CHAPTER XIX.

ON SYSTEMS OF PERIODS AND ON GENERAL JACOBIAN FUNCTIONS.

343. THE present chapter contains a brief account of some general ideas which it is desirable to have in mind in dealing with theta functions in general and more especially in dealing with the theory of transformation.

Starting with the theta functions it is possible to build up functions of p variables which have $2p$ sets of simultaneous periods—as for instance by forming quotients of integral polynomials of theta functions (Chap. XI., § 207), or by taking the second differential coefficients of the logarithm of a single theta function (Chap. XI., § 216, Chap. XVII., § 311 (δ)). Thereby is suggested, as a matter for enquiry, along with other questions belonging to the general theory of functions of several independent variables, the question whether every such multiply-periodic function can be expressed by means of theta functions*. Leaving aside this general theory, we consider in this chapter, in the barest outline, (i) the theory of the periods of an analytical multiply-periodic function, (ii) the expression of the most general single valued analytical integral function of which the second logarithmic differential coefficients are periodic functions.

344. If an uniform analytical function of p independent complex variables $u_1, \ldots, u_p$ be such that, for every set of values of $u_1, \ldots, u_p$, it is unaltered by the addition, respectively to $u_1, \ldots, u_p$, of the constants $P_1, \ldots, P_p$, then $P_1, \ldots, P_p$ are said to constitute a period column for the function. Such a column will be denoted by a single letter, P, and P_k will denote any one of $P_1, \ldots, P_p$. It is clear that if each of $P, Q, R, \ldots$ be period columns for the function, and $\lambda, \mu, \nu, \ldots$ be any definite integers, independent of k, then the column of quantities $\lambda P_k + \mu Q_k + \nu R_k + \ldots$ is also a period column for the function; we shall denote this column by $\lambda P + \mu Q + \nu R + \ldots$, and say that it is a linear function of the columns $P, Q, R, \ldots$, the coefficients $\lambda, \mu, \nu, \ldots$, in this case, but not necessarily

* Cf. Weierstrass, *Crelle*, LXXXIX. (1880), p. 8.

always, being integers. The real parts of the new column are the same linear functions of the real parts of the component columns, as also are the imaginary parts. More generally, when the p quantities $\lambda P_k + \mu Q_k + \nu R_k + \ldots$ are zero for the same values of $\lambda, \mu, \nu, \ldots$, we say that the columns $P, Q, R, \ldots$ are connected by a linear equation; it must be noticed, for the sake of definiteness, that it does not thence follow that, for instance, P is a linear function of the other columns, unless it is known that λ is not zero.

It is clear moreover that any $2p + 1$, or more, columns of periods are connected by at least one linear equation with real coefficients (that is, an equation for each of the p positions in the column—p equations in all, with the same coefficients); for, in order to such an equation, the separation of real and imaginary gives $2p$ linear equations to be satisfied by the $2p + 1$ real coefficients; allowing possible zero values for coefficients these equations can always be satisfied.

For instance the periods $\Omega = \Omega_1 + i\Omega_2$, $\omega = \omega_1 + i\omega_2$, $\omega' = \omega_1' + i\omega_2'$, are connected by an equation

$$t\Omega + x\omega + y\omega' = 0,$$

in which however, if $\omega_1\omega_2' - \omega_2\omega_1' = 0$, also $t = 0$.

Thus, for any periodic function, there exists a least number, r, of period columns, with r lying between 1 and $2p + 1$, which are themselves not connected by any linear equation with real coefficients, but are such that every other period column is a linear function of these columns with real finite coefficients. Denoting such a set* of r period columns by $P^{(1)}, \ldots, P^{(r)}$, and denoting any other period column by Q, we have therefore the p equations

$$Q_k^{(r)} = \lambda_1 P_k^{(1)} + \ldots\ldots + \lambda_r P_k^{(r)}, \qquad (k = 1, 2, \ldots, p),$$

wherein $\lambda_1, \ldots, \lambda_r$ are independent of k, and are real and not infinite. *It is the purpose of what† follows to shew, in the case of an uniform analytical function of the independent complex variables $u_1, \ldots, u_p$,* (I.) *that unless the function can be expressed in terms of less than p variables which are linear functions of the arguments $u_1, \ldots, u_p$, the coefficients $\lambda_1, \ldots, \lambda_r$ are rational numbers,* (II.) *that, $\lambda_1, \ldots, \lambda_r$ being rational numbers, sets of r columns of periods exist in terms of which every existing period column can be linearly expressed with integral coefficients.*

Two lemmas are employed which may be enunciated thus:—

(α) If an uniform analytical function of the variables $u_1, \ldots, u_p$ have a column of infinitesimal periods, it is expressible as a function of less than p variables which are linear functions of $u_1, \ldots, u_p$. And conversely, if such

* It will appear that the number of such sets is infinite; it is the number r which is unique.

† These propositions are given by Weierstrass. *Abhandlungen aus der Functionenlehre* (Berlin, 1886), p. 165 (or *Berlin. Monatsber.* 1876).

uniform analytical function of $u_1, \ldots, u_p$ be expressible as a function of less than p variables which are linear functions of $u_1, \ldots, u_p$, it has columns of infinitesimal periods.

(β) Of periods of an uniform analytical function of the variables $u_1, \ldots, u_p$, which does not possess any columns of infinitesimal periods, there is only a finite number of columns of which every period is finite.

345. To prove the first part of lemma (α) it is sufficient to prove that when the function $f(u_1, \ldots, u_p)$ is *not* expressible as a function of less than p linear functions of $u_1, \ldots, u_p$, then it has *not* any columns of infinitesimal periods.

We define as an *ordinary* set of values of the variables $u_1, \ldots, u_p$ a set $u_1', \ldots, u_p'$, such that, for absolute values of the differences $u_1 - u_1', \ldots, u_p - u_p'$ which are within sufficient (not vanishing) nearness to zero, the function, $f(u_1, \ldots, u_p)$, can be represented by a converging series of positive integral powers of these differences—the possibility of such representation being the distinguishing mark of an analytical function; other sets of values of the variables are distinguished as *singular* sets of values*.

Then if the function be not expressible by less than p linear functions of $u_1, \ldots, u_p$, there can exist no set of constants $c_1, \ldots, c_p$ such that the function

$$c_1 \frac{\partial f}{\partial u_1} + \ldots + c_p \frac{\partial f}{\partial u_p}$$

vanishes for all ordinary sets of values of the variables; for this would require f to be a function of the $p-1$ variables $c_i u_1 - c_1 u_i \; (i = 2, \ldots, p)$. Hence there exist sets of ordinary values such that not all the differential coefficients $\partial f/\partial u_1, \ldots, \partial f/\partial u_p$ are zero; let $u_1^{(1)}, \ldots, u_p^{(1)}$ be such an ordinary set of values; for all values of $u_1, \ldots, u_p$ in the immediate neighbourhoods respectively of $u_1^{(1)}, \ldots, u_p^{(1)}$, the statement remains true that not all the partial differential coefficients are zero.

Then, similarly, the determinants of two rows and columns formed from the array

$$\left| \begin{array}{cccc} \dfrac{\partial f}{\partial u_1^{(1)}}, & \dfrac{\partial f}{\partial u_2^{(1)}}, & \ldots, & \dfrac{\partial f}{\partial u_p^{(1)}} \\ \dfrac{\partial f}{\partial u_1}, & \dfrac{\partial f}{\partial u_2}, & \ldots, & \dfrac{\partial f}{\partial u_p} \end{array} \right|$$

do not all vanish for every ordinary set of values of the variables; let $u_1^{(2)}, \ldots, u_p^{(2)}$ be an ordinary set for which they do not vanish; for all values of

* The ordinary sets of values constitute a continuum of $2p$ dimensions, which is necessarily limited; the limiting sets of values are the singular sets. Cf. Weierstrass, *Crelle*, LXXXIX. (1880), p. 3.

$u_1, \ldots, u_p$ in the immediate neighbourhoods respectively of $u_1^{(2)}, \ldots, u_p^{(2)}$, the statement remains true that not all these determinants are zero.

Proceeding step by step in the way thus indicated we infer that there exist sets of ordinary values of the variables, $(u_1^{(1)}, \ldots, u_p^{(1)}), \ldots, (u_1^{(p)}, \ldots, u_p^{(p)})$, such that the determinant, Δ, of p rows and columns in which the k-th element of the r-th row is $\partial f(u_1^{(r)}, \ldots, u_p^{(r)})/\partial u_k^{(r)}$, does not vanish; and since these are ordinary sets of values of the arguments, this determinant will remain different from zero if (for $r=1, \ldots, p$) the set $u_1^{(r)}, \ldots, u_p^{(r)}$ be replaced by $v_1^{(r)}, \ldots, v_p^{(r)}$, where $v_k^{(r)}$ is a value in the immediate neighbourhood of $u_k^{(r)}$.

This fact is however inconsistent with the existence of a column of infinitesimal periods. For if $H_1, \ldots, H_p$ be such a column, of which the constituents are not all zero, we have

$$0 = f(u_1^{(r)} + H_1, \ldots, u_p^{(r)} + H_p) - f(u_1^{(r)}, \ldots, u_p^{(r)}), \qquad (r = 1, \ldots, p),$$

$$= \sum_{k=1}^{p} H_k \frac{\partial f}{\partial u_k} [u_1^{(r)} + \theta_1 H_1, \ldots, u_p^{(r)} + \theta_p H_p],$$

where $\theta_1, \ldots, \theta_p$ are quantities whose absolute values are $\not> 1$, and the bracket indicates that, after forming $\partial f/\partial u_k$, we are (for $m = 1, \ldots, p$) to substitute $u_m^{(r)} + \theta_m H_m$ for $u_m^{(r)}$; these p equations, by elimination of $H_1, \ldots, H_p$ give zero as the value of a determinant which is obtainable from Δ by slight changes of the sets $u_1^{(r)}, \ldots, u_p^{(r)}$; we have seen above that such a determinant is not zero.

To prove the converse part of lemma (α) we may proceed as follows. Suppose that the function is expressible by m arguments $v_1, \ldots, v_m$ given by

$$v_k = a_{k,1} u_1 + \ldots + a_{k,p} u_p, \qquad (k = 1, \ldots, m),$$

wherein $m < p$. The conditions that $v_1, \ldots, v_m$ remain unaltered when $u_1, \ldots, u_p$ are replaced respectively by $u_1 + tQ_1, \ldots, u_p + tQ_p$ are satisfied by taking $Q_1, \ldots, Q_p$ so that

$$a_{k,1} Q_1 + \ldots\ldots + a_{k,p} Q_p = 0, \qquad (k = 1, \ldots, m),$$

and since $m < p$ these conditions can be satisfied by finite values of $Q_1, \ldots, Q_p$ which are not all zero. The additions of the quantities $tQ_1, \ldots, tQ_p$ to $u_1, \ldots, u_p$, not altering $v_1, \ldots, v_m$, will not alter the value of the function f. Hence by supposing t taken infinitesimally small, the function has a column of infinitesimal periods.

346. As to lemma (β), let $P_k = \rho_k + i\sigma_k$ be one period of any column of periods, $(k = 1, \ldots, p)$, wherein ρ_k, σ_k are real, so that, in accordance with the condition that the function has no column of infinitesimal periods, there

is an assignable real positive quantity ϵ such that not all the $2p$ quantities ρ_k, σ_k are less than ϵ. Then if μ_k, ν_k be $2p$ specified positive integers, there is at most one column of periods satisfying the conditions

$$\mu_k\epsilon \ngtr |\rho_k| < (\mu_k+1)\epsilon, \qquad \nu_k\epsilon \ngtr |\sigma_k| < (\nu_k+1)\epsilon, \qquad (k=1, \dots, p);$$

wherein $|\rho_k|$, $|\sigma_k|$ are the numerical values of ρ_k, σ_k; for if $\rho_k' + i\sigma_k'$ were one period of another column also satisfying these conditions, the quantities $\rho_k' - \rho_k + i(\sigma_k' - \sigma_k)$ would form a period column wherein every one of the $2p$ quantities $\rho_k' - \rho_k$, $\sigma_k' - \sigma_k$ was numerically less than ϵ.

Hence, since, if g be any assigned real positive quantity, there is only a finite number of sets of $2p$ positive integers μ_k, ν_k such that each of the $2p$ quantities $\mu_k\epsilon$, $\nu_k\epsilon$ is within the limits $(-g, g)$, it follows that there is only a finite number of columns of periods $P_k = \rho_k + i\sigma_k$, such that each of ρ_k, σ_k is numerically less than g. And this is the meaning of the lemma.

347. We return now to the expression (§ 344) of the most general period column of the function f by real linear functions of r period columns, of finite periods, in the form

$$Q = \lambda_1 P^{(1)} + \dots\dots + \lambda_r P^{(r)};$$

here the suffix is omitted, and we make the hypothesis that the function is not expressible by fewer than p linear combinations of $u_1, \dots, u_p$.

Consider, first, the period columns Q from which $\lambda_2 = \lambda_3 = \dots = \lambda_r = 0$ and $0 < \lambda_1 \ngtr 1$. Since there are no columns of infinitesimal periods, there is a lower limit to the values of λ_1 corresponding to existing period columns Q satisfying these conditions; and since there is only a finite number of period columns of wholly finite periods, there is an existing period for which λ_1 is equal to this lower limit. Let $\lambda_{1,1}$ be this least value of λ_1, and $Q^{(1)}$ be the corresponding period column Q.

Consider, next, the period columns Q for which $\lambda_3 = \lambda_4 = \dots = \lambda_r = 0$, and $0 \ngtr \lambda_1 \ngtr 1$, $0 < \lambda_2 \ngtr 1$. As before there are period columns of this character in which λ_2 has a least value, which we denote by $\lambda_{2,2}$. If there exist several corresponding values of λ_1, let $\lambda_{1,2}$ denote one of these, and denote $\lambda_{1,2}P^{(1)} + \lambda_{2,2}P^{(2)}$ by $Q^{(2)}$.

In general consider the period columns of the form

$$\lambda_1 P^{(1)} + \dots\dots + \lambda_m P^{(m)}, \qquad (m \ngtr r),$$

wherein

$$0 \ngtr \lambda_1 \ngtr 1, \dots\dots, 0 \ngtr \lambda_{m-1} \ngtr 1, \; 0 < \lambda_m \ngtr 1.$$

Since there are no infinitesimal periods, there is a lower limit to the values of λ_m corresponding to existing period columns satisfying these conditions; since there is only a finite number of period columns of wholly finite periods, there is at least one existing column Q for which λ_m is equal to this lower limit; denote this value of λ_m by $\lambda_{m,m}$, and denote by $\lambda_{1,m}, \ldots, \lambda_{m-1,m}$ values arising in an actual period column $Q^{(m)}$ given by

$$Q^{(m)} = \lambda_{1,m} P^{(1)} + \lambda_{2,m} P^{(2)} + \ldots + \lambda_{m,m} P^{(m)};$$

there may exist more than one period column in which the coefficient of $P^{(m)}$ is $\lambda_{m,m}$.

Thus, taking $m = 1, 2, \ldots, r$, we obtain r period columns $Q^{(1)}, \ldots, Q^{(r)}$. In terms of these any period column $Q, = \lambda_1 P^{(1)} + \ldots + \lambda_r P^{(r)}$, in which $\lambda_1, \ldots, \lambda_r$ are real, can be uniquely written in the form

$$N_1 Q^{(1)} + \ldots + N_r Q^{(r)} + \mu_1 P^{(1)} + \ldots + \mu_r P^{(r)},$$

wherein $N_1, \ldots, N_r$ are integers, and $\mu_1, \ldots, \mu_r$ are real quantities which are zero or positive and respectively less than $\lambda_{1,1}, \ldots, \lambda_{r,r}$. For, putting λ_r into the form $N_r \lambda_{r,r} + \mu_r$, where N_r is an integer and μ_r, if not zero, is positive and less than $\lambda_{r,r}$, we have

$$\begin{aligned} Q &= \lambda_1 P^{(1)} + \ldots + \lambda_r P^{(r)} \\ &= \lambda_1' P^{(1)} + \ldots + \lambda'_{r-1} P^{(r-1)} + N_r Q^{(r)} + \mu_r P^{(r)}, \end{aligned}$$

where

$$\lambda_1' = \lambda_1 - N_r \lambda_{1,r}, \ \ldots, \ \lambda'_{r-1} = \lambda_{r-1} - N_r \lambda_{r-1,r};$$

and herein the column $Q' = \lambda_1' P^{(1)} + \ldots + \lambda'_{r-1} P^{(r-1)}$ can quite similarly be expressed in the form

$$Q' = \lambda_1'' P^{(1)} + \ldots + \lambda''_{r-2} P^{(r-2)} + N_{r-1} Q^{(r-1)} + \mu_{r-1} P^{(r-1)},$$

and so on.

But now, if $N_1 Q^{(1)} + \ldots + N_r Q^{(r)} + \mu_1 P^{(1)} + \ldots + \mu_r P^{(r)}$ be a period column, it follows, as $N_1, \ldots, N_r$ are integers, that also $\mu_1 P^{(1)} + \ldots + \mu_r P^{(r)}$ is a period column; and this in fact is only possible when each of $\mu_1, \ldots, \mu_r$ is zero. For, by our definition of $Q^{(r)}$, the coefficient μ_r is zero; then, by the definition of $Q^{(r-1)}$, the coefficient μ_{r-1} is zero; and so on.

On the whole we have the proposition (II., § 344)—*if*

$$Q^{(m)} = \lambda_{1,m} P^{(1)} + \ldots + \lambda_{m,m} P^{(m)}, \qquad (m = 1, \ldots, r),$$

be that real linear combination of the first m columns from $P^{(1)}, \ldots, P^{(r)}$ *in which the m-th coefficient* $\lambda_{m,m}$ *has the least existing value greater than zero and not greater than unity, or be one such combination for which* $\lambda_{m,m}$ *satisfies the same condition, then every period column is expressible as a linear combination of the columns* $Q^{(1)}, \ldots, Q^{(r)}$ *with integral coefficients.*

It should be noticed that $Q^{(1)}, \ldots, Q^{(r)}$ are not connected by any linear equation with real coefficients, or the same would be true of $P^{(1)}, \ldots, P^{(r)}$. And it should be borne in mind that the expression of any period column by means of *integral* coefficients, in terms of $Q^{(1)}, \ldots, Q^{(r)}$, is a consequence of the fact that the function $f(u_1, \ldots, u_p)$ has only a limited number of period columns which consist wholly of finite periods. Conversely the period columns, of finite periods, obtainable with such integral coefficients, are limited in number.

Another result (I., § 344) is thence obvious—*The coefficients in the linear expression of any period column in terms of* $P^{(1)}, \ldots, P^{(r)}$ *are rational numbers.*

For by the demonstration of the last result it follows that the period columns $P^{(1)}, \ldots, P^{(r)}$ can be expressed with integral coefficients in terms of $Q^{(1)}, \ldots, Q^{(r)}$ in the form

$$P^{(m)} = N_1^{(m)} Q^{(1)} + \ldots + N_r^{(m)} Q^{(r)}, \qquad (m = 1, \ldots, r);$$

from these equations, since the columns $P^{(1)}, \ldots, P^{(r)}$ are not connected by any linear relation with real coefficients, the columns $Q^{(1)}, \ldots, Q^{(r)}$ can be expressed as linear combinations of $P^{(1)}, \ldots, P^{(r)}$ with only rational numbers as coefficients; hence any linear combinations of $Q^{(1)}, \ldots, Q^{(r)}$ with integral coefficients is a linear combination of $P^{(1)}, \ldots, P^{(r)}$ with rational-number coefficients.

It needs scarcely* to be remarked that the set of period columns $Q^{(1)}, \ldots, Q^{(r)}$, in terms of which any other column can be expressed with integral coefficients, is not the only set having this property.

348. We consider briefly the application of the foregoing theory to the case of uniform analytical functions of a single variable which do not possess any infinitesimal periods. It will be sufficient to take the case when the function has two periods which have not a real ratio; this is equivalent to excluding singly periodic functions.

If $2\omega_1$, $2\omega_2$ be two periods of the function, whose ratio is not real, and 2Ω be any other period, it is possible to find two real quantities λ_1, λ_2 such that

$$\Omega = \lambda_1\omega_1 + \lambda_2\omega_2;$$

then of periods of the form $2\lambda_1\omega_1$, in which $0 < \lambda_1 \not> 1$, of which form periods do exist, $2\omega_1$ itself being one, there is one in which λ_1 has a least value, other than zero—as follows because the function has only a finite number of finite periods. Denote this least value by μ_1, and put $\Omega_1 = \mu_1\omega_1$. Of periods of the form $2\lambda_1\omega_1 + 2\lambda_2\omega_2$ in which $0 \not> \lambda_1 \not> 1$, $0 < \lambda_2 \not> 1$, there is a finite number, and therefore one, in which λ_2 has the least value arising, say μ_2; let one of the corresponding values of λ_1 be λ; put $\Omega_2 = \lambda\omega_1 + \mu_2\omega_2$. Then any period $2\Omega = 2\lambda_1\omega_1 + 2\lambda_2\omega_2$ is of the form $2N_1\Omega_1 + 2N_2\Omega_2 + 2\nu_1\omega_1 + 2\nu_2\omega_2$, where ν_1, ν_2 are (zero or) positive and respectively less than μ_1 and μ_2, and N_1, N_2 are integers, such that $\lambda_2 = N_2\mu_2 + \nu_2$, $\lambda_1 - N_2\lambda = N_1\mu_1 + \nu_1$. But the existence of a period $\Omega - 2N_1\Omega_1 - 2N_2\Omega_2 = 2\nu_1\omega_1 + 2\nu_2\omega_2$ with $\nu_1 < \mu_1$, $\nu_2 < \mu_2$ is contrary to the definition of μ_1 and μ_2, unless ν_1 and ν_2 be both zero. Hence every period is expressible in the form

$$\Omega = 2N_1\Omega_1 + 2N_2\Omega_2,$$

where N_1, N_2 are integers.

In other words, *a uniform analytical function of a single variable without infinitesimal periods cannot be more than doubly periodic*†.

* For the argument compare Weierstrass (l. c., § 344), Jacobi, *Ges. Werke*, t. ii., p. 27, Hermite, *Crelle*, XL. (1850), p. 310, Riemann, *Crelle*, LXXI. (1859) or *Werke* (1876), p. 276. See also Kronecker, "Die Periodensysteme von Functionen reeller Variabeln," *Sitzungsber. der Berl. Akad.*, 1884, (Jun. bis Dec.), p. 1071.

† Cf. Forsyth, *Theory of Functions* (1893), §§ 108, 107. It follows from these Articles, in this order, that any three periods of a uniform function of one variable can be expressed, with

It follows also that every period is expressible by $2\omega_1$, $2\omega_2$ with only rational-number coefficients.

349. *Ex.* i. If r quantities be connected by k homogeneous linear equations with integral coefficients ($r>k$), it is possible to find $r-k$ other quantities, themselves expressible as linear functions of the r quantities with integral coefficients, in terms of which the r quantities can be linearly expressed with integral coefficients.

Ex. ii. If $P^{(1)}, \ldots, P^{(r)}$ be r columns of real quantities, each containing $r-1$ constituents, a column $N_1P^{(1)}+\ldots+N_rP^{(r)}$ can be formed, in which $N_1, \ldots, N_r$ are integers, whose $r-1$ constituents are within assigned nearness of any $r-1$ assigned real quantities (cf. Chap. IX., § 166, and Clebsch u. Gordan, Abels. Funct., p. 135).

Ex. iii. An uniform analytical function of p variables, having r period columns $P^{(1)}, \ldots, P^{(r)}$, each of p constituents, and having a further period column expressible in the form $\lambda_1P^{(1)}+\ldots+\lambda_rP^{(r)}$, wherein $\lambda_1, \ldots, \lambda_r$ are real, will necessarily have a column of infinitesimal periods if even one of the coefficients $\lambda_1, \ldots, \lambda_r$ be irrational.

From this result, taken with Ex. i., another demonstration of the proposition of the text (§ 347) can be obtained. The result is itself a corollary from the reasoning of the text.

Ex. iv. If $u_1^{x,a}, \ldots, u_p^{x,a}$ be linearly independent integrals of the first kind, on a Riemann surface, and the periods, $2\omega_{r,s}$, $2\omega'_{r,s}$, of the integral $u_r^{x,a}$ be written $\rho_{r,s}+i\sigma_{r,s}$, $\rho'_{r,s}+i\sigma'_{r,s}$, shew that the vanishing of the determinant of $2p$ rows and columns which is denoted by

$$\left|\begin{matrix} \rho_{r,1}, \ldots, \rho_{r,p}, & \rho'_{r,1}, \ldots, \rho'_{r,p} \\ \sigma_{r,1}, \ldots, \sigma_{r,p}, & \sigma'_{r,1}, \ldots, \sigma'_{r,p} \end{matrix}\right|,$$

would involve* the equation

$$(M_1-iN_1)\,u_1^{x,a}+\ldots+(M_p-iN_p)\,u_p^{x,a}=\text{constant},$$

where M_1, N_1, ..., M_p, N_p are the minors of the elements of the first column of this determinant and are supposed not all zero.

The vanishing of this determinant is the condition that the period columns of the integrals should be connected by a homogeneous linear relation with real coefficients.

350. The argument of the text has important bearings on the theory of the Inversion Problem discussed in Chap. IX. The functions by which the solution of that problem is expressed have $2p$ columns of periods in terms of which all other period columns can be expressed linearly with integral coefficients; these $2p$ columns are not connected by any linear equation with integral coefficients (§ 165), and, therefore, are not connected by any linear equation with real coefficients.

It has been remarked (§ 174, Chap. X.) that the Riemann theta functions whereby the $2p$-fold periodic functions expressing the solution of the Inversion Problem can be built up, are not the most general theta functions possible. The same is therefore presumably true of the $2p$-fold periodic functions themselves. Weierstrass has stated a theorem†

integral coefficients, in terms of two periods. These two periods, and any fourth period of the function, can, in their turn, be expressed integrally by two other periods—and so on. The reasoning of the text shews that when the function has no infinitesimal periods, the successive processes are finite in number, and every period can be expressed, with integral coefficients, in terms of two periods.

* Forsyth, *Theory of Functions* (1893), p. 440, Cor. ii.

† *Berlin, Monatsber.* Dec. 2, 1869, *Crelle*, LXXXIX. (1880). For an application to integrals of radical functions, Cf. Wirtinger, *Untersuchungen über Thetafunctionen* (Leipzig, 1895), p. 77.

whereby it appears that the most general $2p$-fold periodic functions that are possible can be supposed to arise in the solution of a generalised Inversion Problem; this Inversion Problem differs from that of Jacobi in that the solution involves multiform periodic functions*; the theorems of the text possess therefore an interest, so far as they hold, in the case of such multiform functions. The reader is referred to Weierstrass, *Abhandlungen aus der Functionenlehre* (Berlin, 1886), p. 177, and to Casorati, *Acta Mathematica*, t. viii. (1886).

351. We pass now to a brief account of a different theory which is necessary to make clear the position occupied by the theory of theta functions. Considering, *à priori*, uniform integral analytical functions which, like the theta functions, are such that their partial logarithmic differential coefficients of the second order are periodic functions, we investigate certain relations which must necessarily hold among the periods, and we prove that all such functions can be expressed by means of theta functions.

Suppose that to the p variables $u_1, \dots, u_p$ there correspond σ columns of quantities $a_i^{(j)}$ $(i = 1, \dots, p, j = 1, \dots, \sigma)$ and σ columns of quantities $b_i^{(j)}$—according to the scheme

$$\begin{array}{c|c|c|}
u_1 & a_1^{(1)}, \dots, a_1^{(\sigma)} & b_1^{(1)}, \dots, b_1^{(\sigma)} \\
u_2 & a_2^{(1)}, \dots, a_2^{(\sigma)} & b_2^{(1)}, \dots, b_2^{(\sigma)} \\
\cdot & \cdot \quad \cdot\cdot \quad \cdot & \cdot \quad \cdot\cdot \quad \cdot \\
\cdot & \cdot \quad \cdot\cdot \quad \cdot & \cdot \quad \cdot\cdot \quad \cdot \\
u_p & a_p^{(1)}, \dots, a_p^{(\sigma)} & b_p^{(1)}, \dots, b_p^{(\sigma)}
\end{array};$$

and suppose $\phi(u)$ to be an uniform, analytical function of $u_1, \dots, u_p$ which for finite values of $u_1, \dots, u_p$ is finite and continuous—which further has the property expressed by the equations

$$\phi(u + a^{(j)}) = e^{2\pi i b^{(j)}[u + \frac{1}{2}a^{(j)}] + 2\pi i c^{(j)}} \phi(u), \qquad (j = 1, \dots, \sigma), \qquad (\text{I.})$$

wherein $b^{(j)}$ is a symbol for a column $b_1^{(j)}, \dots, b_p^{(j)}$ and $c^{(j)}$ is a single quantity depending only on j. The aggregate of $c^{(1)}, \dots, c^{(\sigma)}$ will be called the characteristic or the parameter of $\phi(u)$; $a_i^{(j)}$ will finally be denoted by $a_{i,j}$. We suppose that the columns $a^{(j)}$ are *independent*, in the sense that there exists no linear equation connecting them of which the coefficients are rational numbers; but it is not assumed that the columns $a^{(j)}$ constitute all the independent columns for which the function ϕ satisfies an equation of the form (I.). Also we suppose that the equation (I.) is not satisfied for any column of wholly infinitesimal quantities put in place of $a^{(j)}$. The reason for this last supposition is that in such case it is possible to express ϕ as the product of an exponential of a quadric function of $u_1, \dots, u_p$, multiplied into a function of less than p variables, these fewer variables being linear functions of $u_1, \dots, u_p$. The function $\phi(u)$ in the most general

* With a finite number of values.

case is a generalisation of a theta function; it will be distinguished by the name of a *Jacobian function*; but, for example, it may be a theta function, for which, when $\sigma < 2p$, the columns $a^{(j)}$ are σ of the $2p$ columns of quasi-periods, $2\omega^{(j)}$.

A consequence of the two suppositions is that in the matrix of σ columns and $2p$ rows, of which the $(2i-1)$th and $2i$-th rows are formed respectively by the real and imaginary parts of the row $a_i^{(1)}, \ldots, a_i^{(\sigma)}$, not every determinant of σ rows and columns can vanish. For if with σ arbitrary real variables $x_1, \ldots, x_\sigma$ we form $2p$ linear functions, the $(2i-1)$th and $2i$-th of these having for coefficients the $(2i-1)$th and $2i$-th rows of the matrix of σ columns and $2p$ rows just described, the condition that every determinant from this matrix with σ rows and columns should vanish, is that all these $2p$ linear functions should be expressible as linear functions of at most $\sigma - 1$ of them. Now it is possible to choose *rational integer* values of $x_1, \ldots, x_\sigma$ to make all of these $\sigma - 1$ linear functions infinitesimally small*; they cannot be made simultaneously zero since the σ columns of periods are independent. Therefore every one of the $2p$ linear functions would be infinitesimally small for the same integer values of $x_1, \ldots, x_\sigma$. Thus there would exist a column of infinitesimal quantities expressible in the form $x_1 a^{(1)} + \ldots + x_\sigma a^{(\sigma)}$. Now it will be shewn to be a consequence of the coexistence of equations (I.) that also an equation of the form (I.) exists when $a^{(j)}$ is replaced by an expression $x_1 a^{(1)} + \ldots + x_\sigma a^{(\sigma)}$, wherein $x_1, \ldots, x_\sigma$ are integers. This however is contrary to our second supposition above.

Hence also the matrix of σ columns and $2p$ rows, wherein the $(2i-1)$th and $2i$-th rows consist of $a_i^{(1)}, \ldots, a_i^{(\sigma)}$ and the quantities which are the conjugate complexes of these respectively, is such that not every determinant of σ rows and columns formed therefrom is zero.

And also, by the slightest modification of the argument, σ cannot be $> 2p$. The case when σ is equal to $2p$ is of especial importance; in fact the case $\sigma < 2p$ can be reduced to this† case.

352. Consider now the equations (I.). We proceed to shew that in order that they should be consistent with the condition that $\phi(u)$ is an uniform function, it is necessary, if a, b denote the matrices of p rows and σ columns which occur in the scheme of § 351, that the matrix of σ rows and columns‡, expressed by

$$\bar{a}b - \bar{b}a, \qquad \text{(A)},$$

should be a skew symmetrical one of which each element is a rational

* Chap. IX., § 166.

† When $\sigma = 2p$, the hypothesis of no infinitesimal periods is a consequence of the other conditions (cf. § 345).

‡ The notation already used for square matrices can be extended to rectangular matrices. See, for example, Appendix II., at the end of this volume (§ 406).

integer. Denote it by k, so that $k_{\alpha\alpha} = 0$, $k_{\alpha\beta} = -k_{\beta\alpha}$. But further also we shew that it is necessary, if x denote a column of σ quantities and x_1 denote the column whose elements are the conjugate complexes of those of x, that for all values, other than zero, satisfying the p equations

$$ax = 0, \qquad \text{(B)},$$

the expression $ikxx_1$ should be positive. We shew that $ikxx_1$ cannot be zero unless, beside ax, also ax_1 be zero: a condition only fulfilled by putting each of the elements of $x = 0$ (as follows because the σ columns of periods are independent and there are no infinitesimal periods). The condition (B) is in general inoperative when $\sigma < p + 1$.

353. Before giving the proof it may be well to illustrate these results by shewing that they hold for the particular case of the theta functions for which (cf. § 284, Chap. XV.)

$$\sigma = 2p, \quad a = |\,2\omega,\ 2\omega'\,|, \quad 2\pi i b = |\,2\eta,\ 2\eta'\,|,$$

and therefore

$$ax = 2\omega X + 2\omega' X' = \Omega_X, \quad bx = \frac{1}{2\pi i} H_X,$$

where X is a column of p quantities, X' a column of p quantities, and $x = \begin{vmatrix} X \\ X' \end{vmatrix}$. Let $y = \begin{vmatrix} Y \\ Y' \end{vmatrix}$, where, similarly, each of Y and Y' is a column of p quantities; then*

$$XY' - X'Y = \frac{1}{2\pi i}(H_X \Omega_Y - H_Y \Omega_X) = ay\,.\,bx - ax\,.\,by = (\bar{a}b - \bar{b}a)\,xy = kxy,$$

but

$$XY' - X'Y = \sum_{i,j}^{1\ldots p}[X_i Y_i' - X_j' Y_j] = \sum_{i,j}^{1\ldots p}(x_i y_{i+p} - x_{j+p} y_j) = \sum_{i,j}^{1\ldots p}[\epsilon_{i+p,i} x_i y_{i+p} + \epsilon_{j,j+p} x_{j+p} y_j],$$

where $\epsilon_{i+p,i} = +1 = -\epsilon_{i,i+p}$ and $\epsilon_{i,j} = 0$ when $i \sim j$ is not equal to p; thus we may write

$$kxy = XY' - X'Y = \epsilon xy,$$

namely, the matrix k is in the case of the theta functions the matrix ϵ, of $2p$ rows and columns, which has already been employed (Chap. XVIII., § 322).

It can be similarly shewn that in the case of theta functions of order r, $k = r\epsilon$.

Next if a, b, h denote the matrices occurring in the exponents of the exponential in the theta series, we have†

$$\mathrm{h}\Omega_X = \pi i X + \mathrm{b}X',$$

namely $\mathrm{h}\,.\,ax = \pi i X + \mathrm{b}X'$. Hence the equations $ax = 0$ give $X = -\frac{1}{\pi i}\mathrm{b}X'$. If X_1, X_1' denote the conjugate complexes of X, X' we have therefore $X_1 = \frac{1}{\pi i}\mathrm{b}_1 X_1'$.

Hence $ikxx_1 = i\epsilon xx_1 = i(XX_1' - X'X_1) = -\frac{1}{\pi}[\mathrm{b}X'X_1' + \mathrm{b}_1 X_1' X'] = -\frac{1}{\pi}(\mathrm{b} + \mathrm{b}_1)X'X_1'$, since $\mathrm{b} = \bar{\mathrm{b}}$ and $\mathrm{b}_1 = \bar{\mathrm{b}}_1$. Thus if $\mathrm{b} = \mathrm{c} + i\mathrm{d}$, $\mathrm{b}_1 = \mathrm{c} - i\mathrm{d}$, the quantity $-\mathrm{c}X'X_1'$ is positive unless each element of X' is zero, namely, the real part of $\mathrm{b}X'X_1'$ is negative for *all* values of X' (except zero). If $X' = m + in$, $\mathrm{b}\,(m^2 + n^2)$ is equal to $\mathrm{b}m^2 + \mathrm{b}n^2$; and the condition that this be negative is just the condition that the theta series converge.

* For the notation see Appendix II.

† Chap. X. § 190, Chap. VII. § 140.

354. Passing from this case to the proof of equations (A), (B) of § 352, we have, from equation (I.),

$$\phi\left[u+a^{(1)}+a^{(2)}\right]=e^{2\pi i b^{(1)}\left[u+a^{(2)}+\frac{1}{2}a^{(1)}\right]+2\pi i c^{(1)}}\phi\left(u+a^{(2)}\right)$$
$$=e^{2\pi i b^{(1)}\left[u+a^{(2)}+\frac{1}{2}a^{(1)}\right]+2\pi i c^{(1)}+2\pi i b^{(2)}\left[u+\frac{1}{2}a^{(2)}\right]+2\pi i c^{(2)}}\phi(u)$$
$$=e^{2\pi i\left[b^{(1)}+b^{(2)}\right]\left[u+\frac{1}{2}a^{(1)}+\frac{1}{2}a^{(2)}\right]+2\pi i\left[c^{(1)}+c^{(2)}\right]}e^{L_{12}}\phi(u),$$

where $L_{12}=\pi i\left[b^{(1)}a^{(2)}-b^{(2)}a^{(1)}\right], =-L_{21}$. Since the left-hand side of the equation is symmetrical in regard to a_1 and a_2, $e^{L_{12}}$ must be $=e^{L_{21}}$, and hence $L_{12}/\pi i$ is a rational integer, $=k_{21}$ say, such that $k_{12}=-k_{21}$.

Obviously, in $k_{12}=a^{(1)}b^{(2)}-a^{(2)}b^{(1)}$, the part $a^{(1)}b^{(2)}$ is formed by compounding the first column of the matrix a (of σ columns and p rows) with the second column of the matrix b. Similarly with $a^{(2)}b^{(1)}$. Namely k_{12} is the (1, 2)th element of $k=\bar{a}b-\bar{b}a$. Since similar reasoning holds for every element, it follows that the matrix k is a skew symmetrical matrix of integers. Conversely, if this be so, it is easy to prove by successive steps the equation

$$\phi\left(u+a^{(1)}m_1+a^{(2)}m_2+\ldots+a^{(\sigma)}m_\sigma\right)/\phi(u)$$
$$=e^{2\pi i\left[b^{(1)}m_1+\ldots+b^{(\sigma)}m_\sigma\right]\left[u+\frac{a^{(1)}m_1+\ldots+a^{(\sigma)}m_\sigma}{2}\right]+2\pi i\left(c^{(1)}m_1+\ldots+c^{(\sigma)}m_\sigma\right)+\pi i L}, \qquad \text{(II.)}$$

where

$$L=\sum_{\substack{\alpha=1,\ldots,\sigma\\ \beta=2,\ldots,\sigma}}^{\alpha<\beta}k_{\alpha\beta}m_\alpha m_\beta,$$

and $m_1, \ldots, m_\sigma$ are integers; this equation may be represented* by

$$\phi(u+am)=\phi(u)\,e^{2\pi i bm\left[u+\frac{am}{2}\right]+2\pi i cm+\pi i\sum^{\alpha<\beta}k_{\alpha\beta}m_\alpha m_\beta}.$$

In fact, assuming the equation (II.) to be true for one set $m_1, \ldots, m_\sigma$, we have, by the equations (I.),

$$\phi\left[u+am+a^{(1)}\right]=e^{2\pi i b^{(1)}\left[u+am+\frac{1}{2}a^{(1)}\right]+2\pi i c^{(1)}}\phi(u+am),$$
$$=e^{2\pi i bm\left[u+\frac{1}{2}am\right]+2\pi i b^{(1)}\left[u+am+\frac{1}{2}a^{(1)}\right]+2\pi i cm+2\pi i c^{(1)}+\pi i\sum^{\alpha<\beta}k_{\alpha\beta}m_\alpha m_\beta}\phi(u),$$
$$=e^{2\pi i\left[bm+b^{(1)}\right]\left[u+\frac{1}{2}am+\frac{1}{2}a^{(1)}\right]+2\pi i\left[cm+c^{(1)}\right]+\pi i\sum^{\alpha<\beta}k_{\alpha\beta}m_\alpha m_\beta+\pi i R}\phi(u),$$

* For the notation see Appendix II.—or thus—

$$bm\,.\,u=\sum_i\left[b_{i1}m_1+\ldots\ldots+b_{i\sigma}m_\sigma\right]u_i$$
$$=\left(\sum_i b_{i1}u_i\right)m_1+\ldots\ldots+\left(\sum_i b_{i\sigma}u_i\right)m_\sigma$$
$$=\left(\sum_i b_i^{(1)}u_i\right)m_1+\ldots\ldots+\left(\sum_i b_i^{(\sigma)}u_i\right)m_\sigma$$
$$=b^{(1)}u\,.\,m_1+\ldots\ldots+b^{(\sigma)}u\,.\,m_\sigma$$
$$=b^{(1)}m_1\,.\,u+\ldots\ldots+b^{(\sigma)}m_\sigma\,.\,u.$$

where R is equal to $b^{(1)}.am - bm.a^{(1)}$, namely equal to

$$\sum_i b_i^{(1)}[a_i^{(1)}m_1 + \ldots + a_i^{(\sigma)}m_\sigma] - \sum_j [b_j^{(1)}m_1 + \ldots + b_j^{(\sigma)}m_\sigma]a_j^{(1)} = k_{21}m_2 + \ldots + k_{\sigma 1}m_\sigma,$$

so that

$$R + \sum^{\alpha<\beta} k_{\alpha\beta}m_\alpha m_\beta$$

$$= k_{21}m_2 + \ldots + k_{\sigma 1}m_\sigma + k_{12}m_1m_2 + \ldots + k_{1\sigma}m_1m_\sigma + k_{23}m_2m_3 + \ldots + k_{2\sigma}m_2m_\sigma + \ldots,$$

$$= 2(k_{21}m_2 + \ldots + k_{\sigma 1}m_\sigma) + k_{12}(m_1+1)m_2 + \ldots + k_{1\sigma}(m_1+1)m_\sigma + k_{23}m_2m_3 + \ldots;$$

hence

$$e^{\pi iR + \pi i \sum_{\alpha<\beta} k_{\alpha\beta}m_\alpha m_\beta} = e^{\pi i \sum_{\alpha<\beta} k_{\alpha\beta}m_\alpha' m_\beta'},$$

where

$$[m_1', \ldots, m_\sigma'] = [m_1 + 1, m_2, \ldots, m_\sigma];$$

therefore

$$\phi[u + am'] = e^{2\pi ibm'[u + \frac{1}{2}am'] + 2\pi icm' + \pi i \sum_{\alpha<\beta} k_{\alpha\beta}m_\alpha' m_\beta'}\phi(u).$$

Similarly we can take the case $\phi(u + am - a^{(1)})$, noticing that equation (I.) can be written

$$\phi(v - a^{(j)}) = \phi(u)\,e^{-2\pi ib^{(j)}[v - \frac{1}{2}a^{(j)}] - 2\pi ic^{(j)}},$$

where $v = u + a^{(j)}$.

355. The theorem (A) is thus proved. The theorem (B) is of a different character, and may be made to depend on the fact that a one-valued function of a single complex variable cannot remain finite for all values of the variable.

Consider the expression

$$L(\xi) = e^{-2\pi ib\xi(v + \frac{1}{2}a\xi) - 2\pi ic\xi}\phi(v + a\xi),$$

wherein $\xi_1, \ldots, \xi_\sigma$ are real quantities.

Then $L(\xi + m)/L(\xi)$, wherein $m_1, \ldots, m_\sigma$ are rational integers, is equal to $e^{\pi ikm\xi + \pi i \sum^{\alpha<\beta} k_{\alpha\beta}m_\alpha m_\beta}$, as immediately follows from equation (I.), and is therefore a quantity whose modulus is unity. Now when $\xi_1, \ldots, \xi_\sigma$ are each between 0 and 1 and v is finite, $L(\xi)$ is finite. Its modulus is therefore finite for *all* real values of ξ; let G be an upper limit to the modulus of $L(\xi)$; G can be determined by considering values of ξ between 0 and 1. Let now $x_1, \ldots, x_\sigma$ be such that $ax = 0$, and let x_1 denote the column of quantities which are the conjugate complexes of the elements of the column x. Put $\xi = x + x_1$, so that $a\xi = ax_1$.

Then

$$\phi(v + ax_1) = \phi(v + a\xi) = e^{\pi ib\xi.a\xi + 2\pi i(c+bv)\xi}L(\xi),$$

wherein an upper limit of the modulus of $L(\xi)$ is a positive quantity G whose value may be taken large enough to be unaffected by replacing x by any other solution of $ax=0$; it is necessary in fact only to consider the modulus of $L(\xi)$ when ξ is between 0 and 1.

We have

$$b\xi\,.\,a\xi=b(x+x_1)\,.\,a(x+x_1)=bx\,.\,ax_1+bx_1\,.\,ax_1$$
$$=bx\,.\,ax_1-bx_1\,.\,ax+bx_1\,.\,ax_1=kxx_1+\bar{a}bx_1^2,$$

$(c+\bar{b}v)\,\xi,\ =w(x+x_1)$, say, $=wx+w_1x_1+(w-w_1)\,x_1$,

where $w=c+\bar{b}v$; therefore

$$e^{\pi ib\xi\,.\,a\xi+2\pi i(c+\bar{b}v)\xi}\,L(\xi)=e^{i\pi kxx_1+i\pi\bar{a}bx_1^2+2\pi i(w-w_1)x_1}\,e^{2\pi i(wx+w_1x_1)}\,L(\xi);$$

this equation is the same as

$$e^{-i\pi\bar{a}bx_1^2-2\pi i(w-w_1)x_1}\,\phi(v+ax_1)=e^{\rho}K,$$

where

$$K,\ =L(\xi)\,e^{2\pi i(wx+w_1x_1)},$$

has the same modulus as $L(\xi)$, less than G, and where

$$\rho=i\pi kxx_1$$
$$=i\pi\Sigma k_{ij}\left[x_j(x_1)_i-x_i(x_1)_j\right]=i\pi\Sigma k_{ij}\begin{vmatrix} y_j+iz_j, & y_j-iz_j \\ y_i+iz_i, & y_i-iz_i\end{vmatrix}=2i\pi\Sigma k_{ij}i\begin{vmatrix} y_j, & -z_j \\ y_i, & -z_i\end{vmatrix}$$
$=2\pi\Sigma k_{ij}(y_jz_i-y_iz_j)=2\pi kyz$, is a real quantity ($x$ being equal to $y+iz$).

Now if x be any solution of the equations $ax=0$, then $\mu_1 x$ is also a solution, μ being any arbitrary complex quantity and μ_1 its conjugate complex. Replace x throughout by $\mu_1 x$, and therefore ξ by $\mu_1 x+\mu x_1$. Then the equation just written becomes

$$e^{-i\pi\bar{a}b\mu^2x_1^2-2\pi i(w-w_1)\mu x_1}\,\phi(v+\mu ax_1)=e^{\rho\mu\mu_1}\,.\,K,$$

K having also its modulus $< G$.

Herein the left side, if not independent of μ, is, for definite constant values of v and x, a one-valued continuous (analytical) function of μ which is finite for all finite values of μ. Hence it must be infinite for infinite values of μ. Hence ρ must be positive, *viz., values of* x *such that* $ax=0$ *are such that the real quantity* $ikxx_1$ *is necessarily positive* provided only the expression

$$e^{-\mu^2 i\pi\bar{a}bx_1^2-2\pi i\mu(w-w_1)x_1}\,\phi(v+\mu ax_1)$$

is not independent of μ.

Now if this expression be independent of μ, it is equal to $\phi(v)$, the value obtained when $\mu=0$, and therefore

$$e^{-i\pi\mu^2\bar{a}bx_1^2}\,\frac{\phi(v+\mu ax_1)}{\phi(v)}=e^{2\pi i\mu(w-w_1)x_1};$$

here the left side is a function of v provided ax_1 be not zero; when ax_1 is zero its value is unity; we take these possibilities in turn:

(i) Suppose first ax_1 is not zero,

then

$$(w - w_1)\, x_1 = (\bar{b}v - \bar{b}_1 v_1)\, x_1 = bx_1 \,.\, v - b_1 x_1 \,.\, v_1$$

must, like the left side, be a function of v and therefore a linear function, say $\frac{1}{2\pi i}(Bv + C)$, so that

$$\phi\,(v + \mu a x_1) = \phi\,(v)\, e^{A\mu^2 + Bv\mu + C\mu}, \text{ where } A = i\pi \bar{a} b x_1^{\,2};$$

hence $\mu a x_1$ represents a column of periods* for the function $\phi\,(v)$—and this for arbitrary values of μ.

In this case however $\phi\,(v)$ would be capable of a column of infinitesimal periods, contrary to our hypothesis.

Hence ρ must be positive for values of x such that $ax = 0$, $ax_1 \neq 0$.

(ii) But in fact as there are σ columns of independent periods we cannot simultaneously have $ax = 0$, $ax_1 = 0$. For the last is equivalent to $a_1 x = 0$; and $ax = 0$, $a_1 x = 0$, together, involve that every determinant of σ rows and columns in the matrix $\begin{vmatrix} a \\ a_1 \end{vmatrix}$ is zero—and thence involve the existence of infinitesimal periods (§ 351).

Hence $ikxx_1$ is necessarily positive for values of x, other than zero, satisfying $ax = 0$; and this is the theorem (B).

Remark i. From the existence of two matrices a, b of p rows and σ columns, for which $\bar{a}b - \bar{b}a$ is a skew symmetrical matrix of integers k such that $ikxx_1$ is positive for values of x other than zero satisfying $ax = 0$, can be inferred that in the matrix of σ columns and $2p$ rows, $\begin{vmatrix} a \\ a_1 \end{vmatrix}$, not every determinant of σ rows and columns can vanish—and also that the σ columns of quantities which form the matrix a are independent, namely that we cannot have the p equations $a_{i_1} x^{(1)} + \dots + a_{i\sigma} x^{(\sigma)} = 0$ satisfied by rational integers $x^{(1)}, \dots, x^{(\sigma)}$. For then, also, $a_1 x = 0$, since $x = x_1$.

Remark ii. In the matrix k, if σ be not less than p, all determinants of $2(\sigma - p)$ rows and columns cannot be zero. In the matrix a, not all determinants of $\frac{1}{2}\sigma$ or $\frac{1}{2}(\sigma + 1)$ rows and columns can be zero. In particular when $\sigma = 2p$, for the matrix k, the determinant is not zero; for the matrix a, not all determinants of p rows and columns can be zero.

Let ξ, η be columns each of σ quantities. Then the coexistence of the 3 sets of equations

$$a\xi = 0, \quad a_1 \eta = 0, \quad \bar{k}(\xi + \eta) = 0$$

is inconsistent with the conditions (A) and (B) (§ 352), except for zero values of ξ and η. The second of them obviously gives also $a\eta_1 = 0$.

For from these equations we infer that $k\eta_1\xi = a\xi \,.\, b\eta_1 - b\xi \,.\, a\eta_1$ is zero, and also

$$\bar{k}(\xi + \eta) \,.\, \eta_1 = k\eta_1 (\xi + \eta) = k\eta_1 \xi + k\eta_1 \eta,$$

and therefore also $k\eta_1\eta$ is zero. But by condition (B) the vanishing of $k\eta_1\eta$ when, as here, $a\eta_1 = 0$, enables us to infer $\eta = 0$.

* We use the word period for the quantities $a^{(j)}$ occurring in our original equation (I.).

Similarly

$$k\xi\xi_1 = \bar{k}\xi_1\xi = \bar{k}(\xi_1+\eta_1)\,.\,\xi - \bar{k}\eta_1\xi = \bar{k}(\xi_1+\eta_1)\,.\,\xi - k\xi\eta_1 = \bar{k}(\xi_1+\eta_1)\,.\,\xi - (\bar{a}b - \bar{b}a)\,\xi\eta_1$$
$$= \bar{k}(\xi_1+\eta_1)\,.\,\xi - (a\eta_1\,.\,b\xi - b\eta_1\,.\,a\xi)$$

is zero when $\bar{k}(\xi+\eta_1)=0$, $a\eta_1=0$, $a\xi=0$. Thence by condition (B), since $a\xi=0$, ξ is zero.

Suppose now that the number of the p linear functions $a\xi$ which are linearly independent is ν, so that all determinants of $(\nu+1)$ rows and columns of the matrix a are zero, but not all determinants of ν rows and columns; and that the number of the σ linear functions $k\xi$ which are linearly independent is 2κ*, so that in the matrix k all determinants of $2\kappa+1$ rows and columns vanish, but not all of 2κ rows and columns. Then we can choose $2\nu+2\kappa$ linearly independent linear functions from the $2p+\sigma$ functions $a\xi$, $a_1\eta$, $\bar{k}(\xi+\eta)$. If this number, $2\nu+2\kappa$, of independent functions, were less than the number 2σ of variables ξ, η, the chosen independent functions could be made to vanish simultaneously for other than zero values of the variables, and then all the linear functions dependent on these must also vanish.

Hence

$$2\nu+2\kappa \geqq 2\sigma \text{ or } \nu+\kappa \geqq \sigma.$$

Now

$$\nu \leqq p,\quad 2\kappa \leqq \sigma\,;\quad \text{hence}\quad \nu \geqq \tfrac{1}{2}\sigma,\quad 2\kappa \geqq 2(\sigma-p).$$

Remark iii. It follows from (ii) that if $k=0$, then $\nu=\sigma$ and $\sigma \leqq p$. Also that a function of p variables which is everywhere finite, continuous and one-valued for finite values of the variables and has no infinitesimal periods cannot be *properly* periodic (without exponential factors) for more than p columns of independent periods; in every set of σ independent periods of such a function the determinants of σ rows and columns are not all zero. The proof is left to the reader.

Remark iv. When $\sigma=2p$ we can put $a=|2\omega, 2\omega'|$, wherein the square matrix 2ω is chosen so that its determinant is not zero. When we write $a=|2\omega, 2\omega'|$ we shall always suppose this done.

356. *Ex.* i. Prove that the exponential of any quadric function of $u_1, \ldots, u_p$ is a Jacobian function of the kind here considered, for which the matrix k is zero.

Ex. ii. Prove that the product of any two or more Jacobian functions, ϕ, with the same number of variables and the same value for σ, is a function of the same character, and that the matrix k of the product is the sum of the matrices k of the separate factors.

Ex. iii. If ϕ be considered as a function of other variables v than u, obtained from them by linear equations of the form $u=\mu+cv$ (μ being any column of p quantities, and c any matrix of p rows and columns), prove that the matrix k of the function ϕ, regarded as a function of v, is unaltered.

Obtain the transformed values of a, b, c and $bm(u+\frac{1}{2}am)+cm$. (Cf. Ex. i., § 190, Chap. X.)

Ex. iv. If instead of the periods a we use $a'=ag$, where g is a matrix of integers with σ rows and columns, prove that $\phi(u+a'm)$ is of the form $e^{2\pi i b'm(u+\frac{1}{2}a'm)+2\pi i cm}\,\phi(u)$, and that $k'=\bar{g}kg$; and also that kxy becomes changed to $k'x'y'$ by the linear equations $x=gx'$, $y=gy'$. In such case the form $k'x'y'$ is said to be *contained* in kxy. When the relation is reciprocal, or $g^2=1$, the forms are said to be equivalent. Thus to any function ϕ there corresponds a class of equivalent forms k. (Cf. Chap. XVIII., § 324, Ex. i.)

Examples iii. and iv. contain an important result which may briefly be summarised by

* That the number must be even is a known proposition, Frobenius, *Crelle*, LXXXII. (1877), p. 242.

saying that for Jacobian functions, *qua* Jacobian functions, there is no theory of transformation of periods such as arises for the theta functions. A transformed theta function is a Jacobian function; the equations of Chap. XVIII. (§ 324) are those which are necessary in order that, for this Jacobian function, the matrix k should be the matrix ϵ, or $r\epsilon$ (cf. § 353).

Ex. v. If A be a matrix of $2p$ rows and σ columns of which the first p rows are the rows of a and the second p rows those of b, prove that

$$\bar{A}\epsilon A = k.$$

In fact if $\xi = Ax$, $\xi' = Ax'$, then

$$kx'x = ax \,.\, bx' - ax' \,.\, bx = \Sigma\,[\xi_i \xi'_{i+p} - \xi_j' \xi_{j+p}] = \epsilon\xi\xi'$$
$$= \epsilon Ax \,.\, Ax' = \bar{A}\epsilon A \,.\, x'x.$$

Hence also when $\sigma = 2p$ the determinant of A is the square root of the determinant of k, which in that case, being a skew symmetrical determinant of even order, is a perfect square.

Ex. vi. Shew that when $\sigma = 2p$ and with the notation $a = |2\omega,\ 2\omega'|$, $2\pi i b = |2\eta,\ 2\eta'|$, that

$$\bar{A}\epsilon A = \frac{2}{\pi i}\begin{vmatrix} \bar{\omega}\,\eta - \bar{\eta}\,\omega, & \bar{\omega}\,\eta' - \bar{\eta}\,\omega' \\ \bar{\omega}'\eta - \bar{\eta}'\omega, & \bar{\omega}'\eta' - \bar{\eta}'\omega' \end{vmatrix},$$

the notation being an abbreviated one for a matrix of $2p$ rows and columns. Thus in the case when $k = \epsilon$, the equation of Ex. v. expresses the Weierstrass equations for the periods (Chap. VII., § 140).

Ex. vii. In the case of the theta functions we shewed (§ 140, and p. 533) that the relations connecting the periods could be written in two different ways, one of which was associated with the name of Weierstrass, the other with that of Riemann. We can give a corresponding transformation of the equations (A), (B) (§ 352) in this case, provided $\sigma = 2p$, the determinant of the matrix k not being zero.

As to the equation (A), writing it in the equivalent form given in Ex. v., we immediately deduce

$$Ak^{-1}\bar{A} = \epsilon, \qquad (A'),$$

which is the transformation of equation (A).

As to the equation (B), let x be a column of $\sigma = 2p$ arbitrary quantities, and determine the column z, of $\sigma = 2p$ elements, so that the $2p$ equations expressed by $az = 0$, $bz = x$, are satisfied. Then

$$\bar{a}x = \bar{a}bz = (\bar{a}b - \bar{b}a)\,z = kz, \; = \mu, \text{ say; so that } k^{-1}\mu = z,\ k^{-1}\mu_1 = z_1;$$

thus

$$ikzz_1 = i\,(\bar{a}b - \bar{b}a)\,zz_1 = i\,(az_1 \,.\, bz - az \,.\, bz_1) = iaz_1 \,.\, bz = iaz_1x = i\bar{a}xz_1 = i\mu z_1$$
$$= ik^{-1}\mu_1\mu = ik^{-1}\bar{a}_1x_1 \,.\, \bar{a}x = iak^{-1}\bar{a}_1x_1x;$$

therefore, the form

$$iak^{-1}\bar{a}_1x_1x \qquad (B'),$$

is positive for all values of the column x, other than zero. This is the transformed form of equations (B).

Ex. viii. When $a = |2\omega,\ 2\omega'|$, $b = \frac{1}{2\pi i}|2\eta,\ 2\eta'|$, $\sigma = 2p$, we have

$$A\epsilon\bar{A} = \begin{vmatrix} 2\omega, & 2\omega' \\ \frac{\eta}{\pi i}, & \frac{\eta'}{\pi i} \end{vmatrix}\begin{vmatrix} 0 & -1 \\ 1 & 0 \end{vmatrix}\begin{vmatrix} 2\bar{\omega}, & \frac{\bar{\eta}}{\pi i} \\ 2\bar{\omega}', & \frac{\bar{\eta}'}{\pi i} \end{vmatrix} = \begin{vmatrix} -4\,(\omega\bar{\omega}' - \omega'\bar{\omega}), & -\frac{2}{\pi i}(\omega\bar{\eta}' - \omega'\bar{\eta}) \\ \frac{2}{\pi i}(\eta'\bar{\omega} - \eta\bar{\omega}'), & -\frac{1}{(\pi i)^2}(\eta\bar{\eta}' - \eta'\bar{\eta}) \end{vmatrix}.$$

Hence when $k=\epsilon$, the equation (A′) of Ex. vii., equivalent to $A\epsilon\bar{A}=-\epsilon$, expresses the Riemann equations for the periods (Chap. VII., § 140). In the same case the equation (B′), of Ex. vii., expresses that

$$ia\epsilon\bar{a}_1 x_1 x=\sum_{\nu=1}^{p}\sum_{\kappa,\lambda=1}^{p}\left[(a_1)_{\kappa,\nu}a_{\lambda,\nu+p}-a_{\lambda,\nu}(a_1)_{\kappa,\nu+p}\right]x_\lambda (x_1)_\kappa$$

is negative for all values of x other than zero.

Ex. ix. When $p=1$, the two conditions (B), (B′), or

$$i\epsilon x x_1=\text{positive for } ax=0,\quad ia\epsilon\bar{a}_1 x_1 x=\text{negative for arbitrary } x,$$

become, for $a=|2\omega,\ 2\omega'|$, if the elements of x be denoted by x and x', and the conjugate imaginaries by x_1, x_1', respectively,

$$i(\omega\omega_1)^{-1}(\omega\omega_1'-\omega'\omega_1)x'x_1'=\text{positive},\quad i(\omega_1\omega'-\omega\omega_1')xx_1=\text{negative},$$

and if $\omega=\rho+i\sigma$, $\omega_1=\rho-i\sigma$, $\omega'=\rho'+i\sigma'$, $\omega_1'=\rho'-i\sigma'$, these conditions are equivalent to

$$\rho\sigma'-\rho'\sigma>0,$$

and express that the real part of $i\omega'/\omega$ is negative.

357. Suppose now that $\sigma=2p$; we proceed (§ 359) to consider how to express the Jacobian function. Two arithmetical results, (i) and (ii), will be utilised, and these may be stated at once: (i) *if k be a skew symmetrical matrix whose elements are integers, with $2p$ rows and columns, and ϵ have the signification previously attached to it, it is possible to find a matrix g, of $2p$ rows and columns, whose elements are integers, such that** $k=\bar{g}\epsilon g$.* For instance when $p=1$, we can find a matrix such that

$$\begin{vmatrix}0 & k_{12}\\ -k_{12} & 0\end{vmatrix}=\begin{vmatrix}g_{11} & g_{21}\\ g_{12} & g_{22}\end{vmatrix}\begin{vmatrix}0 & -1\\ 1 & 0\end{vmatrix}\begin{vmatrix}g_{11} & g_{12}\\ g_{21} & g_{22}\end{vmatrix}=\begin{vmatrix}g_{21}g_{11}-g_{11}g_{21} & g_{21}g_{12}-g_{11}g_{22}\\ g_{22}g_{11}-g_{12}g_{21} & g_{22}g_{12}-g_{12}g_{22}\end{vmatrix},$$

namely, such that $k_{12}=g_{21}g_{12}-g_{11}g_{22}$; for this we can in fact take g_{11}, g_{12} arbitrarily. In general the $4p^2$ integers contained in g are to satisfy $p(2p-1)$ conditions.

Ex. i. If a be a matrix of integers, of p rows and columns, and λ be an integer, and

$$k=\begin{vmatrix}0, & -\lambda\bar{a}\\ \lambda a, & 0\end{vmatrix},$$

g may have either of the two following forms

$$g_1=\begin{vmatrix}\lambda, & 0\\ 0, & \bar{a}\end{vmatrix},\quad g_2=\begin{vmatrix}\lambda a, & 0\\ 0, & 1\end{vmatrix}=\begin{vmatrix}\lambda, & 0\\ 0, & \bar{a}\end{vmatrix}\begin{vmatrix}a, & 0\\ 0, & \bar{a}^{-1}\end{vmatrix},\ =g_1\mu,\text{ say},$$

for we immediately find $\bar{\mu}k\mu=k$.

* For a proof see Frobenius, *Crelle*, LXXXVI. (1879), p. 165, *Crelle*, LXXXVIII. (1880), p. 114.

Ex. ii. If μ be any matrix of integers, with $2p$ rows and columns, such that $\bar{\mu}\epsilon\mu=\epsilon$ (cf. § 322, Chap. XVIII.), we have, if $k=\bar{g}\epsilon g$, also $k=\bar{g}\bar{\mu}^{-1}\epsilon\mu^{-1}g$, and instead of g we may take the matrix $\mu^{-1}g$.

(ii) If g be a given matrix of integers, of $2p$ rows and columns, and x be a column of $2p$ elements, the conditions, for x, that the $2p$ elements gx should be prescribed integers cannot always be satisfied, however the elements of x (which are necessarily rational numerical fractions) are chosen. If for any rational values of x, integral or not, gx be a row of integers, and we put $x=y+L$, where y has all its elements positive (or zero) and less than unity, and L is a row of integers (including zero), then $gx=gy+gL=gy+M$, where M is a row of integers; in this case the row gx will be said to be congruent to gy for modulus g. The result to be utilised* is, that *the number of incongruent rows gx, namely, the number of integers which can be represented in the form gx while each element of x is zero or positive and less than unity, is finite. It is in fact equal to the absolute value of the determinant of g.* For instance when g is $\begin{vmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{vmatrix}$ there are $g_{11}g_{22}-g_{12}g_{21}$ integer pairs which can be written $g_{11}x_1+g_{12}x_2$, $g_{21}x_1+g_{22}x_2$, for (rational) values of x_1, x_2 less than unity. The reader may verify, for instance, that when $g=\begin{vmatrix} 6 & 3 \\ 1 & 2 \end{vmatrix}$, the 9 ways are given (cf. p. 637, *Footnote*) by

	1	2	3	4	5	6	7	8	9
x_1, x_2	0, 0	$\frac{1}{9}$, $\frac{4}{9}$	$\frac{5}{9}$, $\frac{2}{9}$	$\frac{1}{3}$, $\frac{1}{3}$	$\frac{2}{9}$, $\frac{8}{9}$	$\frac{7}{9}$, $\frac{1}{9}$	$\frac{4}{9}$, $\frac{7}{9}$	$\frac{2}{3}$, $\frac{2}{3}$	$\frac{8}{9}$, $\frac{5}{9}$
$6x_1+3x_2$, x_1+2x_2	0, 0	2, 1	4, 1	3, 1	4, 2	5, 1	5, 2	6, 2	7, 2

To prove the statement in general let t be the number required, of integers representable in the form gx, when $x<1$. Consider how many integers could be obtained in the form gX when X is restricted only to have all its elements less than (a positive number) N. Corresponding to any one of the t integers obtained in the former case we can now obtain $N-1$ others by increasing only one of the elements of x in turn by $1, 2, \ldots, N-1$. This can be done independently for each element of x. Hence the number of integers gX is tN^σ where σ, here to be taken $=2p$, is the number of elements in x. Let one of these integers be called M. Then $g\frac{X}{N}=\frac{M}{N}$ or say $gx=\frac{M}{N}$, wherein x is less than unity. Now when N is very great, the

* Cf. Appendix ii, § 418, and the references there given, and Frobenius, *Crelle*, xcvii. (1884), p. 189.

variation of $z=\frac{M}{N}$, as M changes, approaches to that of a continuous quantity, and the number of its values, being the same as the number of values of M, is

$$\iint\ldots(Ndz_1)\ldots(Ndz_\sigma),$$

where $z_1, \ldots, z_\sigma$ vary from zero to all values which give to x, in the equations $gx=z$, a value less than unity. Now this integral is

$$N^\sigma\iint\ldots\frac{\partial(z_1, \ldots, z_\sigma)}{\partial(x_1, \ldots, x_\sigma)}dx_1\ldots dx_\sigma = N^\sigma\,|g|\,.\iint\ldots dx_1\ldots dx_\sigma = N^\sigma\,|g|.$$

Since this is equal to tN^σ, it follows that t is equal to $|g|$, as was stated.

358. Supposing then that the matrix g, with $2p$ rows and columns each consisting of integers, has been determined so that $k=\bar{a}b-\bar{b}a=\bar{g}\epsilon g$, we consider the expression of the Jacobian function when $\sigma=2p$. The determinant of k not being zero, the determinant of g is not zero.

Put $K=ag^{-1}$, so that K is a matrix of p rows and $2p$ columns, and $a=Kg$; put similarly $b=Lg$; also, take a row of $2p$ quantities denoted by C, such that $c=\bar{g}C+\frac{1}{2}[g]$, where c is the parameter (§ 351) of the Jacobian function, and $[g]$ is a row of $2p$ quantities of which one element is

$$[g]_\alpha=\sum_{\kappa=1}^{\kappa=p} g_{\kappa,\alpha}\,g_{p+\kappa,\alpha}, \qquad (\alpha=1, \ldots, 2p);$$

take x, x', X, X', rows of $2p$ quantities such that

$X=gx$, $X'=gx'$, so that $ax=Kgx=KX$, $bx=LX$, $ax'=KX'$, $bx=LX'$;

then as

$$kx'x, \;=ax\,.\,bx'-ax'\,.\,bx, =(\bar{K}L-\bar{L}K)\,X'X,$$

is also equal to

$$\bar{g}\epsilon gx'x=\epsilon gx'\,.\,gx=\epsilon X'X,$$

we have

$$\bar{K}L-\bar{L}K=\epsilon, \qquad \text{(C)},$$

so that

$$KxLx'-Kx'Lx=(\bar{K}L-\bar{L}K)\,x'x=\epsilon x'x=\sum_{i,j}^{1,\ldots,p}(x_i x'_{i+p}-x'_j\,x_{j+p});$$

further, as $ikxx_1$ is positive for $ax=0$, we have

$$i\epsilon XX_1=\text{positive when } KX=0, \qquad \text{(D)};$$

thus, if A denote the matrix $\begin{vmatrix} K \\ L \end{vmatrix}$, we have, from the equation (C),

$$\bar{A}\bar{\epsilon}A=-A\epsilon\bar{A}=\epsilon, \qquad \text{(E)},$$

and, if z be a row of p arbitrary quantities, and X be a row of $2p$ quantities such that $KX = 0$, $LX = z$, so that $\bar{K}z = \bar{K}LX = (\bar{K}L - \bar{L}K)X = \epsilon X$, and therefore $\epsilon\bar{K}z = -X$, $\bar{K}_1 z_1 = \epsilon X_1$, we have

$$iK_1\epsilon\bar{K}zz_1 = \text{positive, for arbitrary } z \text{ other than zero,} \qquad \text{(F)};$$

for

$$iK_1\epsilon\bar{K}zz_1 = -iK_1Xz_1 = -i\bar{K}_1z_1X = -i\epsilon X_1X = i\epsilon XX_1.$$

If we now change the notation by writing $K = |2\omega, 2\omega'|$, $2\pi iL = |2\eta, 2\eta'|$, and introduce the matrices a, b, h of p rows and columns defined by

$$\mathrm{a} = \tfrac{1}{2}\eta\omega^{-1}, \quad \mathrm{h} = \tfrac{1}{2}\pi i\omega^{-1}, \quad \mathrm{b} = \pi i\omega^{-1}\omega',$$

it being assumed, in accordance with Remark iv. (§ 355) *that the determinant of the matrix* ω *is not zero*, then the equation (E) shews (cf. Ex. viii., § 356) that the matrices a, b are symmetrical, and that $\eta' = \eta\omega^{-1}\omega' - \frac{1}{2}\pi i\bar{\omega}^{-1}$, so that we can also write

$$\eta = 2\mathrm{a}\omega, \quad \eta' = 2\mathrm{a}\omega' - \mathrm{h}', \quad 2\mathrm{h}\omega = \pi i, \quad 2\mathrm{h}\omega' = \mathrm{b};$$

also, by actual expansion,

$$iK_1\epsilon\bar{K} = 4i\omega_1[\omega_1^{-1}\omega_1' - \bar{\omega}'\bar{\omega}^{-1}]\,\bar{\omega} = -\frac{1}{\pi}\,\omega_1[\mathrm{b}_1 + \bar{\mathrm{b}}]\,\bar{\omega} = -\frac{1}{\pi}\,\omega_1[\mathrm{b}_1 + \mathrm{b}]\,\bar{\omega}$$

$$= -\frac{2}{\pi}\,\omega_1\mathrm{c}\bar{\omega}, \text{ if } \mathrm{b} = \mathrm{c} + i\mathrm{d};$$

thus

$iK_1\epsilon\bar{K}zz_1 = -\dfrac{2}{\pi}\,\mathrm{c}t_1t$, where $t = \bar{\omega}z$, z and t being rows of p arbitrary quantities; and therefore, by the equation (F), for real values of $n_1, \ldots, n_p$ other than zero, the quadratic form $\mathrm{b}n^2$ has its real part essentially negative.

Hence we can define a theta function by the equation

$$\vartheta\left(u;\ \begin{matrix}\gamma' \\ -\gamma\end{matrix}\right) = \sum_n e^{\mathrm{a}u^2 + 2\mathrm{h}u(n+\gamma') + \mathrm{b}(n+\gamma')^2 - 2\pi i\gamma(n+\gamma')},$$

wherein γ, γ' are rows of p quantities given by $C = (\gamma', \gamma)$, that is, $C_r = \gamma_r'$, $C_{p+r} = \gamma_r$, for $r < p + 1$. Denoting this function by $\vartheta(u;\ C)$ and taking μ for a row of $2p$ integers, the function is immediately seen (§ 190, Chap. X.) to satisfy the equation

$$\vartheta(u + K\mu;\ C) = e^{2\pi iL\mu(u + \frac{1}{2}K\mu) + 2\pi iC\mu + \pi i\sum\limits_{\alpha,\beta}^{\alpha<\beta}\epsilon_{\alpha,\beta}\mu_\alpha\mu_\beta}\,\vartheta(u;\ C),$$

which is the definition equation for a Jacobian function of periods K, L and parameter C, for which the matrix k is ϵ.

Further, if μ be a matrix of integers with $2p$ rows and columns, such that $\bar{\mu}\epsilon\mu = \epsilon$, and (Ex. ii., § 357) we replace g by $\mu^{-1}g$, the matrices K, L are replaced by $K\mu$ and $L\mu$. Thus instead of the theta function $\vartheta(u;\ C)$ we obtain a linear transformation of this theta function (cf. § 322, Chap. XVIII.).

359. Proceeding further to obtain the expression for the general value of the Jacobian function ϕ, let $\phi(u;\nu)$ denote

$$\phi(u+K\nu)\,e^{-2\pi iL\nu(u+\frac{1}{2}K\nu)-2\pi iC\nu+2\pi inn'},$$

where $\nu_i=n_i$, $\nu_{i+p}=n_i'$, for $i<p+1$. Then, since $a=Kg$, and therefore $aN=KgN$, we have

$$\phi(u+aN,\nu)=\phi(u+KgN,\nu)=\phi(u+K\mu,\nu), \qquad (1),$$

where μ denotes the row gN, so that $aN=K\mu$, N being a column of $2p$ integers and therefore μ a column of integers; thus $\phi(u+aN,\nu)$ is equal to

$$\phi(u+aN+K\nu)\,e^{-2\pi iL\nu(u+K\mu+\frac{1}{2}K\nu)-2\pi iC\nu+\pi inn'}=\phi(u+K\nu)\,e^{R},$$

where

$$R=2\pi ibN(u+K\nu+\tfrac{1}{2}aN)+2\pi icN+\pi i\sum^{\alpha<\beta}k_{\alpha\beta}N_\alpha N_\beta$$
$$-2\pi iL\nu(u+K\mu+\tfrac{1}{2}K\nu)-2\pi iC\nu+\pi inn',$$

by the properties of ϕ, N being a column of integers; thus $\phi(u+aN,\nu)$ is equal to

$$\phi(u,\nu)\,e^{2\pi ibN(u+\frac{1}{2}aN)+2\pi icN+\pi i\sum^{\alpha<\beta}k_{\alpha\beta}N_\alpha N_\beta+2\pi i(bN.K\nu-L\nu.K\mu)}.$$

Now $bN=LgN=L\mu$, therefore

$$bN.K\nu-L\nu.K\mu=(\bar{K}L-\bar{L}K)\mu\nu=\epsilon\mu\nu=mn'-m'n,$$

where $\mu_i=m_i$, $\mu_{i+p}=m_i'$, etc. for $i<p+1$. If then we take ν, as well as μ, to consist of integers, it will follow that

$$\phi(u+aN,\nu)=\phi(u,\nu)\,.\,e^{2\pi ibN(u+\frac{1}{2}aN)+2\pi icN+\pi i\sum^{\alpha<\beta}k_{\alpha\beta}N_\alpha N_\beta},$$

and therefore that

$$\frac{\phi(u+aN)}{\phi(u)}=\frac{\phi(u+aN,\nu)}{\phi(u,\nu)}=e^{2\pi ibN(u+\frac{1}{2}aN)+2\pi icN+\pi i\sum^{\alpha<\beta}k_{\alpha\beta}N_\alpha N_\beta}.$$

Next

$$\phi(u,\mu+\nu)=\phi(u+K\mu+K\nu)\,e^{-2\pi i(L\mu+L\nu)(u+\frac{1}{2}K\mu+\frac{1}{2}K\nu)-2\pi i(C\mu+C\nu)+\pi i(m+m')(n+n')} \qquad (2),$$

and this

$$=\phi(u+K\mu,\nu)\,e^{M},$$

where

$$M=2\pi iL\nu(u+K\mu+\tfrac{1}{2}K\nu)+2\pi iC\nu-\pi inn'-2\pi i(L\mu+L\nu)(u+\tfrac{1}{2}K\mu+\tfrac{1}{2}K\nu)$$
$$-2\pi i(C\mu+C\nu)+\pi i(m+m')(n+n');$$

therefore

$$\frac{\phi(u+K\mu,\nu)}{\phi(u,\mu+\nu)}\,e^{-[2\pi iL\mu(u+\frac{1}{2}K\mu)+2\pi iC\mu-\pi imm']}$$
$$=e^{2\pi iL\mu(\frac{1}{2}K\nu)-2\pi iL\nu(\frac{1}{2}K\mu)+\pi imm'+\pi inn'-\pi i(m+m')(n+n')},$$

of which the exponent of the right side is

$$\pi i\,[(\bar{K}L - \bar{L}K)\,\mu\nu - mn' - m'n] = \pi i\,[mn' - m'n - (mn' + m'n)] = -\,2\pi i m'n,$$

so that, since μ, ν consist of integers, the right side is unity.

Hence we have

$$\frac{\phi\,(u + K\mu,\, \nu)}{\phi\,(u,\, \mu + \nu)} = e^{2\pi i L\mu\,(u + \frac{1}{2}K\mu) + 2\pi i C\mu - \pi i mm'}.$$

It is to be carefully noticed that this equation does not require $\mu \equiv 0$ (mod. g).

We suppose now that $\mu \equiv 0$ (mod. g). Then $cN + \frac{1}{2}\overset{\alpha<\beta}{\Sigma} k_{\alpha\beta} N_\alpha N_\beta \equiv C\mu - \frac{1}{2}mm'$ (mod. unity) and $L\mu = bN$, $K\mu = aN$, as will be proved immediately (§ 360); thus

$$\frac{\phi(u+aN)}{\phi\,(u)} = \frac{\phi(u+aN,\nu)}{\phi\,(u,\,\nu)} = \frac{\phi(u + aN,\nu)}{\phi\,(u,\,\mu + \nu)} = e^{2\pi i bN\,(u + \frac{1}{2}aN) + 2\pi i cN + \pi i \overset{\alpha<\beta}{\Sigma} k_{\alpha\beta} N_\alpha N_\beta},$$

and therefore $\phi\,(u,\,\mu + \nu) = \phi\,(u,\,\nu)$ for integer values ν and any integer values μ that can be written in the form gN, for integer N; namely $\phi\,(u,\,\nu)$ is unaltered by adding to ν any set of integers congruent to zero for the matrix modulus g.

The set of $|g|$ integers gr, wherein r has all rational fractional values less than unity will now be denoted by ν, each value of ν denoting a column of $2p$ integers—in particular $r = 0$ corresponds to a set of integers $\equiv \mu$ (mod. g). And ν' shall denote a special one of the sets of integers which are similarly a representative incongruent system for the transposed matrix modulus $\bar{g}$, such that $\nu' = gr'$, the quantities r' being a set of fractions less than 1. With the assigned values for ν, let

$$\psi\,(u) = \sum_{\nu} e^{-2\pi i r'\nu}\,\phi\,(u,\,\nu);$$

then

$$\psi\,(u + K\lambda) = \sum_{\nu} e^{-2\pi i r'\nu}\,\phi\,(u + K\lambda,\,\nu) = \sum_{\nu} e^{2\pi i r'\nu}\,e^{2\pi i L\lambda\,(u + \frac{1}{2}K\lambda) + 2\pi i C\lambda - \pi i ll'}\,\phi\,(u,\,\lambda + \nu)$$

for any set of integers λ, as has been shewn (λ being such that, for $i < p + 1$, $\lambda_i = l_i$, $\lambda_{i+p} = l_i'$).

If now $\nu + \lambda = \rho$, so that ρ also describes, with ν, a set of integers incongruent in regard to modulus g, those for which the necessary fractions s, in $\rho = gs$, are > 1 being replaced, by the theorem proved*, by others for which the necessary fractions are < 1, so that the range of values for ρ is precisely that for ν, then we have

$$\begin{aligned}\psi\,(u + K\lambda) &= \sum_{\nu} e^{-2\pi i r'\rho + 2\pi i r'\lambda}\,e^{2\pi i L\lambda\,(u + \frac{1}{2}K\lambda) + 2\pi i C\lambda - \pi i ll'}\,\phi\,(u,\,\rho),\\ &= e^{2\pi i r'\lambda + 2\pi i L\lambda\,(u + \frac{1}{2}K\lambda) + 2\pi i C\lambda - \pi i ll'}\sum_{\nu} e^{-2\pi i r'\nu}\,\phi\,(u,\,\nu),\\ &= e^{2\pi i r'\lambda + 2\pi i L\lambda\,(u + \frac{1}{2}K\lambda) + 2\pi i C\lambda - \pi i ll'}\,\psi\,(u).\end{aligned}$$

* That $\phi\,(u,\,\nu)$ is unaltered when to ν is added a column $\equiv 0$ (mod. g).

Hence, by the result of § 284, Chap. XV., we have

$$\psi(u) = A_{\nu'}\vartheta(u, C + r'),$$

the theta function depending on the a, b, h derived in this chapter (§ 358).

Now let ν' describe a set of incongruent values for the modulus $\bar{g}$; then

$$\sum_{\nu'} A_{\nu'}\vartheta(u, C + r') = \Sigma\psi(u) = \sum_{\nu}\sum_{\nu'} e^{-2\pi i r'\nu}\phi(u, \nu);$$

and since $\nu' = \bar{g}r'$, we have $\nu' r = \bar{g}r'r = grr' = \nu r'$; thus

$$\sum_{\nu'} e^{-2\pi i r'\nu} = \sum_{\nu'}(e^{-2\pi i r\nu'}) = \sum_{\nu'}(e^{-2\pi i r_1})^{\nu'_1}(e^{-2\pi i r_2})^{\nu'_2}\dots(e^{-2\pi i r_{2p}})^{\nu'_{2p}}:$$

this sum can be evaluated:

when $\nu \equiv 0$ (mod. g), or the numbers r are zero, its value is equal to the number of incongruent columns for modulus $\bar{g}$, $= |g|$. Since $k = \bar{g}\epsilon g$, we have $|k| = (|g|)^2$, so that $|g| = \sqrt{|k|}$.

when $\nu \not\equiv 0$ (mod. g), so that some of $r_1, \dots, r_{2p}$ are fractional, its value is zero, as is easy to prove (see below, § 360).

Hence we have the following fundamental equation:

$$\sqrt{|k|}\,\phi(u) = \sum_{\nu'} A_{\nu'}\vartheta(u, C + \nu'),$$

which was the expression sought.

Thus between $\sqrt{|k|} + 1$ functions ϕ with the same periods and parameters there exists a homogeneous linear relation with constant coefficients.*

Ex. i. Prove that a product of n functions ϕ is a function ϕ for which $\sqrt{|k|}$ is changed into $n^p\sqrt{|k|}$. In fact the periods are na, nb.

Ex. ii. Prove that the number of homogeneous products of n factors selected from $p+2$ functions ϕ of the same periods and parameters is greater than $n^p\sqrt{|k|}$ when n is large enough. And infer that there exists a homogeneous polynomial relation connecting any $p+2$ functions ϕ of the same periods and parameters. (Cf. Chap. XV., § 284, Ex. v.)

360. We now prove the two results assumed.

(a) If $\mu \equiv 0$ (mod. g) or $\mu = gN$, where N are integers, then

$$cN + \tfrac{1}{2}\sum^{\alpha<\beta} k_{\alpha\beta}N_\alpha N_\beta \equiv C\mu - \tfrac{1}{2}mm' \quad \text{(mod. unity)}.$$

For

$$k_{\alpha\beta} = (\bar{g}\epsilon g)_{\alpha\beta} = \sum_{\gamma}(\bar{g})_{\alpha\gamma}(\epsilon g)_{\gamma\beta} = \sum_{\gamma=1}^{2p}(\bar{g})_{\alpha\gamma}\sum_{\lambda=1}^{p}[\epsilon_{\gamma,\lambda}g_{\lambda,\beta} + \epsilon_{\gamma,\lambda+p}g_{\lambda+p,\beta}]$$

$$= \sum_{\gamma=1}^{p} g_{\gamma\alpha}\sum_{\lambda=1}^{p}[\epsilon_{\gamma\lambda}g_{\lambda\beta} + \epsilon_{\gamma,\lambda+p}g_{\lambda+p,\beta}] + \sum_{\gamma=1}^{p} g_{\gamma+p,\alpha}\sum_{\lambda=1}^{p}[\epsilon_{\gamma+p,\lambda}g_{\lambda,\beta} + \epsilon_{\gamma+p,\lambda+p}g_{\lambda+p,\beta}]$$

$$= -\sum_{\gamma=1}^{p} g_{\gamma,\alpha}g_{\gamma+p,\beta} + \sum_{\gamma=1}^{p} g_{\gamma+p,\alpha}g_{\gamma,\beta} = \sum_{\gamma=1}^{p}[g_{\gamma+p,\alpha}g_{\gamma,\beta} - g_{\gamma,\alpha}g_{\gamma+p,\beta}]$$

$$= \sum_{\gamma=1}^{p}[g_{\gamma+p,\alpha}g_{\gamma,\beta} - g_{\gamma,\alpha}g_{\gamma+p,\beta}];$$

* Weierstrass, *Berl. Monatsber.*, 1869; Frobenius, *Crelle*, XCVII. (1884); Picard, Poincaré, *Compt. Rendus*, XCVII. (1883), p. 1284.

therefore

$$\sum^{\alpha<\beta} k_{\alpha,\beta} N_\alpha N_\beta = \sum_{\gamma=1}^{p} \sum^{\alpha<\beta} [g_{\gamma+p,\alpha} N_\alpha \cdot g_{\gamma,\beta} N_\beta - g_{\gamma,\alpha} N_\alpha \cdot g_{\gamma+p,\beta} N_\beta]$$

$$\equiv \sum_{\gamma=1}^{p} \sum^{\alpha<\beta} [g_{\gamma+p,\alpha} N_\alpha \cdot g_{\gamma,\beta} N_\beta + g_{\gamma,\alpha} N_\alpha \cdot g_{\gamma+p,\beta} N_\beta], \quad (\text{mod. } 2),$$

$$\equiv \sum_{\gamma=1}^{p} \left\{ \sum^{\alpha<\beta} g_{\gamma+p,\alpha} N_\alpha \cdot g_{\gamma,\beta} N_\beta + \sum^{\alpha<\beta} g_{\gamma,\beta} N_\beta \cdot g_{\gamma+p,\alpha} N_\alpha \right\}$$

$$\equiv \sum_{\gamma=1}^{p} \Sigma\Sigma g_{\gamma+p,\alpha} N_\alpha \cdot g_{\gamma,\beta} N_\beta, \quad (\text{mod. } 2),$$

where the $\Sigma\Sigma$ indicates that the summation extends to every pair α, β except those for which $\alpha = \beta$; thus

$$\sum^{\alpha<\beta} k_{\alpha\beta} N_\alpha N_\beta + \sum_{\gamma=1}^{p} \sum_{\alpha=1}^{2p} g_{\gamma+p,\alpha} N_\alpha \cdot g_{\gamma,\alpha} N_\alpha$$

$$\equiv \sum_{\gamma=1}^{p} [g_{\gamma,1} N_1 + \ldots\ldots + g_{\gamma,2p} N_{2p}] [g_{\gamma+p,1} N_1 + \ldots\ldots + g_{\gamma+p,2p} N_{2p}]$$

$$\equiv \sum_{\gamma=1}^{p} \mu_\gamma \cdot \mu_{\gamma+p} \equiv mm', \quad (\text{mod. } 2);$$

therefore, since $\tfrac{1}{2} N_\alpha^2 \equiv \tfrac{1}{2} N_\alpha$ (mod. unity), and therefore

$$\tfrac{1}{2} \sum_{\gamma=1}^{p} g_{\gamma+p,\alpha} N_\alpha \cdot g_{\gamma,\alpha} N_\alpha \equiv \tfrac{1}{2} [g] N,$$

we have

$$cN + \tfrac{1}{2} \sum^{\alpha<\beta} k_{\alpha\beta} N_\alpha N_\beta \equiv cN + \tfrac{1}{2} mm' - \tfrac{1}{2}[g] N \equiv \{\bar{g}C + \tfrac{1}{2}[g]\} N + \tfrac{1}{2} mm' - \tfrac{1}{2}[g] N, \quad (\text{mod. } 1),$$

$$\equiv gN \,.\, C + \tfrac{1}{2} mm' \equiv \mu C + \tfrac{1}{2} mm' \equiv C\mu - \tfrac{1}{2} mm', \text{ as required.}$$

(*b*) If $r_1, \ldots\ldots, r_{2p}$ be any set of rational fractions all less than unity and not all zero and such that the row $gr = \nu$ consists of integers, and $(\nu'_1, \ldots\ldots, \nu'_{2p}), = \nu'$, be every integer row in turn which can be represented in the form $\bar{g}r'$ for values of r' less than unity, then

$$\sum_{\nu'} (e^{-2\pi i r_1})^{\nu'_1} \,.\, (e^{-2\pi i r_2})^{\nu'_2} \ldots\ldots (e^{-2\pi i r_{2p}})^{\nu'_{2p}}$$

is zero. Since, as remarked (§ 359), the sum can also be written

$$\sum_{r'} (e^{-2\pi i \nu_1})^{r'_1} \ldots\ldots (e^{-2\pi i \nu_{2p}})^{r'_{2p}},$$

wherein $\nu_1, \ldots, \nu_{2p}$ are integers, the sum is unaffected by the addition of any integers to any one or more of the representants $r'_1, \ldots, r'_{2p}$, namely it has the same value for all sets, ν', of incongruent columns (for the modulus $\bar{g}$). If to each of any set of incongruent columns ν' we add the column $(0, \ldots, 0, \lambda_i, 0, \ldots, 0)$, all of whose elements are zero except that occupying the i-th place, which is an integer, we shall obtain another set of incongruent columns.

Suppose then in the above sum r_i is fractional. Add to every one of the incongruent sets ν' the column $(0, 0, \ldots, 1, 0, \ldots, 0)$, of which every element except the i-th is zero. In the summation everything is unaffected except the powers of $e^{-2\pi i r_i}$, which are multiplied by $e^{-2\pi i r_i}$. Hence the sum is unaffected when multiplied by $e^{-2\pi i r_i}$, and must therefore be zero.

We put down the figures for a simple case given by

$$p=1, \quad g=\begin{vmatrix} 4 & 5 \\ 1 & 2 \end{vmatrix};$$

then $gr=(4r_1+5r_2,\ r_1+2r_2)$ and the equations $gr=\nu$ give

$$\left.\begin{matrix} 4r_1+5r_2=\nu_1 \\ r_1+2r_2=\nu_2 \end{matrix}\right\} \therefore \left\{\begin{matrix} 3r_1=2\nu_1-5\nu_2 \\ 3r_2=4\nu_2-\ \nu_1\, ; \end{matrix}\right.$$

thus the values of r_1, r_2 and ν_1, ν_2 are given by the table

$r_1,\ r_2$	$0,\ 0$	$\frac{1}{3},\ \frac{1}{3}$	$\frac{2}{3},\ \frac{2}{3}$
$\nu_1,\ \nu_2$	$0,\ 0$	$3,\ 1$	$6,\ 2$

Similarly $\bar{g}r'=(4r'_1+r'_2,\ 5r'_1+2r'_2)$, and the equations $\bar{g}r'=\nu'$ give

$$\left.\begin{matrix} 4r'_1+\ r'_2=\nu'_1 \\ 5r'_1+2r'_2=\nu'_2 \end{matrix}\right\} \therefore \left\{\begin{matrix} 3r'_1=2\nu'_1-\ \nu'_2 \\ 3r'_2=4\nu'_2-5\nu'_1\, ; \end{matrix}\right.$$

thus the values of r'_1, r'_2 and ν'_1, ν'_2 are given by the table

$r'_1,\ r'_2$	$0,\ 0$	$\frac{1}{3},\ \frac{2}{3}$	$\frac{2}{3},\ \frac{1}{3}$
$\nu'_1,\ \nu'_2$	$0,\ 0$	$2,\ 3$	$3,\ 4$

Thus the sum in question is

$$\begin{aligned} &\left(e^{-2\pi i r_1}\right)^0\left(e^{-2\pi i r_2}\right)^0+\left(e^{-2\pi i r_1}\right)^2\left(e^{-2\pi i r_2}\right)^3+\left(e^{-2\pi i r_1}\right)^3\left(e^{-2\pi i r_2}\right)^4 \\ &=\left(e^{-2\pi i \nu_1}\right)^0\left(e^{-2\pi i \nu_2}\right)^0+\left(e^{-2\pi i \nu_1}\right)^{\frac{1}{3}}\left(e^{-2\pi i \nu_2}\right)^{\frac{2}{3}}+\left(e^{-2\pi i \nu_1}\right)^{\frac{2}{3}}\left(e^{-2\pi i \nu_2}\right)^{\frac{1}{3}} \\ &=1+e^{-2\pi i\,(2r_1+3r_2)}+e^{-2\pi i\,(3r_1+4r_2)}=1+e^{-\frac{2\pi i}{3}(\nu_1+2\nu_2)}+e^{-\frac{2\pi i}{3}(2\nu_1+\nu_2)}. \end{aligned}$$

For $r_1=r_2=\nu_1=\nu_2=0$, these terms are each unity; for

$$(r_1,\ r_2)=(\tfrac{1}{3},\ \tfrac{1}{3}), \quad (\nu_1,\ \nu_2)=(3,\ 1)$$

these terms are

$$1+e^{-2\pi i\,(\frac{5}{3})}+e^{-2\pi i\,(\frac{7}{3})}=1+e^{-\frac{2\pi i}{3}(2)}+e^{-\frac{2\pi i}{3}(1)}$$

or zero.

For $(r_1,\ r_2)=(\frac{2}{3},\ \frac{2}{3})$, $(\nu_1,\ \nu_2)=(6,\ 2)$, these terms are

$$1+e^{-2\pi i\,(\frac{10}{3})}+e^{-2\pi i\,(\frac{14}{3})}=1+e^{-\frac{2\pi i}{3}(1)}+e^{-\frac{2\pi i}{3}(2)}$$

or zero.

361. We give now an example of the expression of ϕ functions.

Take the case in which $p=1$, and

$$k=\begin{vmatrix} 0 & -3 \\ 3 & 0 \end{vmatrix};$$

the conditions $\bar{a}b - \bar{b}a = k$, and $\bar{g}\epsilon g = k$, if $a = (a, a')$, $b = (b, b')$, become

$$ab' - a'b = -3, \qquad g_{12}g_{21} - g_{11}g_{22} = -3;$$

taking for instance

$$g = \begin{vmatrix} 4 & 5 \\ 1 & 2 \end{vmatrix},$$

we have, if $x = (x, x')$, $x_1 = (x_1, x_1')$, and $ax + a'x' = 0$, the equation

$$ikxx_1 = 3i\,(xx_1' - x'x_1) = -\frac{3ix'x_1'}{aa_1}(a'a_1 - aa_1') = \frac{6x'x_1'}{aa_1}(\alpha\beta' - \alpha'\beta),$$

where $a = \alpha + i\beta$, $a' = \alpha' + i\beta'$. Thus, beside $ab' - a'b = -3$, we must have $\alpha\beta' > \alpha'\beta$. The quantities a, b, a', b' are otherwise arbitrary.

The equations $a = Kg$, $b = Lg$ give $(a, a') = (4K + K', 5K + 2K')$; therefore

$$3K = 2a - a', \quad 3L = 2b - b',$$
$$3K' = 4a' - 5a, \quad 3L' = 4b' - 5b;$$

further the equation $c = gC + \frac{1}{2}[g]$ gives

$$(c, c') = \begin{vmatrix} 4 & 1 \\ 5 & 2 \end{vmatrix}(C, C') + \tfrac{1}{2}(4, 10) = (4C + C' + 2,\ 5C + 2C' + 5),$$

so that

$$3C = 2c - c' + 1, \quad 3C' = 4c' - 5c - 10.$$

Also, from $K = |\,2\omega, 2\omega'\,|$, $2\pi iL = |\,2\eta, 2\eta'\,|$, with

$$\mathrm{a} = \frac{\eta}{2\omega}, \quad \mathrm{h} = \frac{\pi i}{2\omega}, \quad \mathrm{b} = 2\mathrm{h}\omega',$$

we obtain

$$\mathrm{a} = \pi i\,(2b - b')/(2a - a'), \quad \mathrm{b} = \pi i\,(4a' - 5a)/(2a - a'), \quad \mathrm{h} = 3\pi i/(2a - a').$$

If then $\vartheta(u;\, C)$ denote the theta function, with characteristic $\begin{pmatrix} C \\ -C' \end{pmatrix}$, given by

$$\vartheta(u;\, C) = \Sigma e^{\mathrm{a}u^2 + 2\mathrm{h}u\,(n+C) + \mathrm{b}\,(n+C)^2 - 2\pi iC'\,(n+C)},$$

then the Jacobian function, with a, b as periods, and c as parameter, is given by

$$3\phi(u) = \sum_{\nu'} A_{\nu'}\vartheta(u;\, C + r'),$$

where, in the three terms of the right hand, r' is in turn equal to $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 1/3 \\ 2/3 \end{pmatrix}$, $\begin{pmatrix} 2/3 \\ 1/3 \end{pmatrix}$.

The function $\phi(u)$ may in fact be considered as a theta function of the third order; its various expressions, obtainable by taking different forms for the matrix g, are transformations of one another, in the sense of Chap. XVIII. and XX.

362. The theory of the expression of a Jacobian function which has been given is for the case when $\sigma = 2p$. Suppose $\sigma < 2p$, and that we have two matrices a, b, each of p rows and σ columns, such that $\bar{a}b - \bar{b}a$, $= k$, is a skew symmetrical matrix of integers, for which $ikxx_1$ is a positive form for all values satisfying $ax = 0$, other than those for which also $a_1x = 0$, or $x = 0$; then it is possible* to determine other $2p - \sigma$ columns of quantities, and thence to construct matrices, A, B, of $2p$ columns (whereof the first σ columns are those of a, b), such that $\bar{A}B - \bar{B}A = K$ is a skew symmetrical matrix of integers for which $iKxx_1$ is positive when $Ax = 0$, except when $x = 0$ or $A_1x = 0$.

There will then correspond to the set A, B a function Φ, involving $\sqrt{|K|}$ arbitrary coefficients, such that, for integral n,

$$\Phi(u + An) = e^{2\pi iBn(u+\frac{1}{2}An) + 2\pi iCn + \sum\limits_{\alpha<\beta} K_{\alpha,\beta}n_\alpha n_\beta}\,\Phi(u).$$

The function $\phi(u)$, which is subject only to the condition that

$$\phi(u + an) = e^{2\pi ibn(u+\frac{1}{2}an) + 2\pi icn + \sum\limits_{\alpha<\beta} k_{\alpha,\beta}n_\alpha n_\beta}\,\phi(u),$$

is then obtained by regarding $\phi(u)$ as a particular case of $\Phi(u)$, in which the added columns in A, B are arbitrary except that they must be such that the necessary conditions for A, B are satisfied.

For further development the reader should consult Frobenius, *Crelle*, XCVII. (1884), pp. 16, 188, and *Crelle*, CV. (1889), p. 35.

* Frobenius, *Crelle*, XCVII. (1884), p. 24.