Chapter 10

PARALLEL POSTULATES

[Euclid's Fifth Postulate] ought to be struck from the postulates altogether. For it is a theorem — one that invites many questions ... — and requires for its demonstration a number of definitions as well as theorems. ... it lacks the special character of a postulate.

*— Proclus (Greek, 410–485) [**AT**: Proclus], p. 151*

PARALLEL LINES ON THE PLANE ARE SPECIAL

Up to this point we have not had to assume anything about parallel (non-intersecting) lines. No version of a parallel postulate has been necessary, on the plane, on a sphere, or on a hyperbolic plane. We defined the concrete notion of parallel transport and proved in Problem **8.2** that, on the plane (and hyperbolic planes), parallel transported lines do not intersect. Now in this chapter we will look at three important properties on the plane that require further assumptions and that will be needed in later chapters. If you are willing to assume these three statements, you may skip this chapter, but we urge you at least to read them.

> *If two lines on the plane are parallel transports of each other along some transversal, then they are parallel transports along any transversal.* (Contained in Problem **10.1**)
>
> *On the plane, the sum of the interior angles of a triangle is always* 180°. (Contained in Problem **7.3b** or Problem **10.2**)
>
> *On the plane, non-intersecting lines are parallel transports along all transversals.* (Contained in Problem **10.3d**)

We have already seen that none of these properties is true on a sphere or a hyperbolic plane. Thus, all three need for their proofs some property of the plane that does not hold on spheres and hyperbolic planes. The various properties that permit proofs of these three statements are collectively termed the ***Parallel Postulates***.

Only the three statements above are needed from this chapter for the rest of the book. Therefore, it is possible to omit this chapter and assume one of the above three statements and then prove the others. However, parallel postulates have a historical importance and a central position in many geometry textbooks and in many expositions about non-Euclidean geometries. The problems in this chapter are an attempt to help people unravel and enhance their understanding of parallel postulates. Comparing situations on the plane with situations on a sphere and on a hyperbolic plane is a powerful tool for unearthing our hidden assumptions and misconceptions about the notion of "parallel" on the plane.

As we discussed already in Chapter 8, because we have so many (often unconscious) connotations and assumptions attached to the word "parallel," we find it best to avoid using the term "parallel" as much as possible in this discussion. Instead we will use terms such as "parallel transport," "non-intersecting," and "equidistant," which make explicit the meaning that is intended.

PROBLEM 10.1 PARALLEL TRANSPORT ON THE PLANE

Show that if l_1 and l_2 are lines on the plane such that they are parallel transports along a transversal l, then they are parallel transports along any transversal. Prove this using any assumptions you find necessary. Make as few assumptions as you can and make them as simple as possible. Be sure to state your assumptions clearly.

What part of your proof does not work on a sphere or on a hyperbolic plane?

SUGGESTIONS

This problem is by no means as trivial as it at first may appear. In order to prove this theorem, you will have to assume something — there are many possible assumptions, use your imagination. But at the same time, try not to assume any more than is necessary. If you are having trouble deciding what to assume, try to solve the problem in a way that seems natural to you and then see what develops while making explicit any assumptions you are using.

On spheres and hyperbolic planes, try the same construction and proof you used for the plane. What happens? You should find that your proof does not work on these surfaces. What is it about your proof (on a sphere and hyperbolic plane) that creates difficulties?

Problem **10.1** emphasizes the differences between parallelism on the plane and parallelism on spheres and hyperbolic planes. On the plane, non-intersecting lines exist, and one can "parallel transport" everywhere. Yet, as was seen in Problems **8.2** and **8.3**, on spheres and hyperbolic planes two lines are cut at congruent angles if and only if the transversal line goes through the center of symmetry formed by the two lines. That is, on spheres and hyperbolic planes two lines are parallel transports only when they can be parallel transported through the center of symmetry formed by them. Be sure to draw a picture locating the center of symmetry and the transversal. On spheres and hyperbolic planes it is impossible to slide the transversal along two parallel transported lines while keeping both angles constant (something you *can* do on the plane). In Figures 10.1, the line r' is a parallel transport of line r along line l, but it is not a parallel transport of r along l'.

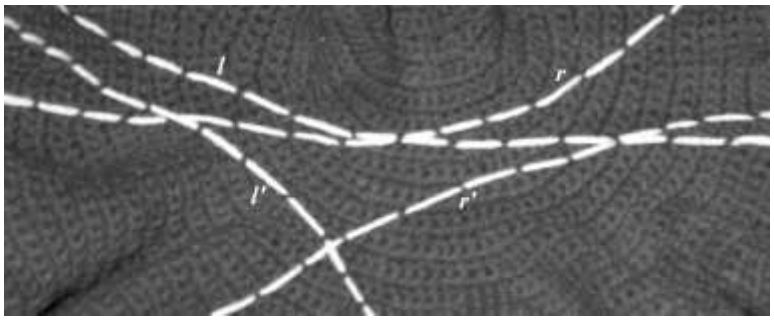

Figure 10.1a Parallel transport on a hyperbolic plane along *l* but not along *l*

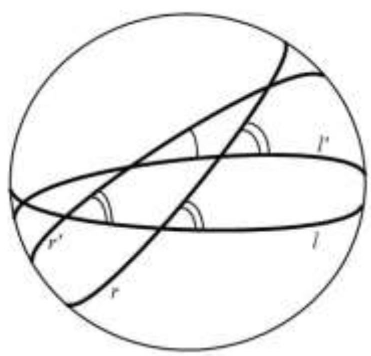

Figure 10.1b Parallel transport on a sphere along *l*, but not along *l′*

**Pause, explore, and write out your ideas before reading further.**

We will now divide parallel postulates into three groups: those involving mostly parallel transport, those involving mostly equidistance, and those involving mostly intersecting or non-intersecting lines. This division is useful even though it is rough and unlikely to fit every conceivable parallel postulate. *Which of these three groups is the most appropriate for your assumption from Problem* **10.1**? We will call the assumption you made in Problem **10.1** "*your parallel postulate*."

PROBLEM 10.2 PARALLEL POSTULATES NOT INVOLVING (NON-)INTERSECTING LINES

Commonly used postulates of this sort are

H = 0: *The holonomy of triangles is zero.*

A = 180: *The sum of the angles of a triangle is equal to 180 °.*

PT!: *If two lines are parallel transports* (*PT*) *along one line then they are PT along ALL transversals.*

Note that these are false on spheres and on hyperbolic planes. The last two properties (**A = 180** and **PT!**) will be needed crucially in almost all the remaining chapters (and you may have already used one of them in Problem **9.4**). These properties are needed to study (on the plane) parallelograms, rectangles, and similar triangles (Chapters 12 and 13), circles and inversion (Chapters 15 and 16), projections of spheres and hyperbolic planes onto the Euclidean plane (Chapters 14 and 17), Euclidean manifolds (Chapters 18 and 24), solutions to quadratic and cubic equations (Chapter 19), trigonometry (Chapter 20), and polyhedra in 3-space (Chapter 23). It is these properties and their uses that we most often associate with consequences of the parallel postulates.

In Problem **7.3b** we proved **A = 180** on the plane, so we can

a. *Ask what assumption about the plane was used in proving* **7.3b.**

b. *Use* **A = 180** *to prove* **PT!.**

Look first at transversals that intersect the line along which the two lines are parallel transports.

c. *Using **PT!** (*Problem **10.1***), prov**e** **A = 180** without using results from* Chapter 7**.**

Start by parallel transporting one side of the triangle.

d. *Show that, on the plane,* **H = 0 ⇔ A = 180 ⇔ PT!.** *If your postulate from* **10.1** *is in this group, then show that it is also equivalent to the others.*

This is usually accomplished most efficiently by proving **H = 0** ⇒ **A = 180** ⇒ **PT!** ⇒ Your Postulate ⇒ **H = 0**, or in any other order.

e. *Prove that, on the plane, two parallel transported lines are equidistant.*

Look for rectangles or parallelograms.

EQUIDISTANT CURVES ON SPHERES AND HYPERBOLIC PLANES

The latitude circles on the earth are sometimes called "parallels of latitude." They are parallel in the sense that they are everywhere equidistant as are concentric circles on the plane. In general, transversals do not cut equidistant circles at congruent angles. However, there is one important case where transversals do cut the circles at congruent angles. Let l and l' be latitude circles the same distance from the equator on opposite sides of it. See Figure 10.2. Then every point on the equator is a center of half-turn symmetry for these pair of latitudes. Thus, as in Problems **8.3** and **10.1**, every transversal cuts these latitude circles in congruent angles, even though these latitude circles are not geodesics. In the first section of Chapter 2 we noted that Euclid, in his *Phenomena*, discussed such equidistant circles.

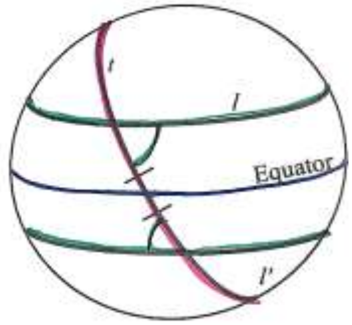

Figure 10.2 Special equidistant circles

The same ideas work on a hyperbolic plane: If g is a geodesic and l and l' are the (two) curves (not geodesics) that are a distance d from g, then l and l' are equidistant from each other and every transversal cuts them at congruent angles. This follows from the fact that g has half-turn symmetry at every point.

PROBLEM 10.3 PARALLEL POSTULATES INVOLVING (NON-)INTERSECTING LINES

One of Euclid's assumptions constitutes ***Euclid's Fifth*** (***or Parallel***) ***Postulate*** (***EFP***), which says

> **EFP:** *If a straight line intersecting two straight lines makes the interior angles on the same side less than two right angles, then the two lines (if extended indefinitely) will meet on that side on which are the angles less than two right angles.*

For a picture of EFP, see Figure 10.3.

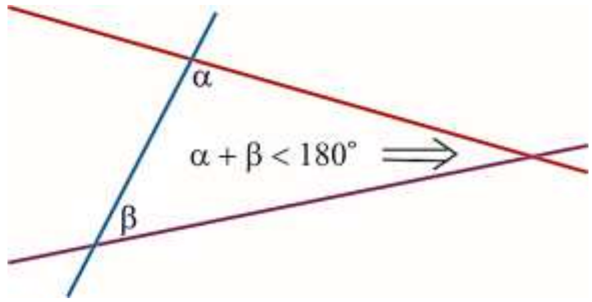

Figure 10.3 Euclid's Parallel Postulate

You probably did not assume EFP in your proof of Problem **10.1**. You are in good company — many mathematicians, including Euclid, have tried to avoid using it as much as possible. However, we will explore EFP because, historically, it is important, and because it has some very interesting properties, as you will see in Problem **10.3**. On a sphere, all straight lines intersect twice, which means that EFP is trivially true on a sphere. But in Problem **10.4**, you will show that EFP is also true in a stronger sense on spheres. You will also be able to prove that EFP is false on a hyperbolic plane.

Thus, EFP does not have to be assumed on a sphere — it can be proved! However, in most high school geometry textbooks, EFP is replaced by another postulate, claimed to be equivalent to EFP. This postulate we will call the ***High School Parallel Postulate* (*HSP*)**, and it can be expressed in the following way:

HSP: For every line **l** and every point **P** not on **l**, there is a unique line **l'** that passes through **P** and does not intersect (is parallel to) **l**.

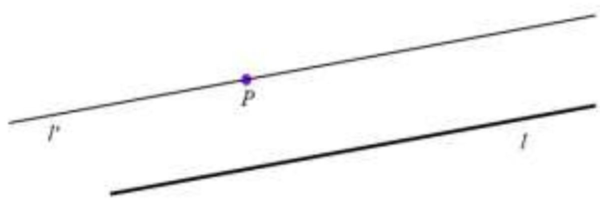

Figure 10.4 High School Parallel Postulate

In many books the **HSP** is called "Playfair's Parallel Postulate", but this is an inappropriate name, as we will show in the historical notes in the last two sections of this chapter.

Note that, because any two great circles on a sphere intersect, there are *no* lines *l'* that are parallel to *l* in the "not intersecting" sense. Therefore, **HSP** is not true on spheres. On the other hand, if we change "parallel" to "parallel transport" then *every* great circle through *P* is a parallel transport of *l* along *some* transversal. What happens on a hyperbolic plane? In Problem **10.3**, you will explore the relationships among **EFP**, **HSP**, and your postulate from Problem **10.1**. In Problem **10.4** we will explore these postulates on spheres and hyperbolic planes.

a. *Show that, on the plane, **EFP** and **HSP** are equivalent. If your postulate from **10.1** is in this group, is it equivalent to the others? Why or why not?*

To show that **EFP** and **HSP** are equivalent on the plane, you need to show that you can prove **EFP** if you assume **HSP** and vice versa. If the three postulates are equivalent, then you can prove the equivalence by showing that **EFP** ⇒ **HSP** ⇒ Your Postulate ⇒ **EFP** or in any other order. It will probably help you to draw lots of pictures of what is going on. Also, remember that we proved in Problem **8.2** (without using any parallel postulate) that parallel (non-intersecting) lines exist. Note that **HSP** is not true on a sphere but **EFP** is true, so your proof that **EFP** implies **HSP** on the plane must use some property of the plane that does not hold on a sphere. Look for it.

b. *Prove that either* **EFP** *or* **HSP** *can be used to prove* (*without using* **10.1** *or* **7.3b**) *one of* **H = 0**, **A = 180**, or **PT!**. (It does not matter which. *Why*?)

So, are all these postulates equivalent to each other on the plane? The answer is almost, but not quite! In order for **A = 180** (or **H = 0** or **PT!**) to imply **EFP** (or **HSP**), we have to make an additional assumption, the ***Archimedean Postulate*** (**AP**) (in some books this is called the *Axiom of Continuity*), named after the Greek mathematician, Archimedes (who lived in Sicily, 287?–212 B.C.):

AP: *On a line, if the segment AB is less than (contained in) the segment AC, then there is a finite (positive) integer, n, such that, if we put n copies of AB end-to-end (see Figure 10.5), then the n^{th} copy will contain the point C.*

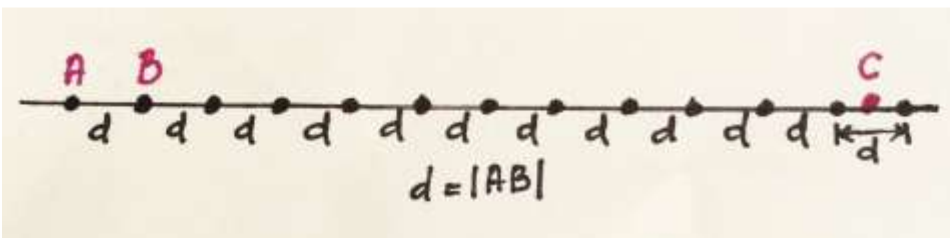

Figure 10.5 The Archimedean Postulate

The Archimedean Postulate can also be interpreted to rule out the existence of infinitesimal lengths. The reason these postulates are "almost but not quite" equivalent to each other on the plane is that, though **AP** is needed, it is assumed by most people to be true on the plane, spheres, and hyperbolic planes. But this is the first time we have needed this assumption in this book.

c. *Show that, on the plane,* **AP** *and* **A = 180** *imply* **EFP**.

Look at the situation of **EFP**, which we redraw in Figure 10.6. Pick a sequence of equally spaced points A, A', A'',... on n (on the side of the angle α). Next, parallel transport l along m to lines l', l'',... that intersect n at the points A', A'',... . And parallel transport m along l to m', m'',..., which also intersect n at the points A', A'',... . Look for congruent angles and congruent triangles. Use **AP** to argue that n and m will eventually intersect.

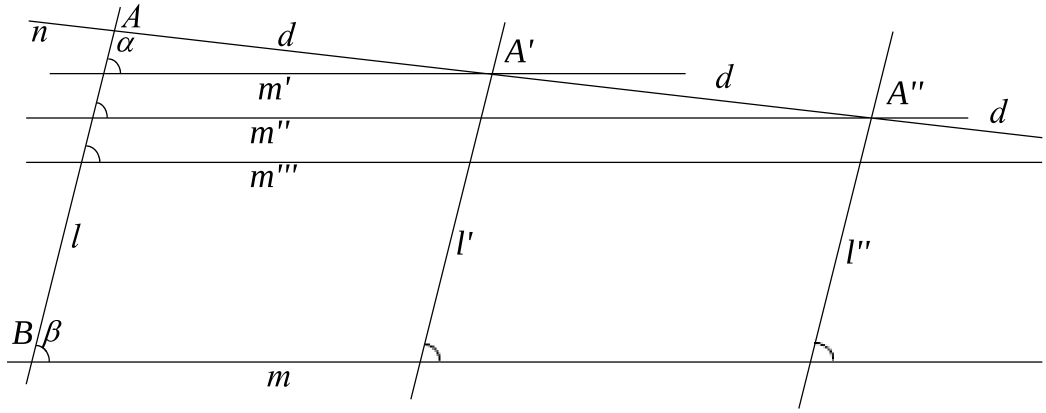

Figure 10.6 **AP** and **A = 180** imply **EFP**

Clearly, this proof feels different from the other proofs in this chapter. And, as already pointed out, most of the applications in this book of parallel postulates only need **A = 180** and **PT!**. Is it possible for us not to bother with **EFP** and **HSP**? Or are they needed? Yes, we need them, but only in a few places, such as in Problem **11.3**, where what we need to know is

d. *Prove using* **EFP**: *On the plane, non-intersecting lines are parallel transports along every transversal.*

Compare with **EFP**.

PROBLEM 10.4 EFP; HSP ON SPHERE AND HYPERBOLIC PLANE

a. *Show that* **EFP** *is true on a sphere in a strong sense; that is, if lines* $\boldsymbol{l}$ *and* $\boldsymbol{l}'$ *are cut by a transversal* $\boldsymbol{t}$ *such that the sum of the interior angles* $\boldsymbol{\alpha} + \boldsymbol{\beta}$ *on one side is less than two right angles, then not only do* $\boldsymbol{l}$ *and* $\boldsymbol{l}'$ *intersect, but they also intersect closest to* $\boldsymbol{t}$ *on the side of* $\boldsymbol{\alpha}$ *and* $\boldsymbol{\beta}$*. You will have to determine an appropriate meaning for "closest."*

To help visualize the postulates, draw these "parallels" on an actual sphere. There are really two parts in this proof — first, you must come up with a definition of "closest," and then prove that **EFP** is true for this definition. The two parts may come about simultaneously as you come up with a proof. This problem is closely related to Euclid's Exterior Angle Theorem but can also be proved without using **EEAT**. One case that you should look at specifically is pictured in Figure 10.7. It is not necessarily obvious how to define the "closest" intersection.

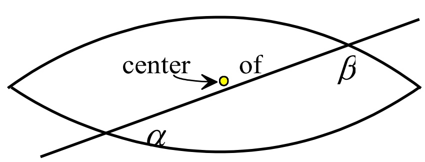

Figure 10.7 Is **EFP** true on a sphere?

b. *On a hyperbolic plane let* ***l*** *be a geodesic and let* ***P*** *be a point not on* ***l****; then show that there is an angle θ with the property that any line* ***l′*** *passing through* ***P*** *is parallel to (not intersecting)* ***l*** *if the line* ***l′*** *does not form an angle less than θ with the line from* ***P*** *that is perpendicular to* ***l*** (Figure 10.8).

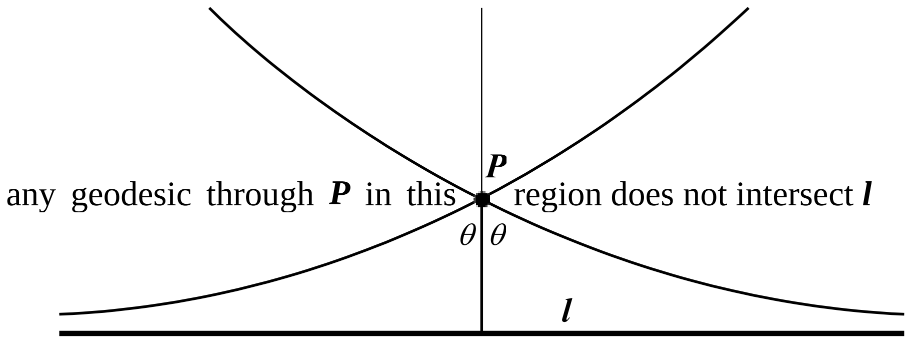

Figure 10.8 Multiple parallels on a hyperbolic plane

Look at a variable line through *P* that does intersect *l* and then look at what happens as you move the intersection point out to infinity.

c. *Using the notion of parallel transport, change the High School Postulate so that the changed postulate is true on both spheres and hyperbolic planes. Make as few alterations as possible and keep some form of uniqueness.*

Try to limit your alterations so that the new postulate preserves the spirit of the old one. You can draw ideas from any of the previous problems to obtain suitable modifications. Prove that your modified versions of the postulate are true on spheres and hyperbolic planes.

d. *Either prove your postulate from Problem* 10.1 *on a sphere and on a hyperbolic plane or change it, with as few alterations as possible, so that it is true on these surfaces. You may need to make different changes for the two surfaces.*

In Problem **10.1** you should have decided whether or not your postulate is true on spheres or on hyperbolic spaces.

COMPARISONS OF PLANE, SPHERES, AND HYPERBOLIC PLANES

Figure 10.9 is an attempt to represent the relationships among parallel transport, non-intersecting lines, **EFP**, and **HSP**. Can you fit your postulate into the diagram? The High School Postulate (**HSP**) assumes both the existence and uniqueness of parallel lines. In Problem **8.2**, you proved that if one line is a parallel transport of another, then the lines do not intersect on the plane or on a hyperbolic plane. Thus, it is not necessary to assume the existence of parallel lines (non-intersecting lines) on the plane or hyperbolic plane. On a sphere any two lines intersect. However, in Problem **8.4** we saw that there *are* non-intersecting lines that are not parallel transports of each other on a hyperbolic plane and on any cone with cone angle larger than 360°.

	Euclidean	**Hyperbolic**	**Spherical**
Parallel transported lines	do not intersect and are equidistant **8.2** & **10.2**	diverge in both directions **8.2** & **8.3**	always intersect **2.1**
Parallel transported lines	are parallel transports along all transversals**10.1**	are parallel transports along any transversal that passes through the center of symmetry of the lines **8.3**	
Non- intersecting lines	are parallel transports along all transversals **10.3d**	sometimes are not parallel transports **8.4b**	do not exist **2.1**
Euclid's 5th Postulate	must be assumed Chapter 10	is false **8.4b**	is provable in a strong sense **10.4a**
High School Parallel Postulate	a unique line through a point not intersecting a given line **10.3**	many lines through a point not intersecting a given line **10.4b**	No non-intersecting lines **2.1**
Your postulate from **10.1**			
Two points determine	a unique line and line segment **6.1d**		at least two line segments **6.1d**
Sum of the angles of a triangle	= 180° **7.3b** & **10.2**	< 180° **7.1**	> 180° **7.2**
Holonomy	= 0° **10.2**	< 0° **7.4**	> 0° **7.4**
VAT & ITT	are always true, **3.2** & **6.2**		
SAS, ASA, and SSS	hold for all triangles **6.4**, **6.5**, & **9.1**		hold for small triangles **6.4**, **6.5**, & **9.1**
AAA	is false, **9.4**, similar triangles	is true, **9.4**, no similar triangles	

Figure 10.9 Comparisons of the three geometries

PARALLEL POSTULATES WITHIN THE BUILDING STRUCTURES STRAND

Basalt pebbles with perpendicular and parallel lines in the title picture of this chapter are from Neolithic period, late 7th millennium B.C.! We cannot know what was puzzling a human who engraved those lines but we do know that the problem of parallels puzzled Greek geometers (see the quote at the beginning of this chapter). The prevailing belief was that it surely follows from the straightness of lines that lines like *n* and *m* in Figure 10.10 *had* to intersect, and so Euclid's Fifth Postulate would turn out to be unnecessary.

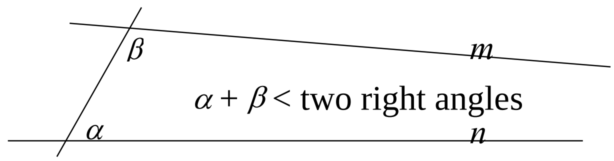

Figure 10.10 Surely these intersect!

During and since the Greek era, there have been attempts to derive the postulate from the rest of elementary geometry; attempts to reformulate the postulate or the definition of parallels into something less objectionable; and descriptions of what geometry would be like if the postulate was in some way denied. Part of these attempts was to study what geometric properties could be proved without using a parallel postulate, but with the other postulates (properties) of the plane. The results of these studies became known as ***absolute geometry***. Both the plane and a hyperbolic plane are examples of absolute geometry. Included in absolute geometry are all the results about the plane and hyperbolic planes that we have discussed in Chapters 1, 3, 5, 6, 8, and 9 (except AAA, Problem **9.4**). In general, absolute geometry includes everything that is true of both the plane and hyperbolic planes.

Claudius Ptolemy (Greek, 100–178) implicitly used what we now call Playfair's Postulate (**PP**) to prove **EFP**. This is reported to us by Proclus (Greek, 410–485) [**AT:** Proclus], who explicitly states **PP**.

PP. *Two straight lines that intersect one another cannot be both parallel to the same straight line.*

Or, in another common wording,

PP. *Given a straight line and a point not on the line, there is at most one straight line through the point that does not intersect the given line.*

Playfair's Parallel Postulate got its current name from the Scottish mathematician John Playfair (1748–1819), who brought out successful editions of Euclid's *Elements* in the years following 1795. After 1800 many commentators referred to Playfair's postulate (**PP**) as the best statement of Euclid's postulate, so it became a tradition in many geometry books to use **PP** instead of **EFP**. The reader can see easily that both **PP** and **EFP** assert the uniqueness of parallel lines and neither asserts the existence of parallel lines. **HSP** seems to have been first stated by Hilbert in the first edition of his *Grundlagen der Geometrie* (but not in the later editions, where he used **PP**) and in **[EG**: Hilbert]**.** But later, apparently starting in the 1960s, many (but not all) textbooks (including first two editions of this book!) started calling **HSP** by the name "Playfair's Postulate". In absolute geometry **HSP** is equivalent to **PP** and **EFP** (as you showed in **10.3a**), and because absolute geometry was the focus of many investigations, the statement "**HSP** is equivalent to **EFP**" was made and is still being repeated in many textbooks and expository writings about geometry even when the context is not absolute geometry. As you showed in **10.4a**, **EFP** is true on spheres and so is **PP**, and so they cannot be equivalent to **HSP**, which is clearly false on the sphere. During the 9th through 12th centuries, parallel postulates were explored by mathematicians in the Islamic world. al-Hasan ibn al-Haytham (Persian, 965–1040) proved **EFP** by assuming, *A quadrilateral with three right angles must have all right angles*. Quadrilaterals with three right angles are known later in Western literature as Lambert quadrilaterals after Johann Lambert (German–Swiss, 1728–1777), who studied them and also was the first to extensively investigate hyperbolic trigonometric functions. The Persian poet and geometer Omar Khayyam (1048–1131) wrote a book that in translation is entitled *Discussion of Difficulties in Euclid* [**AT:** Khayyam 1958]. Khayyam introduces a new postulate (which he attributes to Aristotle, though it has not been found among the surviving works of Aristotle), which says *Two straight lines that start to converge continue to converge*. In his work on parallel lines he studied the Khayyam quadrilaterals (later in the West to be called Saccheri quadrilaterals), which are discussed here in Chapter 12. Nassir al-Din Al-Tusi (Khorasan, 1201–1274) furthered the study of parallel postulates and is credited (though some say it may have been his son) with first proving that **A = 180** *is equivalent to* **EFP**, our **10.3bc**. Al-Tusi's works were the first Islamic mathematical works to be discovered in the Western Renaissance and were published in Rome in 1594.

The assumption *parallel lines are equidistant* (see our **10.2d**) was discussed in various forms by Aristotle (Greek, 383–322 B.C.), Posidonius (Greek, 135–51 B.C.), Proclus, ibn Sina (Uzbek, 980–1037), Omar Khayyam, and Saccheri (Italian, 1667–1733).

John Wallis (English, 1616–1703) proved that **EFP** followed from the assumption *To every triangle, there exists a similar triangle of arbitrary magnitude*. (See our **9.4**.) The works, already mentioned, of Wallis, Saccheri, Lambert in the 17th and 18th centuries were continued by the French school into the early 19th century by Joseph Fourier (French, 1768–1830), who concluded that geometry was a physical science and could not be established a priori, and Adrien-Marie Legendre (1752–1833), who proved that *in absolute geometry the sum of the interior angles of a triangle is always less than or equal to 180°.*

The breakthrough in the study of parallel postulates came in the 19th century when, apparently independently, János Bolyai (Hungarian, 1802–1860), and N. I. Lobachevsky (Russian, 1792–1856) finally developed hyperbolic geometry as an absolute geometry that did not satisfy **EFP**. See Chapters 5 and 17 for more discussions of hyperbolic geometry. These discoveries of hyperbolic geometry showed that the quest for a proof of **EFP** from within absolute geometry is impossible.

For details of the relevant histories, see [**HI**: Gray], [**DG**: McCleary], [**HI**: Rosenfeld], and the Heath's editorial notes in [**AT**: Euclid, *Elements*].

NON-EUCLIDEAN GEOMETRIES WITHIN THE HISTORICAL STRANDS

Building Structures Strand. Because of this long history of investigation into parallel postulates within the Building Structures Strand, many books misleadingly call hyperbolic geometry "*the* non-Euclidean geometry." As pointed out in the introduction to Chapter 2, spherical geometry has been studied since ancient times within the Navigation/Stargazing Strand, but it did not fit into absolute geometry and thus was (and still is) left out of many discussions of non-Euclidean geometry, especially those that take place in the context of the Building Structures Strand. In those discussions that do include spherical geometry, hyperbolic geometry is usually called ***Lobachevskian geometry*** or ***Lobachevskian/Bolyai geometry***, and spherical geometry is usually called ***Riemannian geometry***. Bernhard Riemann (German, 1826–1866) pioneered an intrinsic and analytic view for surfaces and space and suggested that the intrinsic geometry of the sphere could be thought of as a non-Euclidean geometry. He also introduced an intrinsic analytic view of the sphere that became known as the ***Riemann sphere***. The Riemann sphere is usually studied in a course on complex analysis. The use of the term "Riemannian geometry**"** to denote the non-Euclidean geometry of the sphere leads to confusions since "Riemannian geometry" more commonly refers to the part of differential geometry that gives intrinsic analytic descriptions of the local geometry of general surfaces and 3-space.

Art/Pattern Strand. In this strand, artists' experimentations with perspective in art led to the mathematical theory of *projective geometry* that developed initially independently of both spherical and hyperbolic geometries. Then in the last part of the 19th century, projective geometry was used to provide a unified treatment of the three geometries. It was within this development that Felix Klein introduced the names ***double elliptic geometry*** (for the geometry of the sphere), ***elliptic geometry*** (for the geometry of the sphere with antipodal points identified — this the same as the real projective plane, $\mathbf{RP}^2$, discussed in Problem **18.3**), ***parabolic geometry*** (for Euclidean geometry), and ***hyperbolic geometry*** (for the Bolyai/Lobachevsky geometry). In Chapter 11, we will discuss Klein's development of various geometries in terms of their isometries. A geometric connection between spherical and projective geometry can be seen in the image onto the plane of the sphere (with antipodal points identified) under gnomic projection. (See Problems **14.2** and **20.6**.)

For a fuller discussion of spherical and hyperbolic geometry in this strand, see [**HI**: Kline], Chapter 38, and [**HI**: Katz], Section 17.3.

In Kline we also find a statement (page 913) that represented the view of some mathematicians:

> By the early 1870s several basic non-Euclidean geometries, the hyperbolic, and the two elliptic geometries, had been introduced and intensively studied. The fundamental question which had yet to be answered in order to make these geometries legitimate branches of mathematics was whether they were consistent.

So for those, who believe that the only "legitimate" branches of mathematics are the ones included in some rigorous formal system, non-Euclidean geometries did not became a part of mathematics until the late 1800s.

Navigation/Stargazing Strand. In the introduction to Chapter 2 we saw that spherical geometry has been studied within this strand for more than 2000 years, and in this study were developed (spherical) coordinate systems and trigonometric functions. We hope that the reader agrees that this was (and still is) legitimate mathematics.

For more readings (and references) on this history of spherical geometry within the Navigation/Stargazing Strand, see [**HI**: Katz], Chapter 4, and [**HI**: Rosenfeld], Chapter 1. Into the 19th century, spherical geometry occurred almost entirely in the Navigation/Stargazing Strand and was used by Brahe and Kepler in studying the motion of stars and planets and by navigators and surveyors. The popular book (which is also available online) *Spherical Trigonometry: For the use of colleges and schools* (1886) [**SP**: Todhunter] contains several discussions of the use of spherical geometry in surveying and was used in British schools before hyperbolic geometry was widely known in the British Isles. On the continent, C. F. Gauss (1777–1855) was using spherical geometry in various large-scale surveying projects before the advent of hyperbolic geometry. Within this context, Gauss initiated (and Riemann expanded) the analytic study of surfaces that led to what we now call differential geometry. Gauss introduced *Gaussian curvature,* which was discussed in Chapter 7. Within this theory of surfaces, spherical geometry is described locally as the geometry of surfaces with constant positive (Gaussian) curvature and hyperbolic geometry is described locally as the geometry of surfaces with constant negative curvature. As we discussed in Chapter 5, there are no analytic surfaces in 3-space with the complete hyperbolic geometry, but the pseudosphere (Problem **5.3**) locally has hyperbolic geometry. Because of this, in the Navigation/Stargazing strand, hyperbolic geometry is called by some authors ***pseudosphere geometry***.