Chapter 12

DISSECTION THEORY

Oh, come with old Khayyám, and leave the Wise
To talk, one thing is certain, that Life flies;
One thing is certain, and the Rest is Lies;
The flower that once has blown for ever dies.
— Omar Khayyam, Rubaiyat
(from the translation by Edward Fitzgerald)

WHAT IS DISSECTION THEORY?

Figure 12.1 Parallelogram

In showing that the parallelogram in Figure 12.1 has the same area as a rectangle with the same base and height (altitude), we can easily cut the parallelogram into two pieces and rearrange them to form the rectangle in Figure 12.2.

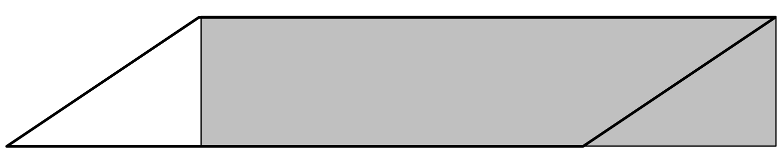

Figure 12.2 Equivalent by dissection to a rectangle

We say that two figures (F and G) are ***equivalent by dissection* ($F =_d G$)** if one can be cut up into a finite number of pieces and the pieces rearranged to form the other. Some authors use the term "*equidecomposable*" instead of "equivalent by dissection".

> **Question:** If two planar polygons have the same area, are they equivalent by dissection?
> **Answer:** Yes! For all (finite) polygons on either the plane, or on a sphere, or on a hyperbolic plane.

You will prove these results about dissections in this chapter and the next and use them to look at the meaning of area. In this chapter you will show how to dissect any triangle or parallelogram into a rectangle with the same base. Then you will do analogous dissections on spheres and hyperbolic planes after first defining an appropriate analog of parallelograms and rectangles. After that you will show that two polygons on a sphere or on a hyperbolic plane that have the same area are equivalent by dissection to each other. The analogous result on the plane must wait until the next chapter.

The proofs and solutions to all the problems can be done using "$=_d$", but if you wish you can use the weaker notion of "$=_s$": We say that two figures (F and G) are ***equivalent by subtraction*** ($F =_s G$) if there are two other figures, S and S', such that $S =_d S'$ and $F \cup S =_d G \cup S'$, where F and S and G and S' intersect at most in their boundaries. Some authors use the term "*of equal content*" instead of "equivalent by subtraction". Saying two figures are equivalent by subtraction means that they can be arrived at by removing equivalent parts from two initially equivalent figures, as in Figure 12.3.

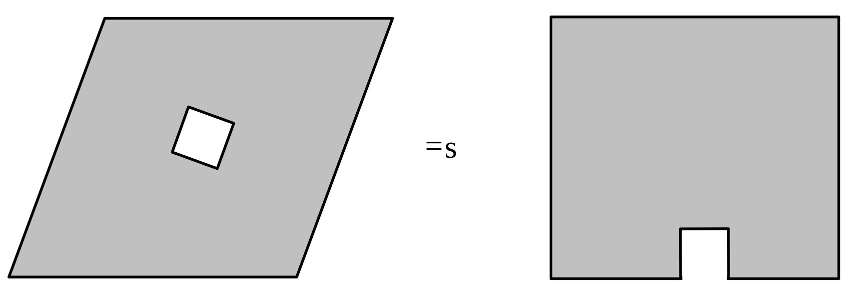

Figure 12.3 Equivalent by subtraction

If we cut out the two small squares as shown in Figure 12.3, we can see that the shaded portions of the rectangle and the parallelogram are equivalent by subtraction, but it is not at all obvious that one can be cut up and rearranged to form the other.

Equivalence by dissection is generally preferable to equivalence by subtraction because it provides a direct way of seeing that two figures have the same area. However, sometimes it is easier to find a proof of equivalence by subtraction. Besides equivalence by subtraction has the advantage (as we will see) that in some situations equivalence by dissection is only true if one assumes the Archimedean Postulate (which we first met in Problem **10.3**), while equivalence by subtraction does not need the Archimedean Postulate.

However, we would urge you to prove equivalence by dissection wherever you can.

The ***Archimedean Postulate*** (in some books this is called the *Axiom of Continuity*), named after the Greek mathematician Archimedes (who lived in Sicily, 287?–212 B.C.), is as follows:

> **AP**: *On a line, if the segment AB is less than* (contained in) *the segment AC, then there is a finite* (positive) *integer, n, such that if we put n copies of AB end to end* (see Figure 10.5), *then the* n^{th} *copy will contain the point C.*

The Archimedean Postulate can also be interpreted to rule out the existence of infinitesimal lengths. It is true that **AP** is needed to prove some results about equivalence by dissection; however, most people assume **AP** to be true on the plane, spheres, and hyperbolic planes.

A DISSECTION PUZZLE FROM 250 B.C. SOLVED IN 2003

Palimpset fragment (Wikimedia Commons)

About 250 B.C., Archimedes wrote a treatise entitled *Stomachion*, which was lost, though from commentaries it was clear that the work discussed the puzzle pictured in Figure 12.4. This is a puzzle, similar to *Tangrams*, that consists of 14 pieces that fit into a square (all of the vertices lie on a 12 × 12 grid). However, Archimedes' interest in the puzzle was not known. Then in 2003, Reviel Netz, a Stanford historian of science, deciphered parts of the *Palimpsest*, which consists of pieces of parchment that originally (about 1000 A.D.) contained several works of Archimedes, but in the 13th century the words of Archimedes were scraped off and the parchment used to write a prayer book. Netz uncovered the introduction to the *Stomachion* treatise and discovered that Archimedes asked the question *How many different ways can these 14 pieces be rearranged to fit exactly in the square?* Then, on November 17, 2003, it was announced (http://mathworld.wolfram.com/Stomachion.html) that Bill Cutler, a puzzle designer with a Ph.D. in mathematics from Cornell University (David was his thesis adviser), found using a computer that there were 17,152 distinct solutions (or only 536 if you counted as the same solutions that varied by rotation or reflection of the square or differed only in the interchange of the congruent pairs of pieces 7 and 14 or 6 and 13). The story was continued

on the front page of the *New York Times*, December 14, 2003, announcing that the University of California–San Diego mathematicians Ronald Graham and Fan Chung had independently solved the problem using combinatorics.
See https://www.nytimes.com/2003/12/14/us/in-archimedes-puzzle-a-new-eureka-moment.html for more fascinating history of Archimedes' *Palimpsest* and the *Stomachion* or explore it on your own http://www.archimedespalimpsest.org/

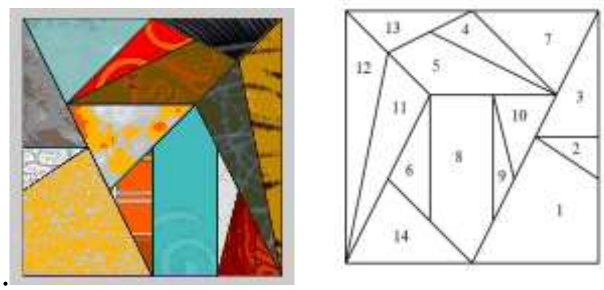

Figure 12.4 Two solutions of Archimedes' Stomachion puzzle

HISTORY OF DISSECTIONS IN THE THEORY OF AREA

Dissections have been the basis, through history, of many proofs for the Pythagorean Theorem, including in Ancient India and China; see also Chapter 13 and Problem **13.2**. In the Plato's *Meno* [**AT:** Plato] there is a Socratic dialogue in which is described the dissection proof that the diagonal of a square is the side of a square of twice the area. Euclid in his *Elements* implicitly used equivalence by dissection and equivalence by subtraction when he proved propositions about the area (he used the term "equal") of polygons. Pythagorean theorem's proof by dissection can be seen in popular tiling pattern.

Proof of Pythagorean theorem in tiling pattern

The use of dissection to find areas continued after Euclid, but it was not until 1902 that David Hilbert [**FO**: Hilbert] developed Euclid's Postulates and dissection theory into a rigorous theory of area. Hilbert discusses (in his Chapter IV) both equivalence by dissection (*zerlegungsgleich*) and equivalence by subtraction (*ergänzungsgleich*) and when the Archimedean Postulate was necessary. In a footnote, Hilbert gives credit for similar discussions of the theory of areas to M. Gérard (in 1895–1898), F. Schur (in 1898), and O. Stolz (in 1894). A recent detailed discussion of the dissection theory of area can be found in [**TX**: Hartshorne]. A recent history of dissections is contained in Chapter 13.

PROBLEM 12.1 DISSECT PLANE TRIANGLE AND PARALLELOGRAM

Some of the dissection problems ahead are very simple, while some are rather difficult. If you think that a particular problem was so easy to solve that you may have missed something, chances are you hit the nail right on the head. Most of the dissection proofs will consist of two parts: First show where to make the necessary cuts, and then prove that your construction works, that is, that all the pieces do in fact fit together as you say they do.

a. *Show that on the plane every triangle is equivalent by dissection to a parallelogram with the same base no matter which base of the triangle you pick.*

Part **a** is fairly straightforward, so don't try anything complicated. You only have to prove it for the plane — a proof for spheres and hyperbolic planes will come in a later problem after we find out what to use in place of parallelograms. Make paper models, and make sure your method works for all possible triangles with any side taken as the base. In particular, make sure that your proof works for triangles whose heights are much longer than their bases. Also, you need to show that the resulting figure actually is a parallelogram.

b. *Show that, if you assume* **AP***, then on a plane every parallelogram is equivalent by dissection to a rectangle with the same base and height. Show equivalence by subtraction without assuming* **AP.**

A partial proof of this was given in the introduction at the beginning of this chapter. But for this problem, your proof must also work for tall, skinny parallelograms, as shown in Figure 12.5, for which the given construction does not work. You may say that you can simply change the orientation of the parallelogram and use a long side as the base; but, as for part **a**, we want a proof that will work no matter which side you choose as the base. Be sure to note where you use **AP**. Again, do not try anything too complicated, and you only have to work on the plane.

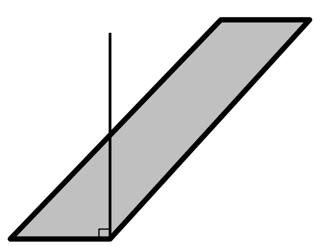

Figure 12.5 Tall, skinny parallelogram

DISSECTION THEORY ON SPHERES AND HYPERBOLIC PLANES

The above statements take on a different flavor when working on spheres and hyperbolic planes because we cannot construct parallelograms and rectangles, as such, on these spaces. We can define two types of polygons on spheres and hyperbolic spaces and then restate the above two problems for these spaces. The two types of polygons are the Khayyam quadrilateral and the Khayyam parallelogram. These definitions were first put forth by the Persian geometer-poet Omar Khayyam (1048–1131) in the 11th century AD [**AT**: Khayyam 1958]. Through a bit of Western chauvinism, geometry books generally refer to these quadrilaterals as Saccheri quadrilaterals after the Italian priest and professor Gerolamo Saccheri (1667–1733), who translated into Latin and extended the works of Khayyam and others.

A ***Khayyam quadrilateral*** (***KQ***) is a quadrilateral such that $AB \cong CD$ and $\angle BAD \cong \angle ADC \cong \pi/2$. A ***Khayyam parallelogram*** (***KP***) is a quadrilateral such that $AB \cong CD$ and AB is a parallel transport of DC along AD. In both cases, BC is called the ***base*** and the angles at its ends are called the ***base angles***. See Figure 12.6.

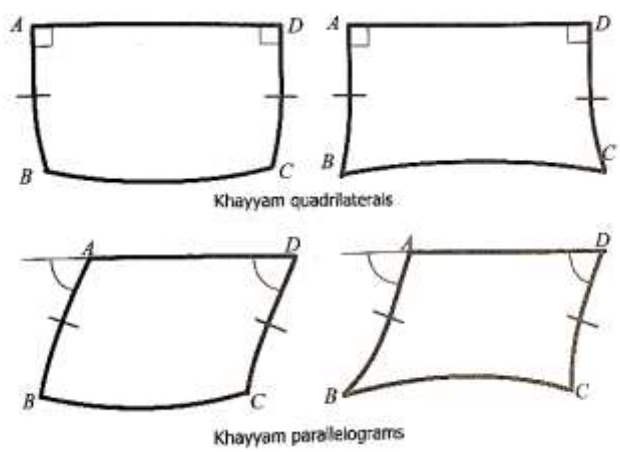

Figure 12.6 Khayyam quadrilaterals and parallelograms

PROBLEM 12.2 KHAYYAM QUADRILATERALS

a. Prove that the base angles of a KQ are congruent.

b. Prove that the perpendicular bisector of the top of a KQ is also the perpendicular bisector of the base.

c. Show that the base angles are greater than a right angle on a sphere and less than a right angle on a hyperbolic plane.

d. A KQ on the plane is a rectangle and a KP on the plane is a parallelogram.

To begin this problem, note that the definitions of KP and KQ make sense on the plane as well as on spheres and hyperbolic planes. The pictures in Figure 12.6 are deliberately drawn with a curved line for the base to emphasize the fact that, on spheres and hyperbolic planes, KPs and KQs do have the same properties as rectangles and parallelograms. You should think of these quadrilaterals and parallelograms in terms of parallel transport instead of parallel lines. Everything you have learned about parallel transport and triangles on spheres and hyperbolic planes can be helpful for this problem. Symmetry can also be useful.

Now we are prepared to modify Problem **12.1** so that it will apply to spheres and hyperbolic planes.

PROBLEM 12.3 DISSECT SPHERICAL AND HYPERBOLIC TRIANGLES AND KHAYYAM PARALLELOGRAMS

a. *Show that every hyperbolic triangle, and every small spherical triangle, is equivalent by dissection to a Khayyam parallelogram with the same base as the triangle.*

Try your proof from Problem **12.1** as a first stab at this problem. You only need to look at a sphere. You should also look at the different proofs given for Problem **12.1**. The only difference between the plane and spheres and hyperbolic planes as far as this problem is concerned is that you must be more careful on spheres and hyperbolic planes because there are no parallel lines; there is only parallel transport. Some of the proofs for Problem **12.1** work well on a sphere or on a hyperbolic plane, and others do not. Remember that the base of a KP is the side opposite the given congruent angles.

b. *Prove that every Khayyam parallelogram is equivalent by dissection (if you assume* **AP***), or equivalent by subtraction (without assuming* **AP***), to a Khayyam quadrilateral with the same base.*

As with part **a**, start with your planar proof and work from there. As before, your method must work for tall, skinny KPs. Once you have come up with a construction, you must then show that the pieces actually fit together as you say they do and prove that the angles at the top are right angles. See Figure 12.7.

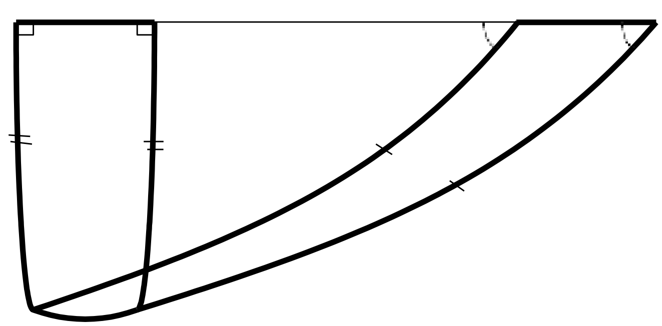

Figure 12.7 Dissecting KP into KQ

PROBLEM 12.4 SPHERICAL POLYGONS DISSECT TO LUNES

In the next chapter you will show (under the assumption of **AP**) that every polygon on the plane is equivalent by dissection to a square, and then we will use this and the Pythagorean Theorem to show that any two polygons with the same area are equivalent by dissection. This does not apply to spheres and hyperbolic planes because there are no squares on these surfaces. However, we have already shown in Problems **7.1** and **7.4** that two polygons (or triangles) on the same sphere have the same area if they have the same holonomy. Thus, every polygon on a sphere must have the same area as some lune with the same holonomy. Now we can show that not only do they have the same area, but they are also equivalent by dissection.

> *Assuming* **AP**, *show that every simple* (sides intersect only at the vertices) *small polygon on a sphere is equivalent by dissection to a lune with the same holonomy.* That is, the angle of the lune is equal to
>
> (½) (2π – sum of the exterior angles of the polygon).
>
> *Consequently, two simple small polygons with the same area on the same sphere are equivalent by dissection.*

OUTLINE OF PROOF

The proof of this result can be completed by proving the following steps (or lemmas). (This proof was first suggested to David by his daughter Becky, now Rebecca Wynne.)

1. *Every simple small polygon can be dissected into a finite number of small triangles, such that the holonomy of the polygon is the sum of the holonomies of the triangles.*

See Problem **7.5**, but what is needed here is easier than **7.5**.

2. *Each small triangle is equivalent by dissection to a KQ with the same base and same holonomy.*

Check your solutions for Problems **12.2** and **12.3**.

3. *Two KQs with the same base and the same holonomy* (*or base angles*) *are congruent.*

Match up the bases and see what you get.

4. *If two triangles have the same base and the same holonomy, then they are equivalent by dissection.*

Put together the previous steps.

5. *Any triangle* Δ *is equivalent by dissection to a lune with* $\mathcal{H}(\Delta) = \mathcal{H}(\text{lune})$ = (*twice the angle of the lune*).

Hint: A lune can also be considered as a triangle.

6. *Two simple small polygons on a sphere with the same area are equivalent by dissection to the same lune and therefore are equivalent by dissection to each other.*

What is the union of two lunes?

The first four steps above will also work (with essentially the same proofs) on a hyperbolic plane. But there is no clear replacement for the biangles (which do not exist on a hyperbolic plane). There is a proof of the following:

> **THEOREM 12.4.** *On a hyperbolic space, two simple* (the sides intersect only at the vertices) *polygons with the same area are equivalent by dissection*.

Two published proofs in English are in [**TX**: Millman & Parker], page 267, and [**DI**: Boltyanski 1978], page 62. These proofs are similar, and both use the first four steps above and use the completeness of the real numbers (in the form of a version of the Intermediate Value Theorem). You can check that Becky's proof above does not use completeness. In addition, the proof of the same result on the plane (see the discussion between Problems **13.2** and **13.3**) also does not need the use of completeness axiom.

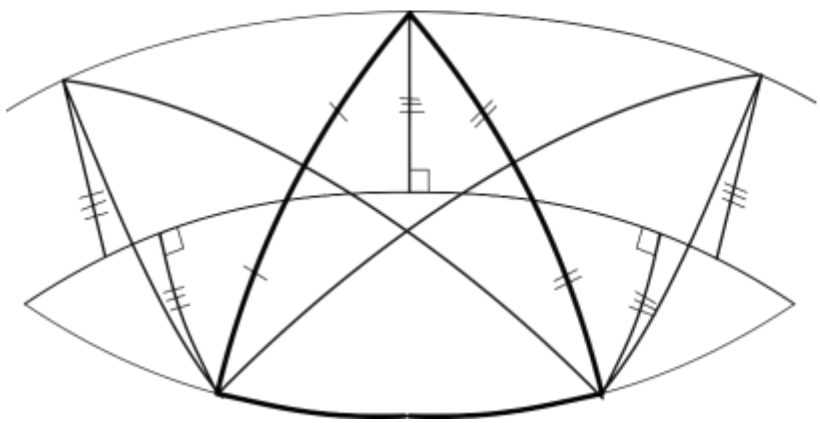

Figure 12.8 Triangles with same base and same area

In the plane all the triangles with the same base and the same height have the same area and the vertices opposite the base of these triangles form a (straight) line not intersecting the line determined by the base. On a sphere, the situation is different. Your proof above should show that midpoints of the (non-base) edges lie on a great circle and the vertices opposite the base must lie on a curve equidistant from this great circle. See Figure 12.8.