves at that point. If dist(k)(p) denotes the distortion of the function k at the point p, then argue that

$$\text{dist}(g)(p) = \text{dist}(\boldsymbol{z}^{-1})(p) \times \text{dist}(f)(\boldsymbol{z}^{-1}(p)) \times \text{dist}(\boldsymbol{z})(f(\boldsymbol{z}^{-1}(p))).$$

2. If $\boldsymbol{z}(a, b) = p$, then show (using Problem **5.1c**) that $\text{dist}(\boldsymbol{z}^{-1})(p) = b/\rho$, where ρ is the radius of the annuli.
3. Show (using part **a**) that $\text{dist}(f)(\boldsymbol{z}^{-1}(p)) = r^2/s^2$, where r is the radius of the circle C that defines f and s is the distance from the center of C to (a, b).
4. Then show that $\text{dist}(\boldsymbol{z})(f(\boldsymbol{z}^{-1}(p)) = \frac{p}{b\frac{r^2}{s^2}}$

We call these inversions (or the corresponding transformations in the annular hyperbolic plane) ***hyperbolic reflections***. We also call reflections through vertical half-lines (corresponding to radial geodesics) hyperbolic reflections.

Now you can prove that

c. *If γ is a semicircle in the upper half-plane with center on the x-axis or a straight half-line in the upper half-plane perpendicular to x-axis, then* $\boldsymbol{z}(\gamma)$ *is a geodesic in the annular hyperbolic plane.*

Because of this, we say that such γ are ***geodesics in the upper-half- plane model***. Since the compositions of two isometries is an isometry, we see immediately that any composition of inversions in semicircles (whose centers are on the x-axis) is an isometry in the upper-half-plane model (that is, the corresponding transformation in the annular hyperbolic plane is an isometry).

PROBLEM 17.3 PROPERTIES OF HYPERBOLIC GEODESICS

a. *Any similarity* (dilations) *of the upper half-plane corresponds to an isometry of an annular hyperbolic. Such similarities must have their centers on the x-axis.* (*Why*?)

Look at the composition of inversions in two concentric semicircles.

b. *If γ is a semicircle in the upper half-plane with center on the x-axis, then there is an inversion* (*in another semicircle*) *that takes γ to a vertical line that is tangent to γ.*

Hint: An inversion takes any circle through the center of the inversion to a straight line (see Problem **16.2**).

Each of the following three parts is concerned with finding a geodesic. Each problem should be looked at in both the annular hyperbolic plane and in the upper-half-plane model. In a crocheted annular hyperbolic plane, we can construct geodesics by folding much the same way we can on a piece of (planar) paper. Geodesics in the upper-half- plane model can be constructed using properties of circles and inversions (see Problem **16.2**). You will also find part **b** very useful.

c. *Given two points A and B in a hyperbolic plane, there is a unique geodesic joining A to B; and there is an isometry that takes this geodesic to a radial geodesic (or vertical line in the upper-half-plane model).*

In the upper-half-plane model, construct a circle with center on the *x*-axis that passes through *A* and *B*. Then use part **b**.

We use *AB* to denote the unique geodesic segment joining *A* to *B*.

d. *Given a geodesic segment AB with endpoints points A and B in a hyperbolic plane, there is a unique geodesic that is a perpendicular bisector of AB.*

Use appropriate folding in annular hyperbolic plane. In the upper-half- plane model, make use of the properties of a reflection through a perpendicular bisector.

e. *Given an angle ∠ABC in a hyperbolic plane, there is a unique geodesic that bisects the angle.*

In the upper-half-plane model, again use the properties of a reflection through the bisector of an angle.

f. *Any two geodesics on a hyperbolic plane either intersect, are asymptotic, or have a common perpendicular.*

Look at two geodesics in the upper-half-plane model that do not intersect in the upper half-plane nor on the bounding *x*-axis.

PROBLEM 17.4 HYPERBOLIC IDEAL TRIANGLES

In Problem **7.2** we investigated the area of triangles in a hyperbolic plane. In the process we looked at ideal triangles and 2/3-ideal triangles. We can look more analytically at the ideal triangles. It is impossible to picture the whole of an ideal triangle in an annular hyperbolic plane, but it is easy to picture ideal triangles in the upper-half-plane model. In the upper-half-plane model an ***ideal triangle*** is a triangle with all three vertices either on the *x*-axis or at infinity. See Figure 17.3.

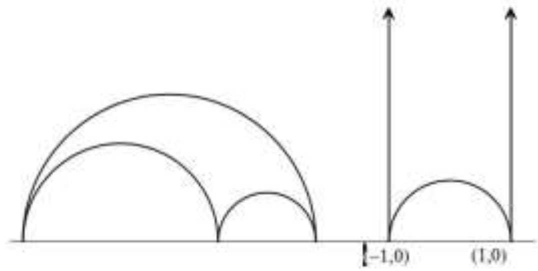


Figure 17.3 Ideal triangles in the upper-half-plane model

At first glance it appears that there must be many different ideal triangles. However,

a. *Prove that all ideal triangles on the same hyperbolic plane are congruent.*

Review your work on Problem **16.2**. Perform an inversion that takes one of the vertices to infinity and the two sides from that vertex to vertical lines. Then apply a similarity to the upper half-plane, taking it to the standard ideal triangle with vertices (–1,0), (0,1), and ∞.

b. *Show that the area of an ideal triangle is* $\pi\rho^2$. (Remember this ρ is the radius of the annuli.)

Hint: Because the distortion dist(**z**)(a, b) is ρ/b, the desired area is

$$\int_{-1}^{1}\int_{\sqrt{1-x^2}}^{\infty}\left(\frac{\rho}{y}\right)^2 dydx.$$

Figure 17.4 shows **2/3-*ideal triangles*** in the upper-half- plane model.

c. *Prove that all* 2/3-*ideal triangles with angle* θ *are congruent and have area* $(\pi - \theta)\rho^2$.

Show, using Problem **16.2**, that all 2/3-ideal triangles with angle θ are congruent to the standard one at the right of Figure 17.4 and show that the area is the double integral

$$\int_{-1}^{\cos\theta}\int_{\sqrt{1-x^2}}^{\infty}\left(\frac{\rho}{y}\right)^2 dydx.$$

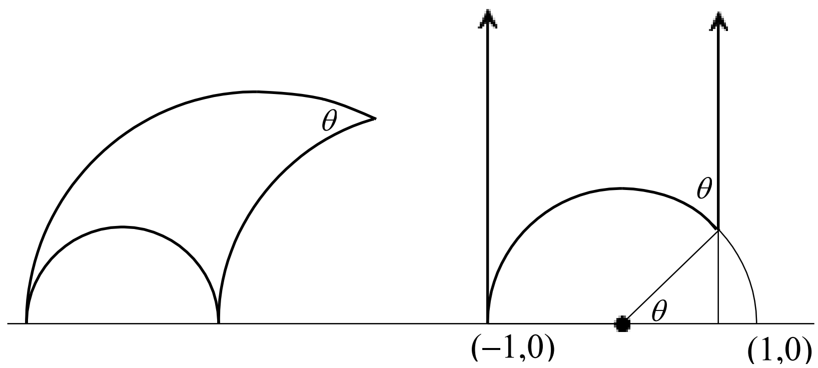


Figure 17.4 2/3-ideal triangles in the upper-half-plane model

PROBLEM 17.5 POINCARÉ DISK MODEL

You showed in Problem **17.1c** that the coordinate map $\boldsymbol{x}$ from a hyperbolic plane to the upper half-plane preserves angles (is conformal); this we called the *upper-half-plane model*. Now we will study other models of the hyperbolic plane.

Let $\boldsymbol{z}$: $\boldsymbol{R}^{2+} \to H^2$ be the coordinate map defined in Problem **17.2** that defines the upper-half-plane model. We will now transform the upper- half-plane model to a disk model that was first discussed by Poincaré in 1882.

a. *Show that any inversion through a circle whose center is in the lower half-plane (that is, $y < 0$) will transform the upper half- plane onto an open (without its boundary) disk. Show that the hyperbolic geodesics in the upper half-plane are transformed by this inversion into circular arcs (or line segments) perpendicular to the boundary of the disk.*

Review the material on inversions discussed in Problem **16.2**.

b. *If $\boldsymbol{w}$: $D^2 \to \boldsymbol{R}^{2+}$ is the inverse of a map from the upper half plane to a (open) disk from part* **a**, *then show that the composition*

$$z \circ w : D^2 \to H^2$$

is conformal. We call this the (***Poincaré***) ***disk model***, after Henri Poincaré (1854–1912, French).

Review the material on inversions in **16.2** and on the upper-half-plane model in **17.2**.

c. *Show that any inversion through a circular arc (or line segments) perpendicular to the boundary of D^2 takes D^2 to itself. Show that these inversions correspond to isometries in the (annular) hyperbolic plane.* Thus, we call these circular arcs (or line segments) ***hyperbolic geodesics*** and call the inversions ***hyperbolic reflections*** in the Poincaré disk model.

Review Problem **17.2**.

See Figure 17.5 for a drawing of geodesics and a triangle in the Poincaré disk and the projective disk model (Problem **17.6**).

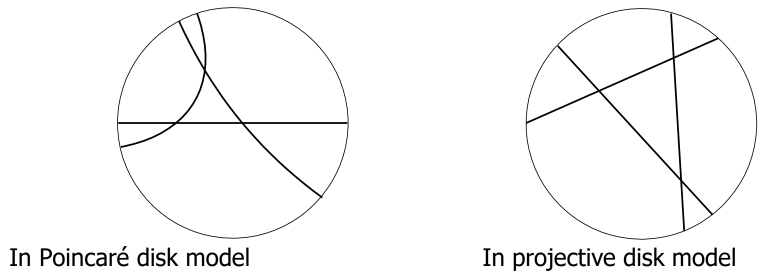


Figure 17.5 Geodesics and a triangle

PROBLEM 17.6 PROJECTIVE DISK MODEL

Let D^2 be the disk model of a hyperbolic plane and assume its radius is 2. Then place a sphere of radius 1 tangent to the disk at its center. Call this point of tangency the south pole *S*. See Figure 17.6.

Now let ***s*** be the stereographic projection from the sphere to the plane containing D^2. Note that ***s***(equator) is the boundary of D^2, and thus ***s*** takes the Southern Hemisphere onto D^2. Now let ***h*** be the orthogonal projection of the southern hemisphere onto the disk, B^2, of radius 1.

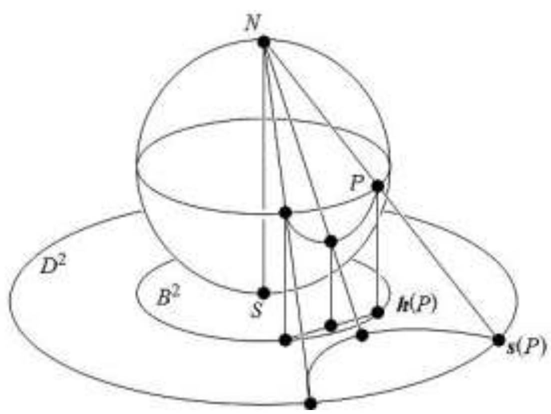


Figure 17.6 Obtaining the projective disk model from the Poincaré disk model

Show that the mapping $\mathbf{h} \circ \mathbf{s}^{-1}$ takes D^2 to B^2 and takes each circle (or diameter) of D^2 to a (straight) chord of B^2. Thus $\mathbf{h} \circ \mathbf{s}^{-1} \circ (\mathbf{z} \circ \mathbf{w})^{-1}$ is a map from the hyperbolic plane to B^2, which takes geodesics to straight line segments (chords) in B^2.

We call this the ***projective disk model***, it is also called the ***Beltrami/Klein model*** or the ***Klein model***, named after Eugenio Beltrami (1835–1900, Italian), who described the model in 1868, and Felix Klein (1849–1925, German), who fully developed it in 1871.