Chapter 6

# Gaussian Curvature Extrinsically Defined

## *Pep Talk to the Reader*

I think that the material in this chapter is very difficult. Don't give up and don't lose hope. What is happening here is that we are standing at the interface between things that we can see and argue about geometrically, and things that are given formally. It is difficult to hold these two aspects together—one is alive and one is dead, but both are important. It sometimes feels easier to jump headlong into the formal stuff—forgetting about what it means geometrically and just following everything through mechanically. Don't do that! Unfortunately that is often the tendency—it is also my tendency! Resist it and persevere in trying to see what the meanings of these formal things are geometrically as you go along. This is hard to do, but the effort will be well worth it. On the other hand, there exists a tendency to ignore the formal stuff and rely only on our geometric intuition. But, if we ignore the formal stuff, we would miss out on the incredibly powerful tools contained in the formalism. We need to use both the formal analytic tools and our geometric intuition; and we need to look for their interrelations. Relate everything in this chapter to the example of surfaces you already know, such as the sphere, cylinder, cone, ribbon, and strake.

In Chapter 5 we developed an intrinsic description of the intrinsic curvature of a surface. In this chapter we start with the more common extrinsic description of the Gaussian curvature of a surface, which is based on the normal curvature introduced in Problem **4.7.a**. The Gaussian and intrinsic curvatures are easily seen to be the same on a sphere. Then we use a mapping (called the ***Gauss map***) from the surface to the sphere, which then allows us to show that the Gaussian curvature and intrinsic curvature coincide on all $C^2$ surfaces.

In Chapter 7 we will use these results to express the Gaussian (intrinsic) curvature in local coordinates and to derive several more intrinsic descriptions of Gaussian curvature.

At the end of this chapter we will explore *mean curvature* and *minimal surfaces*.

## *Problem 6.1. Gaussian Curvature, Extrinsic Definition*

Let $p$ be a point on the smooth $C^2$ surface $M$ in $\mathbf{R}^3$, and let $\mathbf{n}(p)$ be one of the two choices of unit normal to the surface at $p$, so that $\mathbf{n}$ is differentiable in a neighborhood of $p$. Let $\mathbf{T}_p$ be a unit tangent vector at $p$. If $\gamma$ is a curve on $M$, which passes through $p$ and has $\mathbf{T}_p$ as unit tangent vector, then, according to Problem **4.7.a**, the normal curvature of $\gamma$ at $p$ satisfies

$$\boldsymbol{\kappa}_{\mathbf{n}}(p) = \langle \mathbf{T}_p, -\mathbf{T}_p\mathbf{n} \rangle\, \mathbf{n}(p).$$

Since $\mathbf{n}(p)$ is a unit vector, $\langle \mathbf{T}_p, -\mathbf{T}_p\mathbf{n} \rangle$ is the magnitude of the normal curvature vector, and thus we define the (***scalar***) ***normal curvature*** of $M$ at $p$ in the direction $\mathbf{T}_p$ as

$$\kappa_{\mathbf{n}}(\mathbf{T}_p) \equiv \langle \mathbf{T}_p, -\mathbf{T}_p\mathbf{n} \rangle$$

$$\equiv \text{the length of the projection of } -\mathbf{T}_p\mathbf{n} \text{ onto the direction of } \mathbf{T}_p.$$

Note that $\kappa_{\mathbf{n}}(\mathbf{T})$ is positive when $\boldsymbol{\kappa}_{\mathbf{n}}$ is in the direction of $\mathbf{n}$ and negative otherwise. In Problem **4.7.a** we learned that $\kappa_{\mathbf{n}}(\mathbf{T}_p)$ is the normal curvature of *any* unit speed curve through $p$, which has $\mathbf{T}_p$ as velocity vector at $p$. Some books use the name ***Weingarten map*** to indicate the map $\mathsf{L}(\mathbf{X}_p) = -\mathbf{X}_p\mathbf{n}$.

Recall from Chapter 3 that the normal curvature of a curve on the surface $M$ is the curvature of the curve that is due to its being on the surface. So, the normal curvature $\kappa_{\mathbf{n}}(\mathbf{T}_p)$ tells us how the surface is curving in the direction of $\mathbf{T}_p$. Then $\kappa_{\mathbf{n}}$ is a real-valued function defined on the unit vectors (unit circle) in the tangent space $T_pM$; as such, if $\kappa_{\mathbf{n}}$ is continuous, then it has a maximum value and a minimum value, which we shall denote $\kappa_1$ and $\kappa_2$.

These are called the ***principal curvatures***, and the directions in which they occur are called the ***principal directions***. We then define the ***Gaussian curvature*** of the surface at the point $p$ to be the product of $\kappa_1$ and $\kappa_2$. Note that this is an extrinsic definition of the Gaussian curvature. In Problem **6.4** we will show that this (extrinsically defined) Gaussian curvature coincides with the intrinsic curvature on $C^2$ surfaces.

**a.** *What are the principal directions, and principal curvatures of the cylinder, cone, and sphere? Give geometric or analytic reasons.*

If $\mathbf{A}$, $\mathbf{B}$ are linearly independent vectors based at the point $p$, then the ***span of*** $\mathbf{A}$, $\mathbf{B}$, denoted by "span[$\mathbf{A}$,$\mathbf{B}$]", is the plane (through $p$) determined by the two vectors.

**b.** *Show that if $\gamma$ is a unit speed curve in the smooth surface $M$, and*

$$\gamma^* = \{\text{span}[\gamma'(0),\mathbf{n}(\gamma(0))] \cap M\},$$

(see Figure 6.1) *then, at $\gamma(0) = p$, the curvature vectors on $M$ satisfy*:

$$\boldsymbol{\kappa}(\gamma^*) = \boldsymbol{\kappa}_{\mathbf{n}}(\gamma^*) = \boldsymbol{\kappa}_{\mathbf{n}}(\gamma) = \langle\, \gamma'(0),-\gamma'(0)\mathbf{n}\,\rangle\, \mathbf{n},$$

*and the scalar curvatures satisfy:*

$$\pm\kappa(\gamma^*) = \kappa_{\mathbf{n}}(\gamma^*) = \kappa_{\mathbf{n}}(\gamma) = \langle\, \gamma'(0),-\gamma'(0)\mathbf{n}\,\rangle,$$

*where $\kappa_{\mathbf{n}}(\gamma^*) = \kappa_{\mathbf{n}}(\gamma)$ is negative if the curves curve away from the normal $\mathbf{n}$, as in Figure 6.1.*

Note also that $\boldsymbol{\kappa}(\gamma^*) = \boldsymbol{\kappa}_{\mathbf{n}}(\gamma^*)$ is only asserted to hold at the one point $\gamma(0)=p$. This does not imply that $\gamma^*$ is a geodesic, even though $\boldsymbol{\kappa}_g = 0$, because it is only zero at that one point. Look at an example of this by looking at the tangent vector to a helix along a cylinder. The corresponding $\gamma^*$ on the cylinder will be an ellipse tangent to the (intrinsically straight) helix but will not coincide with it.

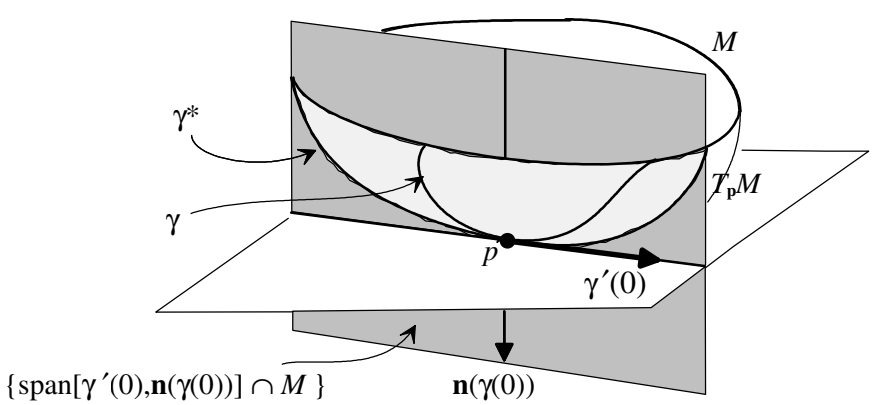


**Figure 6.1. Finding the normal curvature.**

***c.** *Look at the surface which is the graph of*

$$z = (1 - \cos 4\theta)\, r^2.$$

*Find the principal directions and principal curvatures at* (0,0,0), *and note that the principal directions are not perpendicular. Note that similar examples are possible in the general form*

$$z = f(\theta)\, r^2,$$

*where f is any twice differentiable function that satisfies:*

$$f(-x) = f(x).$$

Computer Exercise 6.1 will allow you to display and view these surfaces. When there is a $C^2$ local coordinate patch, the principal directions are orthogonal, as you shall see in the next problem. Thus it must be that *the surfaces in part* ***c*** *must not have any* $C^2$ *coordinate patch.*

***d.** *Show that every closed smooth surface* (see Problem **4.2.c**) *has at least one point at which all the normal curvatures are positive with respect to the inward pointing normal.*

[Hint: Start with a sphere that contains the surface in its interior and then gradually shrink the sphere until it first touches the surface. What can you say about the surface at this point of first touching?]

## *Problem 6.2. Second Fundamental Form*

If $M$ is a smooth surface in $\mathbf{R}^3$, and $\mathbf{V}_p$ and $\mathbf{W}_p$ are orthogonal unit tangent vectors at a point $p$ on $M$, then any other unit tangent vector $\mathbf{T}_p$ at $p$ can be written as a linear combination: $\mathbf{T}_p = a\mathbf{V}_p + b\mathbf{W}_p$. If we are to use local coordinates then we would like to be able to calculate $\kappa_{\mathbf{n}}(\mathbf{T}_p)$, knowing only $\kappa_{\mathbf{n}}(\mathbf{V}_p)$, $\kappa_{\mathbf{n}}(\mathbf{W}_p)$ and $a$ and $b$. However, $\kappa_{\mathbf{n}}$ is not a linear function. In fact:

**a.** *Show that:*

$$\kappa_{\mathbf{n}}(a\mathbf{V}_p + b\mathbf{W}_p) =$$

$$= a^2\, \kappa_{\mathbf{n}}(\mathbf{V}_p) + b^2\, \kappa_{\mathbf{n}}(\mathbf{W}_p) + ab\, \langle \mathbf{V}_p, -\mathbf{W}_p\mathbf{n}\rangle + ab\, \langle \mathbf{W}_p, -\mathbf{V}_p\mathbf{n}\rangle.$$

Thus, it is important to look at the quantities such as

$$\langle \mathbf{V}_p, -\mathbf{W}_p\mathbf{n}\rangle \text{ with } \mathbf{V}_p \neq \mathbf{W}_p.$$

So, if $\mathbf{X}_p, \mathbf{Y}_p \in T_pM$ for a smooth surface $M$ in $\mathbf{R}^3$, then we define the ***second fundamental form*** to be:

$$\mathrm{II}(\mathbf{X}_p, \mathbf{Y}_p) = \langle \mathbf{X}_p, -\mathbf{Y}_p\mathbf{n}\rangle,$$

where $\mathbf{n}(q)$ is a differentiable choice of unit normal to $M$ at all points $q$ near $p$. We are interested, in the end, only in the normal curvature in any direction

$$\mathbf{T}_p\colon \kappa_{\mathbf{n}}(\mathbf{T}_p) = \mathrm{II}(\mathbf{T}_p, \mathbf{T}_p).$$

The general Second Fundamental Form and its mixed terms $\langle \mathbf{X}_p, -\mathbf{Y}_p\mathbf{n}\rangle$ will be needed only when we want to express the normal curvature in terms of local coordinates. Now,

**b.** *Show that the second fundamental form is bilinear (linear in each variable).*

[Hint: Use Problem **4.8**.]

Let $M$ be a smooth surface in $\mathbf{R}^3$ with $C^2$ local coordinates $\mathbf{x}(u^1, u^2)$ with $\mathbf{x}(a,b) = p$.

**c.** *Show that*

$$\mathrm{II}(\mathbf{x}_1,\mathbf{x}_2) = \langle \mathbf{x}_{21},\mathbf{n}\rangle = \langle \mathbf{x}_{12},\mathbf{n}\rangle = \mathrm{II}(\mathbf{x}_2,\mathbf{x}_1)$$

*and that*

$$\mathrm{II}(\mathbf{X}_p,\mathbf{Y}_p) = \mathrm{II}(\mathbf{Y}_p,\mathbf{X}_p).$$

[Hint: Look at $\mathbf{x}_1\langle \mathbf{x}_2,\mathbf{n}\rangle = 0$, using the local coordinates $\mathbf{x}(u^1,u^2)$, and then write

$$\mathbf{X}_p = \sum X^i\mathbf{x}_i(a,b) \text{ and } \mathbf{Y}_p = \sum Y^i\mathbf{x}_i(a,b),$$

and use Problem **4.8**.]

Part **c** is the only place that the assumption $C^2$ is used in a crucial way in all its power (that is, $C^2$ requires that all first and second partial derivatives exist and are continuous *and* that the mixed partials are equal).

**d.** *Show that the maximum and minimum of*

$$\{\mathrm{II}(\mathbf{T},\mathbf{T}) \mid \mathbf{T} \text{ a unit tangent vector at } p\}$$

*occur when* $-\mathbf{T}\mathbf{n} = \lambda\mathbf{T}$. *Thus, in this case, the rate of change of the normal* **n** *with respect to* **T** *is in a direction parallel to* **T.**

[Hint: Note that $\mathrm{II}(-\mathbf{T},-\mathbf{T}) = \mathrm{II}(\mathbf{T},\mathbf{T})$. Part **d** may be solved in at least three ways:

1. using from **analysis** the theory of Lagrange multipliers to maximize/minimize $\mathrm{II}(\mathbf{X},\mathbf{X})$, subject to the constraint that $\langle \mathbf{X},\mathbf{X}\rangle = 1$.
2. expressing $\mathrm{II}(\mathbf{T},\mathbf{T})$ in terms of local coordinates and then using (from **linear algebra)** the theory of eigenvalues and eigenvectors of symmetric matrices or quadratic forms.
3. arguing **geometrically** that, if **T** is a maximum or minimum of $\mathrm{II}(\mathbf{T},\mathbf{T})$, then $\frac{d}{dh}\mathrm{II}(\mathbf{T}+h\mathbf{T}^{\perp}, \mathbf{T}+h\mathbf{T}^{\perp})_{h=0} = 0$ (where $\mathbf{T}^{\perp}$ is a unit vector in $T_pM$ perpendicular to **T**) and then differentiate.]

**e.** *Show that, if* $\mathbf{T}_1$ *and* $\mathbf{T}_2$ *are unit tangent vectors such that*

$$-\mathbf{T}_1\mathbf{n} = \lambda_1\mathbf{T}_1 \textit{ and } -\mathbf{T}_2\mathbf{n} = \lambda_2\mathbf{T}_2,$$

*then either*

$$\lambda_1 = \lambda_2 \textit{ or } \mathbf{T}_1 \textit{ is perpendicular to } \mathbf{T}_2.$$

[Hint: Consider $\mathrm{II}(\mathbf{T}_1, \mathbf{T}_2)$ and remember that $\mathrm{II}(\mathbf{T},\mathbf{T})$ is the normal curvature of the surface in the direction **T**.]

It follows that, if $\lambda_1 \neq \lambda_2$, then $\mathbf{T}_1$ and $\mathbf{T}_2$ must be the principal directions at p. If $\lambda_1 = \lambda_2$ then the normal curvature is the same in all directions and any two perpendicular unit vectors can be chosen as the principal directions. Note also that

$$\kappa_i = \mathrm{II}(\mathbf{T}_i,\mathbf{T}_i) = \langle \mathbf{T}_i,-\mathbf{T}_i\mathbf{n}\rangle = \langle \mathbf{T}_i,\lambda_i\mathbf{T}_i\rangle = \lambda_i\langle \mathbf{T}_i,\mathbf{T}_i\rangle = \lambda_i.$$

**f.** *What are the principal directions on a* $C^2$ *surface of revolution expressed in rectangular coordinates as*

$$(r(z)\cos\theta,\ r(z)\sin\theta,\ z)?$$

*Show that in these directions the principal curvatures (with respect to the inward pointing normal) are:*

$$\kappa_1 = \frac{-r''(z)}{[1+(r'(z))^2]^{\frac{3}{2}}} \quad \textit{and} \quad \kappa_2 = \frac{1}{r(z)\sqrt{1+(r'(z))^2}}.$$

[Hint: Argue geometrically (using **6.2.d**) which are the principal directions. Then calculate the normal curvatures of curves in those directions. Do NOT use the second fundamental form. Note that a surface of revolution is $C^2$ whenever $r(z)$ is positive and $C^2$. You may find helpful the discussion in the section *Curvature of the Graph of a Function*, preceding Problem **2.4**.]

To summarize the above discussion: The directions, $\mathbf{T}_1$, $\mathbf{T}_2$, in which the maximum and minimum of II($\mathbf{T}$,$\mathbf{T}$) occur, are called the ***principal directions at*** $p$ and the values of II($\mathbf{T}$,$\mathbf{T}$) in these directions, $\kappa_1$, $\kappa_2$, are called ***the principal curvatures at*** $p$. Note that, $\kappa_1$, $\kappa_2$, are (by Problem **6.1**) the normal curvatures of unit speed curves in the principal directions. The product $\kappa_1\kappa_2$ is called the ***Gaussian curvature at*** $p$. The above (because it involves the unit normal to the surface) is an extrinsic description of the Gaussian curvature; but below we will show that, in fact, the Gaussian curvature is intrinsic and that it coincides with the intrinsic curvature defined in Chapter 5. In Chapter 7 we can provide an intrinsic description of Gaussian curvature in terms of local coordinates.

Thus in the principal directions, $\mathbf{T}_1$, $\mathbf{T}_2$, at a point $p$, we can write $-\mathbf{T}_i\mathbf{n} = \kappa_i\mathbf{T}_i$. In these principal directions the rate of change of the normal to the surface is equal in magnitude to normal curvature. By **6.1**, for the curve $\gamma^*$, $\boldsymbol{\kappa}(\gamma^*) = \boldsymbol{\kappa}_{\mathbf{n}}(\gamma^*)$ (see Figure 6.1), and thus, by Problem **2.3** the normal to $\gamma^*$ is parallel to the normal to the surface (at $p$), and their rates of change along $\gamma^*$ are equal in both magnitude and direction at $p$. It is worthwhile spending as much time as needed to understand this situation because it will keep coming up and will be crucial later on.

## *PROBLEM 6.3. The Gauss Map*

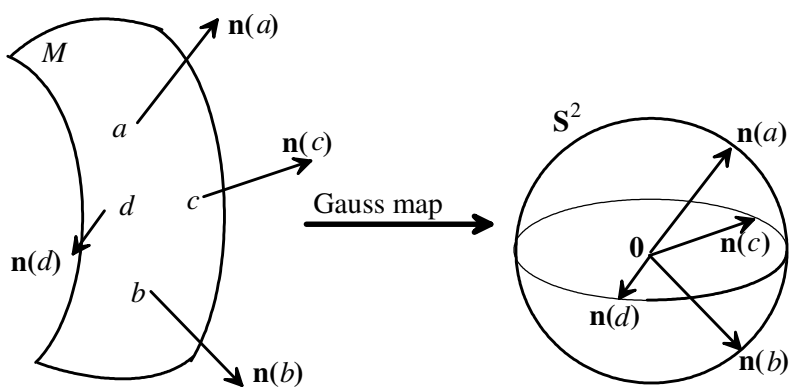


**Figure 6.2. Gauss map.**

We now want to connect what we know from Chapter 5 about the connections between curvature and holonomy on a sphere to similar connections on an arbitrary smooth surface $M$ in $\mathbf{R}^3$. The connection will be via the ***Gauss map*** (or ***normal spherical image***), which is a function from the surface to the unit sphere $\mathbf{S}^2$. The Gauss map is defined as follows: Start with a point $p$ on $M$ and choose a unit normal $\mathbf{n}(p)$ to the surface at that point. Then, if $\mathbf{n}(p)$ is considered as a vector bound at the origin, its head is on a point of the unit sphere. We define this point to be the image of $p$ under the Gauss map. (See Figure 6.2.) We usually write the Gauss map as $\mathbf{n}$: $M \to \mathbf{S}^2$. There are at every point two possible unit normals, and we assume that we can pick one of them at each point so that $\mathbf{n}$ is continuously defined. The surfaces for which choice is possible are called *orientable*. There are non-orientable surfaces (such as a Moebius Strip) but any sufficiently small region on a surface is always orientable.

You may find helpful Computer Exercise 6.3, which will allow you to display of images of the Gauss map. But it is more important for you to argue through specific examples, such as in Part **a**, below. You probably will find it helpful to use physical models of the surface and a sphere.

***a.** *What is the spherical image of cylinders and spheres of different radii? What is the spherical image of a normal circular cone and how does it depend on the cone angle? Describe the spherical image of a strake. Describe the spherical image of a torus and divide the torus into 8 regions each congruent to either A or B in Problem* ***5.5.c*** *and Figure 5.9. What do you notice as you shift from the inside saddle-shaped A-regions on the torus to the outside B-regions? As you move around the boundary curves in a counterclockwise direction how do the images on the unit sphere move?*

**b.** *If* $\mathbf{P}(s)$ *is a parallel vector field along the curve* $\gamma(s)$ *on the orientable surface M, then* $\mathbf{P}(s)$ *is also a parallel vector field along the curve* $\mathbf{n}(\gamma(s))$ *on the sphere.*

[Hint: First show that if a vector $\mathbf{V}$ is tangent to the surface at $\gamma(s)$, then $\mathbf{V}$ is also tangent to the sphere at $\mathbf{n}(\gamma(s))$. Thus $\mathbf{P}(s)$ is a vector field along $\mathbf{n}(\gamma(s))$. Now look at the rate of change of $\mathbf{P}(s)$ with respect to arclength at $\gamma(s)$ and then at $\mathbf{n}(\gamma(s))$. Remember that, if $s$ is arclength on $\gamma$ then it will, in general, not be arclength on $\mathbf{n}(\gamma)$; however, arclength on $\mathbf{n}(\gamma)$ is a function of $s$, specifically $\int_0^s |\frac{d}{ds}\mathbf{n}(\gamma(s))| ds$.]

**c.** *If* $\gamma$ *is a "small" piecewise smooth simple closed curve on the orientable M, then* $\mathcal{H}(\gamma) = \mathcal{H}(\mathbf{n}(\gamma))$. *You must decide what "small" means.*

[Hint: Use the definition of holonomy as an angle between a vector and its parallel transport. (See Chapter 5.) You need to specify "small" in order to avoid the $n2\pi$ ambiguity in the measure of the angle.]

**d.** *If* $\lambda$ *is a piecewise smooth curve on an orientable* $C^2$ *surface M such that the velocity vector* $\lambda'$ *is in a principal direction* $\mathbf{T}_1$ *with principal curvature* $\kappa_1$, *then the velocity vector of the curve* $s \to \mathbf{n}(\lambda(s))$ *is equal to* $-\kappa_1 \lambda'(s)$.

[Hint: Use Problem **6.2.d-e**.]

## *Problem 6.4. Gauss-Bonnet and Intrinsic Curvature*

**a.** *If V is a "small" region on a* $C^2$ *surface M with Gauss map* $\mathbf{n}: M \to \mathbf{S}^2$, *then*

$$\iint_V \kappa_1\kappa_2 \, dA \;=\; \text{Area}\,(\,\mathbf{n}(V)\,),$$

*where the area is a signed area with the same sign as the curvature* $\kappa_1\kappa_2$.

[Hint: At the point $p$ in $M$, choose any local $C^2$ coordinates $\mathbf{x}$ such that $\mathbf{x}_1$ and $\mathbf{x}_2$ are in the principal directions and use Problem **6.3.d**. You must decide what you need "small" to mean.]

**b.** *If V is a "small" region bounded by a piecewise smooth simple closed curve* $\gamma$ *on a* $C^2$ *surface M, then show that*

$$\mathcal{H}(V) = 2\pi - \int_\gamma \kappa_g \, ds - \Sigma\alpha_i = \iint_V \kappa_1\kappa_2 \, dA,$$

*where the double integral is the (surface) integral over V.*

[Hint: Use Problem **5.4.d**.]

**c.** *Now, show that Gaussian curvature* $\kappa_1(p)\kappa_2(p)$ *at a point p on any* $C^2$ *surface is equal to the intrinsic curvature defined as:*

$$K(p) = \lim_{n\to\infty} \mathcal{H}(V_n) \,/\, A(V_n),$$

*where* $\{V_n\}$ *is a sequence of small (geodesic) polygons that converge to p.*

[Hint: Use Problem **6.4.b**.]

This leads to the famous result of Gauss, which is contained in translation in [**DG**: Gauss]. "Theorema Egregium" means "Remarkable Theorem" in Latin, the original language of Gauss' paper.

**d.** (Gauss' Theorema Egregium) *If $M$ and are two* $C^2$ *surfaces in* $\mathbf{R}^3$*, and* $f : M \to N$ *is an isometry* (that is, $\langle \gamma', \beta' \rangle_{T_pM} = \langle (h \cdot \gamma)', (h \cdot \beta)' \rangle_{T_{fp)}N}$ for every point $p$ in $M$ and for every $\gamma', \beta'$ in $T_pM$ ), *then the Gaussian curvature of $M$ at $p$ equals the Gaussian curvature of $N$ at $f(p)$.*

## *Problem 6.5. Second Fundamental Form in Coordinates*

Let $M$ be a smooth surface in $\mathbf{R}^3$ with $C^2$ local coordinates $\mathbf{x}(u^1,u^2)$.

**a.** *Show that if*

$$\mathbf{X}_p = \Sigma\, X^i \mathbf{x}_i(a,b) \text{ and } \mathbf{Y}_p = \Sigma\, Y^i \mathbf{x}_i(a,b),$$

*then*

$$\mathrm{II}(\mathbf{X}_p, \mathbf{Y}_p) =$$

$$= (X^1 X^2) \begin{pmatrix} \mathrm{II}(\mathbf{x}_1(a,b), \mathbf{x}_1(a,b)) & \mathrm{II}(\mathbf{x}_1(a,b), \mathbf{x}_2(a,b)) \\ \mathrm{II}(\mathbf{x}_2(a,b), \mathbf{x}_1(a,b)) & \mathrm{II}(\mathbf{x}_2(a,b), \mathbf{x}_2(a,b)) \end{pmatrix} \begin{pmatrix} Y^1 \\ Y^2 \end{pmatrix} =$$

$$= (X^1 X^2) \begin{pmatrix} \langle \mathbf{x}_{11}(a,b), \mathbf{n}(a,b) \rangle & \langle \mathbf{x}_{12}(a,b), \mathbf{n}(a,b) \rangle \\ \langle \mathbf{x}_{21}(a,b), \mathbf{n}(a,b) \rangle & \langle \mathbf{x}_{22}(a,b), \mathbf{n}(a,b) \rangle \end{pmatrix} \begin{pmatrix} Y^1 \\ Y^2 \end{pmatrix}.$$

The above matrix is called the ***matrix of the second fundamental form*** (in local coordinates $\mathbf{x}(u^1,u^2)$). Some books call the matrix simply the second fundamental form.

**b.** *If we choose local coordinates such that at $p = \mathbf{x}(a,b)$ we have*

$$\mathbf{x}_1(a,b) = \mathbf{T}_1 \text{ and } \mathbf{x}_2(a,b) = \mathbf{T}_2,$$

*then show that the matrix of the second fundamental form is*

$$\begin{pmatrix} \kappa_1 & 0 \\ 0 & \kappa_2 \end{pmatrix},$$

*and if* $\mathbf{T}(\theta)$ *denotes the unit vector that is in a direction at an angle of* $\theta$ *away from the first principal direction* $\mathbf{T}_1$*, then show that the normal curvature is given by*

$$\kappa_{\mathbf{n}}(\mathbf{T}(\theta)) = \kappa_1 \cos^2\theta + \kappa_2 \sin^2\theta.$$

Note on a sphere that the normal curvature is the same in all directions, and thus, any orthogonal local coordinates on the sphere will have their Second Fundamental Form matrix be a diagonal matrix. This is also true for the standard local coordinates on the cylinder and cone. However, it is not true for the standard local coordinates on the strake.

***c.** *Suppose that* $\mathbf{x}$ *expresses $M$ as the graph of a function*:

$$\mathbf{x}(x,y) = (x,y,f(x,y).$$

*Show that, at $p = \mathbf{x}(a,b)$,*

$$\mathbf{x}_1(a,b) = (1,0,f_x(a,b)), \ \mathbf{x}_2(a,b) = (0,1,f_y(a,b))$$

*and*

$$\mathbf{n}(a,b) = \frac{\mathbf{x}_1(a,b)\times\mathbf{x}_2(a,b)}{|\mathbf{x}_1(a,b)\times\mathbf{x}_2(a,b)|} = \frac{(-f_x,-f_y,1)}{\sqrt{1+(f_x)^2+(f_y)^2}},$$

*where*

$$f_x = \tfrac{\partial}{\partial x}f(x,b)_{x=a} \;\; \textit{and} \;\; f_y = \tfrac{\partial}{\partial y}f(a,y)_{y=b} \quad .$$

*Find the matrix of the second fundamental form in these local coordinates at* $\mathbf{x}(0,0)$.

[Note that the tangent vectors $\mathbf{x}_1$ and $\mathbf{x}_2$ are *not* partial derivatives of $f$.]

## *PROBLEM 6.6. Mean Curvature and Minimal Surfaces

Using Problem **6.5.b**, we can calculate (using first year calculus) the ***mean curvature of*** $M$ ***at*** $p$:

$$H = \tfrac{1}{2\pi}\int_0^{2\pi}(\mathbf{T}(\theta))d\theta = \tfrac{1}{2\pi}\int_0^{2\pi}[\kappa_1\cos^2\theta+\kappa_2\sin^2\theta]\,d\theta = \tfrac{1}{2}(\kappa_1+\kappa_2).$$

[Note: Some texts *define* the "mean" curvature as $\kappa_1 + \kappa_2$, but this goes against the meaning of "mean" as "average".]

If the mean curvature is zero, then, either

$$\kappa_1 = 0 = \kappa_2 \text{ and } K = 0,$$

as in the case of the plane, or

$$\kappa_1 = -\kappa_2 \text{ and } K = -(\kappa_1)^2.$$

**a.** *Show geometrically (or by directly calculating) that the helixes that spiral up the strake and the horizontal lines on the strake all have zero normal curvature in the strake. Use this to show that the strake has zero mean curvature H.*

[Hint: The curves with zero normal curvature on the strake are not in the principal directions. Use Problem **6.5.b**.]

**b.** *Show that an element of area dA on the surface that is pushed in the direction of the normal will have its area change at the rate of* $-2H\,dA$.

[Hint: To get a feel for this, first show it directly for the sphere and cylinder by expressing their areas in terms of the radius $r$, and then (since the normal is in the direction of $r$) finding the rate of change of the areas by differentiating with respect to $r$. Use local orthonormal coordinates $(x, y)$ in the principal directions. In each of the principal directions, draw a picture of the osculating circle with radius of 1/(normal curvature). Then we have the picture in Figure 6.3. In this picture we see that the derivative $\frac{d}{dh}l_h = -\kappa\,dx$. Then set $dA=dxdy$ and let $A(h)$ be the area after $dA$ is pushed a distance $h$ in the direction of $\mathbf{n}$.

Find $\frac{d}{dh}A$.]

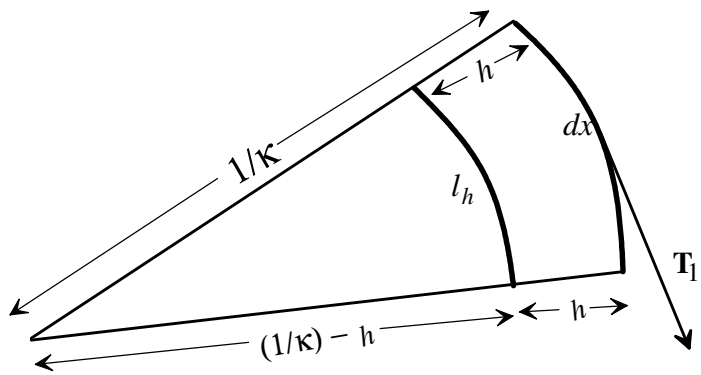


**Figure 6.3. Change of arclength in the direction of the curvature.**

Thus, a perturbation of a surface with zero mean curvature does not change its area. Traditionally, a surface $M$ with zero mean curvature is called a ***minimal surface***. A soap film with equal pressures on both sides is an example of a minimal surface.

**c.** *Show that the surface of revolution*

$$\mathbf{x}(\theta,z) = ((1/a)\cosh(az+b)\cos\theta,\ (1/a)\cosh(az+b)\sin\theta,\ z)$$

*is a minimal surface. This surface is called a* ***catenoid****.*

[Hint: Use Problem **6.2.f**. As a reminder: $\cosh(x) = \frac{1}{2}(e^x + e^{-x})$.]

**d.** *Show that the catenoids are the only surfaces of revolution,*

$$(r(z)\cos\theta,\ r(z)\sin\theta,\ z),$$

*which are minimal surfaces.*

[Hint: Find a (second order, nonlinear) differential equation that $r(z)$ must satisfy in order for a surface of revolution

$$(r(z)\cos\theta,\ r(z)\sin\theta,\ z)$$

to be a minimal surface. Then use the fact that this differential equation has a unique solution for given initial conditions.]

Note that the plane is a minimal surface and can also be considered as a surface of revolution, but it is not of the form

$$(r(z)\cos\theta,\ r(z)\sin\theta,\ z).$$

**e.** *Show that the catenoid, M, and the helicoid, N, are locally isometric. That is there is a map* $f: M \to N$ *such that,* $\langle \gamma', \beta' \rangle_{\mathrm{T}_p M} = \langle (h\cdot\gamma)', (h\cdot\beta)' \rangle_{\mathrm{T}_{fp)}N}$ *for every point p in M and for every* $\gamma', \beta'$ *in* $\mathrm{T}_p M$ *.*

[Hint: Express both the catenoid and the helicoid in geodesic rectangular coordinates. For the catenoid, set $b = 0$ and use the circle $z = 0$ (minus a point) as the base curve. For the helicoid, use the center line (the $z$-axis) as the base curve. Then use the result of Problem **4.9** to express the respective Riemannian metrics.]

Computer Exercise 6.6 will allow you to display and observe a transition from the helicoid to the catenoid.

For more discussion and further bibliography about minimal surfaces see [**Mi**: Osserman(1986)], [**Mi**: Osserman(1989)], and [**Mi**: Morgan].

## *Celebration of Our Hard Work*

You have just traversed some difficult territory that mathematical pioneers struggled with for about 80 years. The results in Problems **6.1**, **6.2** and **6.5** were mostly proved by Euler in 1760. Note that these define curvature (which of course was not yet called *Gaussian* curvature) extrinsically. There seemed to be no suspicion that the curvature could be intrinsic. Thus, when Gauss first discovered this fact, he called the result "egregium Theorema", "remarkable Theorem" in Latin, the original language of Gauss' paper.

The results in Problems **6.3** and **6.4** were developed by Gauss before 1827. In [**DG**: Gauss] Gauss derived his Theorema Egregium as a corollary of results similar to those in Problem **7.1**, using local coordinates. However, there is evidence in unpublished papers (which are also included in translation in [**DG**: Gauss]) that he originally arrived at this result in much the same way we do in Problem **6.4**.

The theory of minimal surfaces dates back at least to Euler in 1744, and the results in Problem **6.6** were mostly known through the work of Lagrange and Meusnier before 1785. However, research on minimal surfaces in $\mathbf{R}^3$ is still active, (see [**Mi**: Hoffman] for a description of the discovery of a new minimal surface in 1987).

We now have under our control powerful ideas that combine the intuitive geometric ideas with formal analytic ideas. In Chapter 7 we will use our new knowledge and power to derive a number of applications of Gaussian curvature and in the process find other intrinsic descriptions.