Chapter 15

CIRCLES

... the Power of the World always works in circles, and everything tries to be round.

— Black Elk in Black Elk Speaks [GC: Neihardt]

Now we will study some important properties (Problem **15.1**) of circles in the plane that are stated and proved in Euclid's *Elements*. These planar results will be used in later chapters, and they are also often studied for their own interest. In Problem **15.2**, we will explore an extension of these results to circles on spheres (and later to hyperbolic planes). We will end the chapter with applications of these properties of circles (Problem **15.3**) to the ancient problem of trisecting angles (Problem **15.4**).

For Chapter 15, the only results needed from Chapters 9–14 are

> **PROBLEM 13.4a**: The **AAA similarity criterion** for triangles on the (Euclidean) plane: *If two triangles are similar* (have congruent angles), *then the corresponding sides of the triangles are in the same* proportion *to one another.* [Needed throughout this and later chapters.]

> **PROBLEM 9.1**: **Side-Side-Side**: *If two triangles* (small triangles if on a sphere) *have congruent corresponding sides, then the triangles are congruent.* [Needed throughout this and later chapters.]

> **PROBLEM 14.4**: *Stereographic projection of a sphere onto a plane preserves angles, takes circles to circles (or to straight lines).* [Needed only for Problem **15.2**.]

If you are willing to assume these results, then you can work through Chapter 15 without Chapters 9–14.

PROBLEM 15.1 ANGLES AND POWER OF POINTS FOR CIRCLES IN THE PLANE

These results are all stated and proved in Euclid's *Elements*. They are contained in Propositions 27, 32, and 35–37 of Euclid's Book III, which is entirely devoted to properties of planar circles.

a. *If an arc of a circle subtends an angle 2α from the center of the circle, then the same arc subtends an angle α from any point on the circumference. In particular, two angles that subtend (from different points on the circumference) the same arc are congruent.*

Use Figures 15.1 and 15.2. Draw a segment from the center of the circle to the point A and use ITT. Note the four different locations for A.

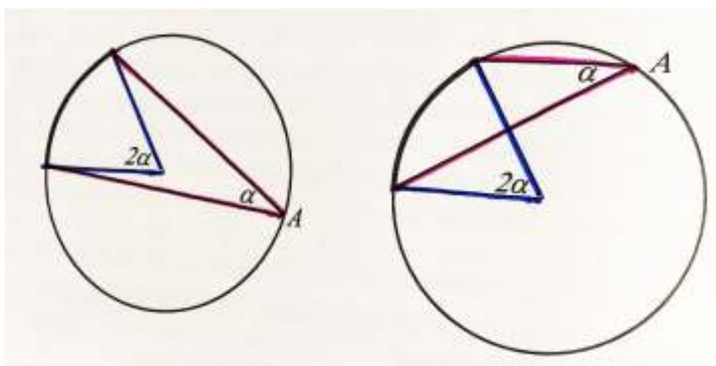


Figure 15.1 Angles subtended from outside the arc

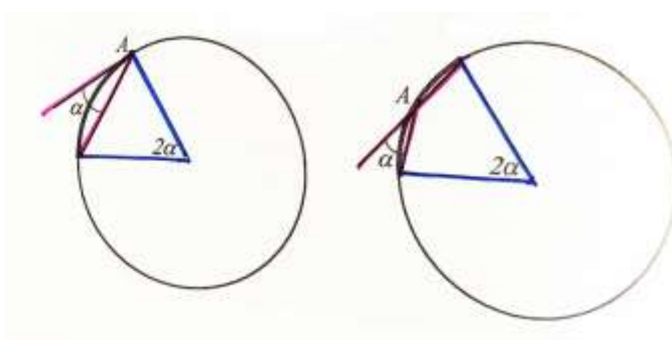


Figure 15.2 Angles subtended from on the arc

b. *On a plane, if two lines through a point P intersect a circle at points A, A′ (possibly coincident) and B, B′ (possibly coincident), then*

$$|PA| \times |PA'| = |PB| \times |PB'|.$$

This product is called the ***power of the point P with respect to the circle.***

Use Figures 15.3 and 15.4 and draw the segment joining A to B' and the segment joining A' to B. Then apply part **a** and look for similar triangles.

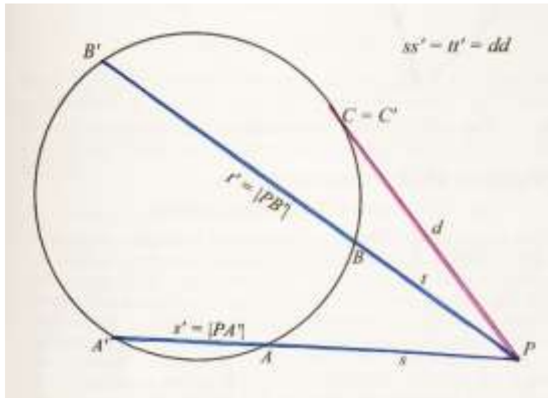
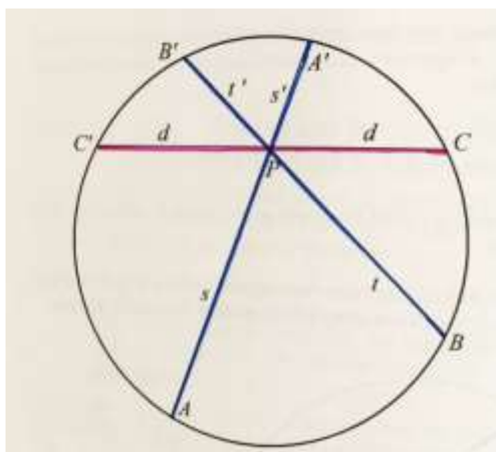


Figure 15.3 Power of a point outside with respect to a circle

For more discussion on power of a point on the plane, see the delightful little book [EG: Coxeter & Greitzer].

In Problem **15.2** we will state and prove results about the power of a point on spheres and hyperbolic planes that were discovered by Robin Hartshorne and published in 2003.



$ss' = tt' = dd$

Figure 15.4 Power of a point inside with respect to a circle

PROBLEM 15.2 POWER OF POINTS FOR CIRCLES ON SPHERES

There are no similar triangles on spheres and hyperbolic planes, so it seems surprising that there could be a notion of power of a point for circles on spheres or hyperbolic planes. However, Robin Hartshorne, [**SP**: Hartshorne], found a way to provide a unified definition of power of a point for circles in non-Euclidean geometries — this problem is based on his paper. Hartshorne starts by pointing out that the equality of the products $|PA| \times |PA'| = |PB| \times |PB'|$ in **15.1b** that we used to define the power of a point for plane circles can be written as an equality of areas of the rectangles with sides, PA, PA', and PB, PB', respectively — this is the way Euclid considered the power of a point. Of course, there are no rectangles on spheres and hyperbolic planes. What can we use in their place? In Chapter 12, for our work in dissection theory, we introduced Khayyam quadrilaterals as an appropriate analogue of rectangles on spheres and hyperbolic planes. Hartshorne instead introduces another analogue of a rectangle: A ***semi-rectangle*** is a quadrilateral with opposite sides equal and at least one right angle.

a. *On the plane, semi-rectangles are rectangles. On spheres and hyperbolic planes, semi- rectangles have the angle opposite the right angle also right and the two remaining angles are congruent.* See Figure 15.5.

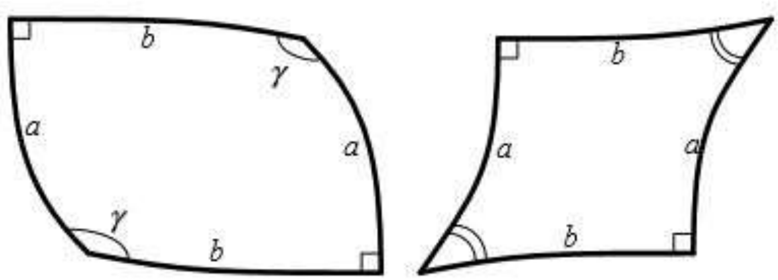


Figure 15.5 Semi-rectangles

There is a close relationship between semi-rectangles and Khayyam quadrilaterals. In fact, the interested reader can show that a semi- rectangle can be dissected, with only one straight cut, into a Khayyam quadrilateral whose base angles are the non-right angles in the semi-rectangle.

Using semi-rectangles, we can now restate **15.1b** as follows:

THEOREM 15.2 *On a plane, sphere, or hyperbolic plane, if two lines through a point P intersect a circle at points A, A′ (possibly coincident) and B, B′ (possibly coincident), then the area of the semi-rectangle with side lengths PA and PA′ is equal to the area of the semi-rectangle with side lengths PB and PB′. This area is called the* ***power of the point P with respect to the circle****.*

On the plane this theorem reduces directly to **15.1b**. Do you see why? We will now prove Theorem 15.2 on spheres (part **c**), but first we need to prove the following result, which is interesting in its own right:

b. *On a sphere with radius ρ, a spherical right triangle with excess δ and legs (sides adjacent to the right angle) a and b satisfies the following:*

$$\tan(\delta/2) = \tan(a/2\rho)\tan(b/2\rho).$$

Thus, if R is a semi-rectangle with sides a and b and with non-right angles equal to γ, (as in Figure 15.5)*, and if we set let $\delta = \gamma - \pi/2$, then*

$$\text{Area}(R) = 2\delta\rho^2 \text{ and } \tan(\delta/2) = \tan(a/2\rho)\tan(b/2\rho).$$

The proofs of parts **b** and **c** (below) rely on properties of stereographic projection which is discussed in Problem **14.4**. The properties we need here are that stereographic projection preserves all angles and takes circles on the sphere to circles (or lines) on the plane.

OUTLINE OF PROOF OF 15.2B:

1. Rotate the sphere until the right-angled vertex on the triangle is at the south pole, S. Now, using stereographic projection from the north pole, project Δ onto the plane. The image of the triangle on the plane is the figure SHI depicted in Figure 15.6. The sides SH and SI are straight. (*Why?*) Let Λ be the image of the great circle that contains the hypotenuse of Δ and let C be the center of Λ. Let Γ be the image of the equator; then Γ is a circle with center S and radius 2ρ. (*Why?*) The circle Λ (that contains the arc HI) intersects the circle Γ at diametrically opposite points DE. (*Why?*)

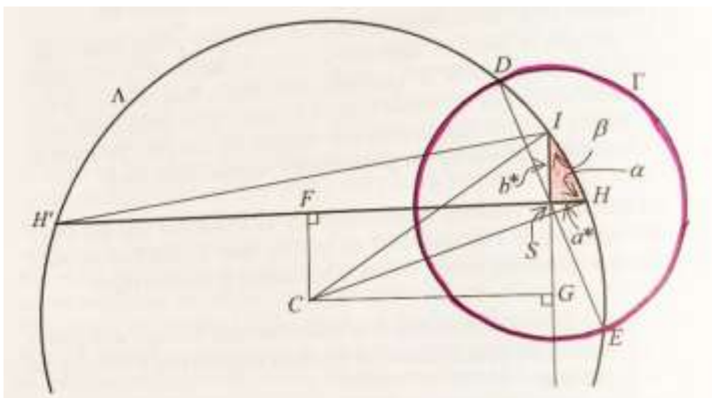


Figure 15.6 Stereographic image of a spherical right triangle

2. Referring to Figure 15.6, show that $\angle ICH = \delta$ and that $\angle IH'H = \delta/2$.
   (Hint: Show that $\angle FCI = \pi/2 - \beta$.)
3. Show that $|SH| \times |SH'| = |SD| \times |SE|$ and thus $|SH'| = 4\rho^2/a^*$. Conclude that $4\rho^2 \tan(\delta/2) = a^*b^*$. Remember that a^* and b^* are the projections of the sides of our original semi-rectangle R and that $\delta\rho^2$ is the area of the triangle.
4. Let R be a semi-rectangle (as in Figure 15.5) whose non-right angles are both γ, then use a diagonal to divide R into two congruent right triangles with legs a and b and non-right angles α and β. Note that $\alpha+\beta = \gamma$. This is enough to conclude (*Why?*) the last equations in part **b**.

Now we are ready to

c. *Prove that Theorem 15.2 is true on a sphere.*

OUTLINE OF PROOF OF 15.2c

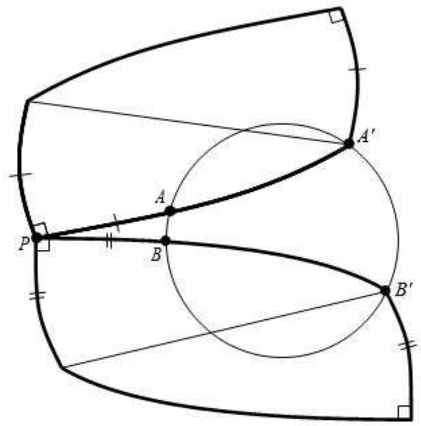


Figure 15.7 Power of a point on a sphere

1. Now let P, A, A', B, B' be as in Theorem **15.2** and Figure 15.7. By rotating the sphere, if necessary, we may consider P to be the south pole S. Let Γ denote the given circle on sphere. Project stereographically everything onto the plane and let A^*, A'^*, B^*, B'^*, and Γ^* denote the corresponding images on the plane after stereographic projection. Note that the image of $P = S$ is the same S and that A^*-S-A'^*, and B^*-S-B'^* are straight lines. (*Why*?) Using **15.1b**, conclude that

$$|SA^*| \times |SA'^*| = |SB^*| \times |SB'^*|.$$

2. Putting this together with Steps 3 and 4 of the proof of **15.2b**, we conclude (*Why?*) that

$$\tan(\delta/2) = \tan(\delta'/2),$$

where the area of the semi-rectangle on PA and PA' is $2\delta\rho^2$ and the area of the semi-rectangle on PB and PB' is $2\delta'\rho^2$. See Step 1. Thus, the areas are equal. (*Why?*)

There is a proof of Theorem 15.3 on a hyperbolic plane that is very similar to the proof above on a sphere. Instead of stereographic projection, the proof on a hyperbolic plane uses the Poincaré disk model thought of as a projection (see Problem **17.5**), which also preserves angles and takes circles on the hyperbolic plane to circles (or lines) on the plane.

PROBLEM 15.3 APPLICATIONS OF POWER OF A POINT

Here are a few applications of the notion of power of a point that hold on the plane, spheres, and hyperbolic planes. We will meet other applications later, especially in Chapter 16 and 19, but these will be applied only in the case of the Euclidean plane. The applications here relate to the notion of ***radical axis of two circles***, which we define to be the locus of points P such that the power of the point P is the same with respect to both circles. You may assume that Theorem 15.2 is true on hyperbolic planes.

a. *If two circles intersect in two points, then their radical axis is the full line determined by the two points of intersection. If two circles are tangent, then the radical axis is their common tangent line.*

b. *If P is a point on the radical axis of two circles, C and D, then any circle with center P that intersects C at right angles also intersects D at right angles.*

c. *If three circles intersect each of the other two circles in two points, then the three chords so defined intersect in a common point.* See Figure 15.8

d. *What happens if, in part* **c**, *some of the pairs of circles are tangent instead of intersecting in two points?*

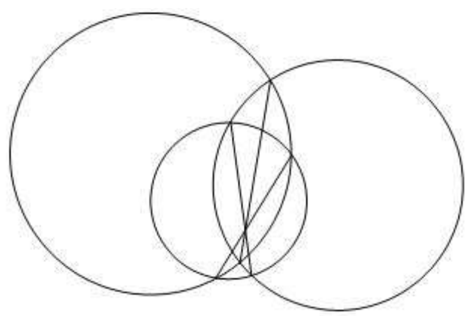


Figure 15.8 Intersecting chords

PROBLEM 15.4 TRISECTING ANGLES AND OTHER CONSTRUCTIONS

In Problem **6.3** you showed how to bisect angles and find the perpendicular bisector of a line segment by only using a compass (for drawing circles) and an unmarked straightedge (for drawing line segments joining two points but not used for measuring). We will now extend these constructions and discuss in what sense angles can and cannot be trisected.

a. *Show, using a compass and unmarked straightedge, how to*

i. *Construct a line from a given point perpendicular to a given line.*
ii. *Construct the line tangent to a given circle at a given point.*
iii. *Construct the two lines tangent to a given circle from a given point outside the circle.* Hint: Use **15.1b**.]
iv. *Construct a line that is a parallel transport of a given line along a given transversal.*

b. *On the plane, show how to n-sect a given line segment using only a compass and (unmarked) straightedge. Will your construction work on spheres or hyperbolic planes?*

Hint: Look at Figure 15.9, where the given segment is *AB* and *AC1* is any segment forming an acute angle with *AB*. Duplicate *AC1* n times. (How?) Draw a line through *C1* parallel to *BCn*. In the figure, $n = 5$.

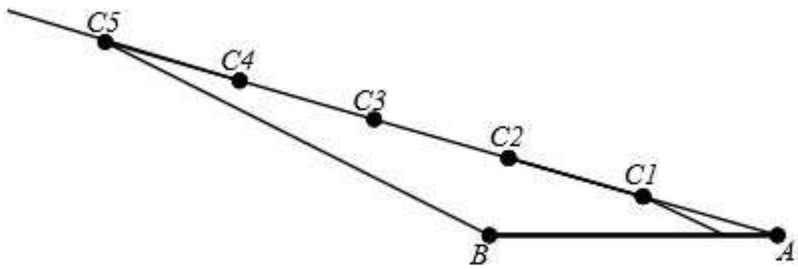


Figure 15.9 Dividing a given line segment into 5 congruent pieces

It is often stated in popular literature that it is impossible to trisect angles with straightedge and compass. See, for example, [**TX**: Martin], page 49. However, it has been known since ancient Greek times that any angle can be trisected using only a (marked) straightedge and compass. For example, Archimedes (287–212 B.C.) showed how to trisect any angle (less than 135°) using a ***marked straightedge*** (a straightedge with two points marked on it).

c. Archimedes Construction (for angles less than 90°): *Referring to Figure* 15.10, *let* $\angle ABC$ *be an angle less than* 90°. *Assume you have a straightedge with two marks on it that are a distance r apart. Draw the circle of radius r with center B and, keeping the angle the same, move A to the intersection of BA with the circle. Lay the straightedge on the figure so that it passes through A; and one mark, D, is on the circle and the other mark, E, is on the extension of BC. Show that* $\angle AEC$ *is 1/3 of* $\angle ABC$.

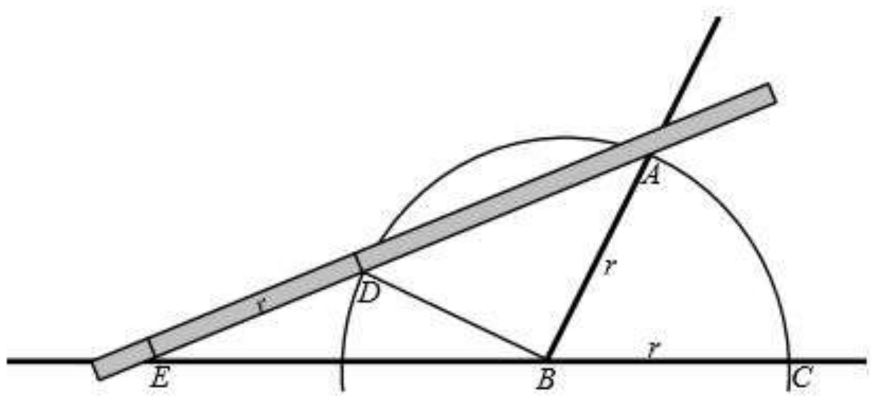


Figure 15.10 Archimedes' construction: $\angle AEC$ is 1/3 of $\angle ABC$

d. Show *that the mechanism in Figure 15.11 will trisect an angle. Will it work on spheres?*

Figure 15.11 Mechanism to trisect an angle

Well, maybe it is impossible to trisect angles with a compass and an *unmarked* straightedge?

e. Construct *on a transparent sheet the figure on the left in* Figure 15.12 *with compass and unmarked straightedge starting with any two points AB. Lay the transparent sheet on the angle* α (see the figure on the right in Figure 15.12) *so the vertex, V, of the angle lies on DB, one side contains C, and the other is tangent to the semicircle with center A. Prove that* $\angle BVC$ *trisects angle* α.

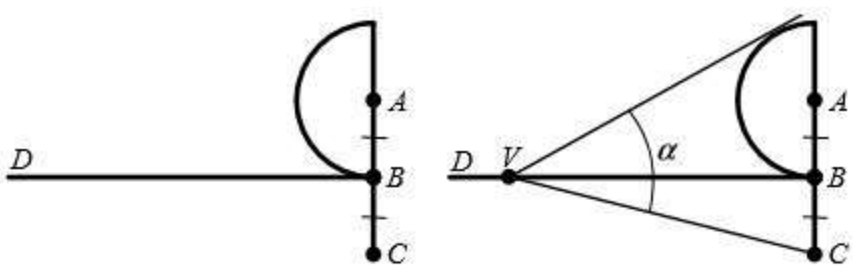


Figure 15.12 "Shoemaker's knife" or "tomahawk" trisector

In order to give a correct statement of what is impossible, it is necessary first to define a ***compass and unmarked straightedge sequence*** to be a finite sequence of points, lines, and circles that starts with two distinct given points and such that (1) Each of the other points in the sequence is the intersection of lines or circles that occur before it in the sequence. (2) Each circle has its center and one point on the boundary occurring before it in the sequence. (3) Each line contains two distinct points that occur before it in the sequence.

Theorem 15.4. *It is impossible to trisect a 60° angle with a compass and unmarked straightedge sequence.*

For a proof of this theorem and related discussions about various compass and straightedge constructions, see [**TX**: Martin] or [**TX**: Hartshorne], Section 28. If you get interested to read more comprehensive history of four the most famous problems in mathematics, look for *Tales of Impossibility: The 2000-year Quest to solve the Mathematical Problems of Antiquity* by David S. Richeson (Princeton University Press, 2019).

Model of an angle trisector which David made for his class