Chapter 14

PROJECTIONS OF A SPHERE ONTO A PLANE

Geography is a representation in picture of the whole known world together with the phenomena which are contained therein.

... The task of Geography is to survey the whole in its proportions, as one would the entire head. For as in an entire painting we must first put in the larger features, and afterward those detailed features which portraits and pictures may require, giving them proportion in relation to one another so that their correct measure apart can be seen by examining them, to note whether they form the whole or a part of the picture. ... Geography looks at the position rather than the quality, noting the relation of distances everywhere, ...

It is the great and the exquisite accomplishment of mathematics to show all these things to the human intelligence ... — Claudius Ptolemy, Geographia, Book One, Chapter I

A major problem for map makers (cartographers) since Ptolemy (approx. 85–165 A.D., Alexandria, Egypt) and before is how to represent accurately a portion of the surface of a sphere on the plane. It is the same problem we have encountered when making drawings to accompany our discussions of the geometry of the sphere. We shall use the terminology used by cartographers and differential geometers to call any one-to-one function from a portion of a sphere onto a portion of a plane a ***chart***. As Ptolemy states in the quote above, we would like to represent the sphere on the plane so that proportions (and thus angles) are preserved and the relative distances are accurate. For a history and mathematical descriptions of charts of the sphere, see [**CE**: Snyder]. For a history and discussion of the political, social, and ethnic controversies that have been and continue to be connected with maps and map making, see [**CE:** Monmonier]. For a discussion of how maps and the stars were used to determine the position of the earth, see the delightful book [**CE:** Sobel]. History of specific map projections is given in the last section of this chapter.

In this chapter we will study various charts for spheres. We will need properties of similar triangles that are investigated in Problem **13.4**.

PROBLEM 14.1 CHARTS MUST DISTORT

It is impossible to make a chart without some distortions.

> *Which results that you have studied so far show that there must be distortions when attempting to represent a portion of a sphere on the plane*?

Nevertheless, there are projections (charts) from a part of a sphere to the plane that do take geodesic segments to straight lines, that is, that preserve the shape of straight lines. There are other projections that pre- serve all areas. There are still other projections that preserve the measure of all angles. In this chapter, we will study these three types of projections on a sphere, and in Chapter 17 we will look at projections of hyperbolic planes.

PROBLEM 14.2 GNOMIC PROJECTION

Imagine a sphere resting on a horizontal plane. See Figure 14.1. A ***gnomic projection*** is obtained by projecting from the center of a sphere onto the plane. Note that only the lower open hemisphere is projected onto the plane; that is, if X is a point in the lower open hemisphere, then its gnomic projection is the point, $\boldsymbol{g}(X)$, where the ray from the center through X intersects the plane.

a. *Show that a gnomic projection takes the portions of great circles in the lower hemisphere onto straight lines in the plane.* (A mapping that takes geodesic segments to geodesic segments is called a ***geodesic mapping***.)

b. *Gnomic projection is often used to make navigational charts for airplanes and ships. Why would this be appropriate*?

Hint: Start with our extrinsic definition of "great circle.

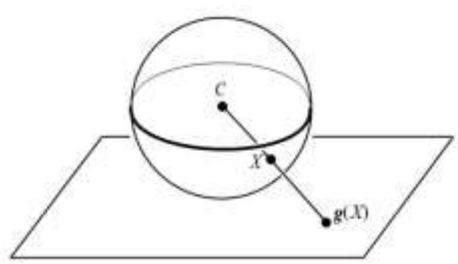

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Figure 14.1 Ancient gnomon and gnomic projection

PROBLEM 14.3 CYLINDRICAL PROJECTION

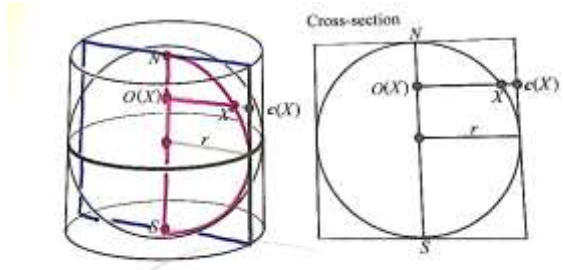

Figure 14.2 Cylindrical projection

Imagine a sphere of radius r, but this time center it in a vertical cylinder of radius r and height $2r$. The ***cylindrical projection*** is obtained by projecting from the axis of the cylinder, which is also a diameter of the sphere; that is, if X is a point (not the north or south poles) on the sphere and $O(X)$ is the point on the axis at the same height as X, then X is projected onto the intersection of the cylinder with the ray from $O(X)$ to X. See Figure 14.2.

a. *Show that cylindrical projection preserves areas.* (Mappings that preserve area are variously called ***area-preserving*** or ***equiareal***.)

Geometric Approach: Look at an infinitesimal piece of area on the sphere bounded by longitudes and latitudes. Check that when it is projected onto the cylinder the horizontal dimension becomes longer but the vertical dimension becomes shorter. Do these compensate for each other?

Analytic Approach: Find a function $\boldsymbol{f}$ from a rectangle in the (z,θ)-plane onto the sphere and a function $\boldsymbol{h}$ from the same rectangle onto the cylinder such that $\boldsymbol{c}(\boldsymbol{f}(z,\theta)) = \boldsymbol{h}(z,\theta)$. Then use the techniques of finding surface area from vector analysis. (For two vectors $\boldsymbol{A}$, $\boldsymbol{B}$, the magnitude of the cross product $|\boldsymbol{A} \times \boldsymbol{B}|$ is the area of the parallelogram spanned by $\boldsymbol{A}$ and $\boldsymbol{B}$. An element of surface area on the sphere can be represented by $|\boldsymbol{f}_z \times \boldsymbol{f}_\theta|\, dz\, d\theta$, the cross product of the partial derivatives.)

We can easily flatten the cylinder onto a plane and find its area to be $4\pi r^2$. We thus conclude the following:

b. *The (surface) area of a sphere of radius r is* $4\pi r^2$.

Anamorphosis – distorted watercolor that can be seen in the cylindrical mirror

PROBLEM 14.4 STEREOGRAPHIC PROJECTION

Imagine the same sphere and plane, only this time project from the uppermost point (north pole) of the sphere onto the plane. This is called ***stereographic projection***.

a. *Show that stereographic projection preserves the sizes of angles.*
(Mappings that preserve angles are variously called ***angle- preserving***, ***isogonal***, or ***conformal***.)

SUGGESTIONS

There are several approaches for exploring this problem. Using a purely geometric approach requires visualization but only very basic geometry. An analytic approach requires knowledge of the differential of a function from $\mathbf{R}^2$ into $\mathbf{R}^3$. See Figure 14.3.

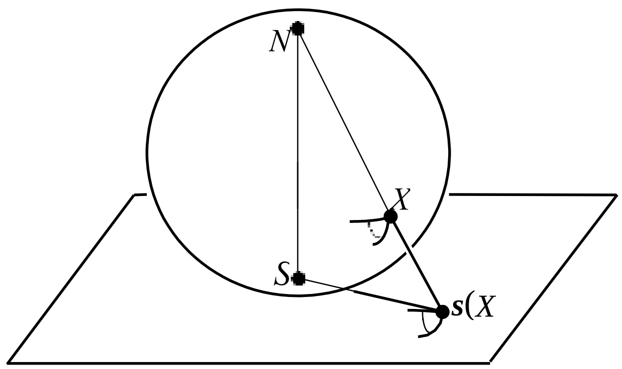

Figure 14.3 Stereographic projection is angle-preserving

Geometric Approach: An angle at a point X on the sphere is determined by two great circles intersecting at X. Look at the two planes that are determined by the north pole N and vectors tangent to the great circles at X. Notice that the intersection of these two planes with the horizontal image plane determines the image of the angle. Because the 3-dimensional figure is difficult for many of us to imagine in full detail, you may find it helpful to consider what is contained in various 2-dimensional planes. In particular, consider the plane determined by X and the north and south poles, the plane tangent to the sphere at X, and the planes tangent to the sphere at the north and south poles. Determine the relationships among these planes.

Analytic Approach: Introduce a coordinate system and find a formula for the function $\boldsymbol{s}^{-1}$ from the plane to the sphere, which is the inverse of the stereographic projection $\boldsymbol{s}$. Use the differential of $\boldsymbol{s}^{-1}$ to examine the effect of $\boldsymbol{s}^{-1}$ on angles. You will need to use the dot (inner) product and the fact that the differential of $\boldsymbol{s}^{-1}$ is a linear transformation from the (tangent) vectors at $\boldsymbol{s}(X)$ to the tangent vectors at X.

b. *Show that stereographic projection takes circles through N to straight lines and circles not through N to circles.* (Such mappings are called ***circle-preserving***.)

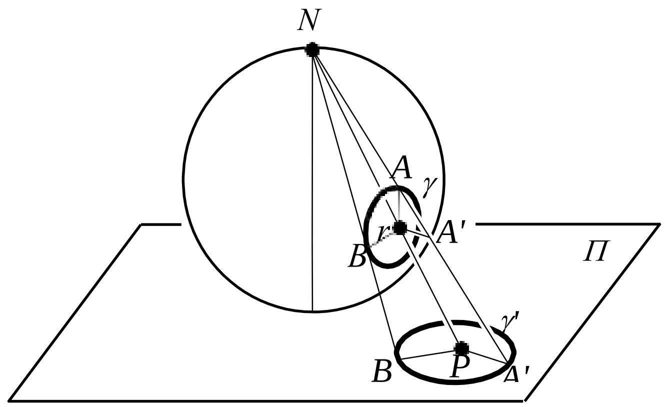

Figure 14.4 Stereographic projection is circle-preserving

SUGGESTIONS

Let γ be a circle on the sphere with points A and B and let γ', A', B' be their images under stereographic projection. Form the cone that is tangent to the sphere along the circle γ and let P be its cone point (note that P is not on the sphere). See Figure 14.4. Thus, the segments BP and AP are tangent to the sphere and have the same length r. Look in the plane determined by N, A, and P and show that $\angle PAA'$ is congruent to $\angle AA'P'$. You probably have already proved this in part **a**; if not, look at the plane determined by N, A, and P and its intersections with the plane tangent to the north pole N and the image plane Π. In this plane draw line PA'' parallel to $P'A'$. Then use similar triangles and Problem **6.2c** to show

$$|A'P'| = r\left(\frac{|NP'|}{|NP|}\right),$$

and thus γ' is a circle with center at P'.

HISTORY OF STEREOGRAPHIC PROJECTION AND ASTROLABE

The earliest known uses of projections of a sphere onto the plane were in Greece and were for purposes of map making (as described in the quote at the beginning of this chapter from Ptolemy's *Geographia*). As we discussed in Chapter 2, the Greeks (following the Babylonians) in 4th century B.C. considered the visible cosmos to be a sphere with three different coordinate systems: celestial, ecliptic, and horizon. At least as early as 2nd century B.C., the Greek and later the Arab mathematicians used the sphere projections that we have studied to represent these three spheres on the plane. The Greeks called it "unfolding the sphere".

The earliest references to stereographic projection in literature are given by Vitruvius (Roman, ~100 B.C.), *Ten Books on Architecture*, and in Ptolemy's *Representation* *of the Sphere in the Plane*. The ancients knew how to prove (using propositions from Apollonius' *Conics* [**AT:** Appollonius]) some properties of stereographic projection, such as the following: Circles through the pole are mapped onto straight lines and all other circles are mapped onto circles. (See Problem **14.4b**.)

In Ptolemy's work the description of stereographic projection was used for a horoscopic instrument for determining time. Later the word *horoscope* denoted the point of intersection of the ecliptic and the eastern part of the horizon determined by means of this instrument. Probably Theon of Alexandria (Greek, ?335–?405) was the first to combine stereo- graphic projections of the three-sphere model onto a single compact planar instrument called an ***astrolabe***, as pictured in Figure 13.15. Theon's work has not survived, but we know of it because of two surviving works on the astrolabe: *On the Construction and Use of the Astrolabe* by Philoponus (6th century Alexandria) and the *Treatise on the Astrolabe* by Severus Sebokt (7th century Syria).

The astrolabe was widely known in the medieval East and Europe from the 9th century until the 19th century and used to solve problems concerning the apparent positions of the stars, sun, moon, and planets for use in navigation, time-telling, and astrology. In the Middle Ages stereo- graphic projection was often called "astrolabe projection." The astrolabe allowed for determining relative positions to about 1° accuracy.

The term "stereographic projection" was apparently first introduced by Francois D'Aguillon (1566–1617) in his *Six Books of Optics*. The earliest exposition of the theory of stereographic projection with proofs was the *Book on the Construction of the Astrolabe* by the 9th-century Baghdad scholar al-Farghani. Mathematicians in the medieval East also tried to use other geometric transformations for constructing astrolabes. The 10th-century scholar al-Saghani suggested a projection from an arbitrary point on the axis — if that point is the center of the sphere, then this projection is the gnomic projection of Problem **14.2**. In such a projection circles on the sphere are mapped onto conics. Al-Biruni (973–1048, now Uzbekistan) described several ways of constructing an astrolabe; including the use of a cylindrical projection (Problem **14.3**).

Figure 14.5 Astrolabe

In the Figure 14.5 the concentric circles on the background plate about the center represent stereographic projection of the celestial sphere from its south pole — the center of the astrolabe represents the north star, the outer rim represents the southern tropic (the furthest the sun goes south, about –24º), and the concentric circle about two-thirds of the way out is the celestial equator. The metal annulus that is off-center is the stereographic projection of the ecliptic (the path of the sun and planets) — this projection is a circle but the equal divisions of the ecliptic (the 360 degrees of the ecliptic) are not projected to equal arcs. The tight system of circles on the background plate of the astrolabe (and mostly above the center) is the projection of the coordinates of the visible hemisphere with the horizon (not complete) on the outside and the image of the zenith (the point in the sky directly overhead) where the coordinate circles converge. Since the relationship between the North Star and the zenith changes with latitude, a given astrolabe is only accurate at one latitude (about 46º north for the one in the photo).

For more about astrolabes and map making, see [**HI**: Rosenfeld], pp. 121–130, [**HI**: Evans], pp. 141–162, and [**HI**: Berggren], pp. 165–186. See also *Divided Spheres: Geodesics and the Orderly Subdivision of the Sphere*, Edward S. Popko, CRC Press, 2012.

Planispheric Astrolabe by Muhammad Zaman al-Munajim al Asturlabi, Iran, 1654-55 (the Metropolitan Museum of Art)

This astrolabe is composed of five stacked circular plates engraved with terrestrial latitudes, rotating around the axis of a central pin. The uppermost pierced brass plate points toward several fixed stars to indicate their celestial positions.