CHAPTER I.

THE SUBJECT OF INVESTIGATION.

1. THIS book is concerned with a particular development of the theory of the algebraic irrationality arising when a quantity y is defined in terms of a quantity x by means of an equation of the form

$$a_0 y^n + a_1 y^{n-1} + \dots + a_{n-1} y + a_n = 0,$$

wherein $a_0, a_1, \dots, a_n$ are rational integral polynomials in x. The equation is supposed to be irreducible; that is, the left-hand side cannot be written as the product of other expressions of the same rational form.

2. Of the various means by which this dependence may be represented, that invented by Riemann, the so-called Riemann surface, is throughout regarded as fundamental. Of this it is not necessary to give an account here*. But the sense in which we speak of a *place* of a Riemann surface must be explained. To a value of the independent variable x there will in general correspond n distinct values of the dependent variable y—represented by as many *places*, lying in distinct sheets of the surface. For some values of x two of these n values of y may happen to be equal: in that case the corresponding sheets of the surface may behave in one of two ways. Either they may just touch at one point without having any further connexion in the immediate neighbourhood of the point†: in which case we shall regard the point where the sheets touch as constituting two places, one in each sheet. Or the sheets may wind into one another: in which case we shall regard this winding point (or branch point) as constituting one place: this place belongs then indifferently to either sheet; the sheets here merge into one another. In the first case, if a be the value of x for which the sheets just touch, supposed for convenience of statement to be finite, and x a value

* For references see Chap. II. § 12, note.

† Such a point is called by Riemann "ein sich aufhebender Verzweigungspunkt": *Gesammelte Werke* (1876), p. 105.

very near to a, and if b be the value of y at each of the two places, also supposed finite, and y_1, y_2 be values of y very near to b, represented by points in the two sheets very near to the point of contact of the two sheets, each of $y_1 - b$, $y_2 - b$ can be expressed as a power-series in $x - a$ with integral exponents. In the second case with a similar notation each of $y_1 - b$, $y_2 - b$ can be expressed as a power-series in $(x-a)^{\frac{1}{2}}$ with integral exponents. In the first case a small closed curve can be drawn on either of the two sheets considered, to enclose the point at which the sheets touch: and the value of the integral $\frac{1}{2\pi i}\int d\log(x-a)$ taken round this closed curve will be 1; hence, adopting a definition given by Riemann*, we shall say that $x-a$ is an infinitesimal of the first order at each of the places. In the second case the attempt to enclose the place by a curve leads to a curve lying partly in one sheet and partly in the other; in fact, in order that the curve may be closed it must pass twice round the branch place. In this case the integral $\frac{1}{2\pi i}\int d\log[(x-a)^{\frac{1}{2}}]$ taken round the closed curve will be 1: and we speak of $(x-a)^{\frac{1}{2}}$ as an infinitesimal of the first order at the place. In either case, if t denote the infinitesimal, x and y are uniform functions of t in the immediate neighbourhood of the place; conversely, to each point on the surface in the immediate neighbourhood of the place there corresponds uniformly a certain value of t†. The quantity t effects therefore a conformal representation of this neighbourhood upon a small simple area in the plane of t, surrounding $t = 0$.

3. This description of a simple case will make the general case clear. In general for any finite value of x, $x = a$, there may be several, say k, branch points‡; the number of sheets that wind at these branch points may be denoted by $w_1 + 1$, $w_2 + 1$, ..., $w_k + 1$ respectively, where

$$(w_1 + 1) + (w_2 + 1) + \ldots + (w_k + 1) = n,$$

so that the case of no branch point is characterised by a zero value of the corresponding w. For instance in the first case above, notwithstanding that two of the n values of y are the same, each of $w_1, w_2, \ldots, w_k$ is zero and k is equal to n: and in the second case above, the values are $k = n - 1$, $w_1 = 1$, $w_2 = 0$, $w_3 = 0$, ..., $w_k = 0$. *In the general case each of these* k *branch points is called a place,* and at these respective places the quantities $(x-a)^{\frac{1}{w_1+1}}, \ldots, (x-a)^{\frac{1}{w_k+1}}$

* *Gesammelte Werke* (1876), p. 96.

† The limitation to the immediate neighbourhood involves that t is not necessarily a rational function of x, y.

It may be remarked that a rational function of x and y *can* be found whose behaviour in the neighbourhood of the place is the same as that of t. See for example Hamburger, *Zeitschrift f. Math. und Phys.* Bd. 16, 1871; Stolz, *Math. Ann.* 8, 1874; Harkness and Morley, *Theory of Functions*, p. 141.

‡ Cf. Forsyth, *Theory of Functions*, p. 171. Prym, *Crelle*, Bd. 70.

are infinitesimals of the first order. For the infinite value of x we shall similarly have n or a less number of places and as many infinitesimals, say $\left(\frac{1}{x}\right)^{\frac{1}{w_1+1}}, \ldots, \left(\frac{1}{x}\right)^{\frac{1}{w_r+1}}$, where $(w_1+1)+\ldots+(w_r+1)=n$. And as in the particular cases discussed above, the infinitesimal t thus defined for every place of the surface has the two characteristics that for the immediate neighbourhood of the place x and y are uniquely expressible thereby (in series of integral powers), and conversely t is a uniform function of position on the surface in this neighbourhood. Both these are expressed by saying that t effects a reversible conformal representation of this neighbourhood upon a simple area enclosing $t=0$. It is obvious of course that quantities other than t have the same property.

A place of the Riemann surface will generally be denoted by a single letter. And in fact a place (x, y) will generally be called the place x. When we have occasion to speak of the (n or less) places where the independent variable x has the same value, a different notation will be used.

4. We have said that the subject of enquiry in this book is a certain algebraic irrationality. We may expect therefore that the theory is practically unaltered by a rational transformation of the variables x, y which is of a reversible character. Without entering here into the theory of such transformations, which comes more properly later, in connexion with the theory of correspondence, it is necessary to give sufficient explanations to make it clear that the functions to be considered belong to a whole class of Riemann surfaces and are not the exclusive outcome of that one which we adopt initially.

Let ξ be any one of those uniform functions of position on the fundamental (undissected) Riemann surface whose infinities are all of finite order. Such functions can be expressed rationally by x and y*. For that reason we shall speak of them shortly as the rational functions of the surface. The order of infinity of such a function at any place of the surface where the function becomes infinite is the same as that of a certain *integral* power of the inverse $\frac{1}{t}$ of the infinitesimal at that place. The sum of these orders of infinity for all the infinities of the function is called the order of the function. The number of places at which the function ξ assumes any other value α is the same as this order: it being understood that a place at which $\xi-\alpha$ is zero in a finite ratio to the rth order of t is counted as r places at which ξ is equal to α†. Let ν be the order of ξ. Let η be another rational function of

* Forsyth, *Theory of Functions*, p. 370.

† For the integral $\frac{1}{2\pi i}\int d\log(\xi-\alpha)$, taken round an infinity of $\log(\xi-\alpha)$, is equal to the order of zero of $\xi-\alpha$ at the place, or to the negative of the order of infinity of ξ, as the case may be. And the sum of the integrals for all such places is equal to the value round the boundary of the surface—which is zero. Cf. Forsyth, *Theory of Functions*, p. 372.

order μ. Take a plane whose real points represent all the possible values of ξ in the ordinary way. To any value of ξ, say $\xi = \alpha$, will correspond ν positions $X_1, \ldots, X_\nu$ on the original Riemann surface, those namely where ξ is equal to α: it is quite possible that they lie at less than ν places of the surface. The values of η at $X_1, \ldots, X_\nu$ may or may not be different. Let H denote any definite rational symmetrical function of these ν values of η. Then to each position of α in the ξ plane will correspond a perfectly unique value of H, namely, H is a one-valued function of ξ. Moreover, since η and ξ are rational functions on the original surface, the character of H for values of ξ in the immediate neighbourhood of a value α, for which H is infinite, is clearly the same as that of a *finite* power of $\xi - \alpha$. Hence H is a rational function of ξ. Hence, if H_r denote the sum of the products of the values of η at $X_1, \ldots, X_\nu$, r together, η satisfies an equation

$$\eta^\nu - \eta^{\nu-1} H_1 + \eta^{\nu-2} H_2 - \ldots + (-)^\nu H_\nu = 0,$$

whose coefficients are rational functions of ξ.

It is conceivable that the left side of this equation can be written as the product of several factors each rational in ξ and η. If possible let this be done. Construct over the ξ plane the Riemann surfaces corresponding to these irreducible factors, η being the dependent variable and the various surfaces lying above one another in some order. It is a known fact, already used in defining the order of a rational function on a Riemann surface, that the values of η represented by *any* one of these superimposed surfaces include all possible values—each value in fact occurring the same number of times on each surface. To any place of the original surface, where ξ, η have definite values, and to the neighbourhood of this place, will correspond therefore a definite place (ξ, η) (and its neighbourhood) on each of these superimposed surfaces. Let $\eta_1, \ldots, \eta_r$ be the values of η belonging, on one of these surfaces, to a value of ξ: and $\eta_1', \ldots, \eta_s'$ the values belonging to the same value of ξ on another of these surfaces. Since for each of these surfaces there are only a finite number of values of ξ at which the values of η are not all different, we may suppose that all these r values on the one surface are different from one another, and likewise the s values on the other surface. Since each of the pairs of values $(\xi, \eta_1), \ldots, (\xi, \eta_r)$ must arise on both these surfaces, it follows that the values $\eta_1, \ldots, \eta_r$ are included among $\eta_1', \ldots, \eta_s'$. Similarly the values $\eta_1', \ldots, \eta_s'$ are included among $\eta_1, \ldots, \eta_r$. Hence these two sets are the same and $r = s$. Since this is true for an infinite number of values of ξ, it follows that these two surfaces are merely repetitions of one another. The same is true for every such two surfaces. Hence r is a divisor of ν and the equation

$$\eta^\nu - H_1 \eta^{\nu-1} + \ldots + (-)^\nu H_\nu = 0,$$

when reducible, is the ν/rth power of a rational equation of order r in η. It will be sufficient to confine our attention to one of the factors and the (ξ, η)

surface represented thereby. Let now $X_1, \ldots, X_\nu$ be the places on the original surface where ξ has a certain value. Then the values of η at $X_1, \ldots, X_\nu$ will consist of ν/r repetitions of r values, these r values being different from one another except for a finite number of values of ξ. Thus to any place (ξ, η) on one of the ν/r derived surfaces will correspond ν/r places on the original surface, those namely where the pair (ξ, η) take the supposed values. Denote these by $P_1, P_2, \ldots$. Let Y be any rational symmetrical function of the ν/r pairs of values $(x_1, y_1), (x_2, y_2), \ldots$, which the fundamental variables x, y of the original surface assume at $P_1, P_2, \ldots$. Then to any pair of values (ξ, η) will correspond only one value of Y—namely, Y is a one-valued function on the (ξ, η) surface. It has clearly also only finite orders of infinity. Hence Y is a rational function of ξ, η. In particular $x_1, x_2, \ldots$ are the roots of an equation whose coefficients are rational in ξ, η—as also are $y_1, y_2, \ldots$.

There exists therefore a correspondence between the (ξ, η) and (x, y) surfaces—of the kind which we call a $\left(1, \frac{\nu}{r}\right)$ correspondence: to every place of the (x, y) surface corresponds one place of the (ξ, η) surface; to every place of this surface correspond $\frac{\nu}{r}$ places of the (x, y) surface.

The case which most commonly arises is that in which the rational irreducible equation satisfied by η is of the νth degree in η: then only one place of the original surface is associated with any place of the new surface. In that case, as will appear, the new surface is as general as the original surface. Many advantages may be expected to accrue from the utilization of that fact. We may compare the case of the reduction of the general equation of a conic to an equation referred to the principal axes of the conic.

5. The following method* is theoretically effective for the expression of x, y in terms of ξ, η.

Let the rational expression of ξ, η in terms of x, y be given by

$$\phi(x, y) - \xi\psi(x, y) = 0, \quad \vartheta(x, y) - \eta\chi(x, y) = 0,$$

and let the rational result of eliminating x, y between these equations and the initial equation connecting x, y be denoted by $F(\xi, \eta) = 0$, each of $\phi, \ldots, \chi, F$ denoting integral polynomials. Let two terms of the expression $\phi(x, y) - \xi\psi(x, y) = 0$ be $ax^ry^s - \xi bx^{r'}y^{s'}$. This expression and therefore all others involved will be unaltered if a, b be replaced by such quantities $a+h$, $b+k$, that $hx^ry^s = \xi kx^{r'}y^{s'}$. In a formal sense this changes $F(\xi, \eta)$ into

$$F + \frac{1}{\underline{|\lambda}}\left[\frac{\partial^\lambda F}{\partial a^\lambda} h^\lambda + \binom{\lambda}{1} \frac{\partial^\lambda F}{\partial a^{\lambda-1}\partial b} h^{\lambda-1}k + \binom{\lambda}{2} \frac{\partial^\lambda F}{\partial a^{\lambda-2}\partial b^2} h^{\lambda-2}k^2 + \ldots + \frac{\partial^\lambda F}{\partial b^\lambda} k^\lambda\right] + \ldots\ldots$$

where $\lambda \geqq 1$, and F is such that all differential coefficients of it in regard to a and b of order less than λ are identically zero.

Hence the term within the square brackets in this expression must be zero. If it is possible, choose now $r = r' + 1$ and $s = s'$, so that $k = hx/\xi$.

* Salmon's *Higher Algebra* (1885), p. 97, § 103.

Then we obtain the equation

$$x^\lambda \frac{\partial^\lambda F}{\partial b^\lambda} - \binom{\lambda}{1} x^{\lambda-1}\xi \frac{\partial^\lambda F}{\partial b^{\lambda-1}\partial a} + \ldots\ldots + (-)^\lambda \xi^\lambda \frac{\partial^\lambda F}{\partial a^\lambda} = 0.$$

This is an equation of the form above referred to, by which x is determinate from ξ and η. And y is similarly determinate.

It will be noticed that the rational expression of x, y by ξ, η, when it is possible from the equations

$$\phi(x, y) - \xi\psi(x, y) = 0, \quad \vartheta(x, y) - \eta\chi(x, y) = 0, \quad f(x, y) = 0,$$

will not be possible, in general, from the first two equations: it is only the places x, y satisfying the equation $f(x, y) = 0$ which are rationally obtainable from the places ξ, η satisfying the equation $F(\xi, \eta) = 0$. There do exist transformations, rationally reversible, subject to no such restriction. They are those known as Cremona-transformations*. They can be compounded by reapplication of the transformation $x : y : 1 = \eta : \xi : \xi\eta$.

We may give an example of both of these transformations—

For the surface

$$y^5 - 5y^3(x^2+x+1) + 5y(x^2+x+1)^2 - 2x(x^2+x+1)^2 = 0$$

the function $\xi = y^2/(x^2+x+1)$ is of order 2, being infinite at the places where $x^2+x+1=0$, in each case like $(x-a)^{-\frac{1}{5}}$, and the function $\eta = x/y$ is of order 4, being infinite at the places $x^2+x+1=0$, in each case like $(x-a)^{-\frac{2}{5}}$, a being the value of x at the place.

From the given equation we immediately find, as the relation connecting ξ and η,

$$2\eta - \xi^2 + 5\xi - 5 = 0,$$

and infer, since the equation formed as in the general statement above should be of order 2 in η, that this general equation will be

$$(2\eta - \xi^2 + 5\xi - 5)^2 = 0.$$

Thence in accordance with that general statement we infer that to each place (ξ, η) on the new surface should correspond two places of the original surface: and in fact these are obviously given by the equations

$$\eta^2\xi = x^2/(x^2+x+1), \quad y = x/\eta.$$

If however we take

$$\xi = y^2/(x^2+x+1), \quad \eta = y/(x-\omega^2),$$

where ω is an imaginary cube root of unity, so that η is a function of order 3, these equations are reversible independently of the original equation, giving in fact

$$x = (\omega\xi - \omega^2\eta^2)/(\xi - \eta^2), \quad y = (\omega - \omega^2)\,\xi\eta/(\xi - \eta^2),$$

and we obtain the surface

$$\eta^2 - \tfrac{1}{2}(1-\omega^2)\,\eta\xi\,(\xi^2 - 5\xi + 5) - \omega^2\xi = 0,$$

having a (1, 1) correspondence with the original one.

It ought however to be remarked that it is generally possible to obtain reversible transformations which are not Cremona-transformations.

6. When a surface (x, y) is (1, 1) related to a (ξ, η) surface, the deficiencies of the surfaces, as defined by Riemann by means of the connectivity, must clearly be the same.

* See Salmon, *Higher Plane Curves* (1879), § 362, p. 322.

It is instructive to verify this from another point of view*.—Consider at how many places on the original surface the function $\frac{d\xi}{dx}$ is zero. It is infinite at the places where ξ is infinite: suppose for simplicity that these are separated places on the original surface or in other words are infinities of the first order, and are not at the branch points of the original surface. At a pole of ξ, $\frac{d\xi}{dx}$ is infinite twice. It is infinite like $\frac{1}{t^w}$ at a branch place (α) where $x - \alpha = t^{w+1}$: namely it is infinite $\Sigma w = 2n + 2p - 2$ times† at the branch places of the original surface. It is zero $2n$ times at the infinite places of the original surface. There remain therefore $2\nu + 2n + 2p - 2 - 2n = 2\nu + 2p - 2$ places where $\frac{d\xi}{dx}$ is zero. If a branch place of the original surface be a pole of ξ, and ξ be there infinite like $\frac{1}{t}$, $\frac{d\xi}{dx}$ is infinite like $\frac{1}{t^2 . t^w}$, namely $2 + w$ times: the total number of infinities of $\frac{d\xi}{dx}$ will therefore be the same as before. Now at a finite place of the original surface where $\frac{d\xi}{dx} = 0$, there are two consecutive places for which ξ has the same value. Since $\frac{\nu}{r} = 1$ they can only arise from consecutive places of the new surface for which ξ has the same value. The only consecutive places of a surface for which this is the case are the branch places. Hence† there are $2\nu + 2p - 2$ branch places of the new surface. This shews that the new surface is of deficiency p.

When ν/r is not equal to 1, the case is different. The consecutive places of the old surface, for which ξ has the same value, may either be those arising from consecutive places of the new surface—or may be what we may call accidental coincidences among the ν/r places which correspond to *one* place of the new surface. Conversely, to a branch place of the new surface, characterised by the same value for ξ for consecutive places‡, will correspond ν/r places on the old surface where ξ has the same value for consecutive places. In fact to two very near places of the new surface will correspond ν/r pairs each of very near places on the old surface. If then C denote the number of places on the old surface at which two of the ν/r places corresponding to a place on the new surface happen to coincide, and w' the number of branch points of the new surface, we have the equation

$$w' \frac{\nu}{r} + C = 2\nu + 2p - 2,$$

* Compare the interesting geometrical account, Salmon, *Higher Plane Curves* (1879), p. 326, § 364, and the references there given.

† Forsyth, *Theory of Functions*, p. 349.

‡ Namely, near such a branch place $\xi = a$, $\xi - a$ is zero of higher order than the first.

and if p' be the deficiency of the new surface (of r sheets), this leads to the equation

$$(2r + 2p' - 2)\frac{\nu}{r} + C = 2\nu + 2p - 2,$$

from which

$$C = 2p - 2 - (2p' - 2)\frac{\nu}{r}.$$

*Corollary**. If $p = p'$, then $C = (2p-2)\left(1 - \frac{\nu}{r}\right)$. Thus $\frac{\nu}{r} \ngtr 1$, so that $C = 0$, and the correspondence is reversible.

We have, herein, excluded the case when some of the poles of ξ are of higher than the first order. In that case the new surface has branch places at infinity. The number of finite branch places is correspondingly less. The reader can verify that the general result is unaffected.

Ex. In the example previously given (§ 5) shew that the function ξ takes any given value at *two* points of the original surface (other than the branch places where it is infinite), η having the same value for these two points, and that there are *six* places at which these two places coincide. (These are the place ($x=0$, $y=0$) and the five places where $x = -2$.)

There is one remark of considerable importance which follows from the theory here given. We have shewn that the number of places of the (x, y) surface which correspond to one place of the (ξ, η) surface is $\frac{\nu}{r}$, where ν is the order of ξ and r is not greater than ν, being the number of sheets of the (ξ, η) surface; hence, *if there were a function ξ of order* 1 *the correspondence would be reversible and therefore the original surface would be of deficiency* 1.

7. This notion of the transformation of a Riemann surface suggests an inference of a fundamental character.

The original equation contains only a finite number of terms: the original surface depends therefore upon a finite number of constants, namely, the coefficients in the equation. But conversely it is not necessary, in order that the equation be reversibly transformable into another given one, that the equation of the new surface contain as many constants as that of the original surface. For we may hope to be able to choose a transformation whose coefficients so depend on the coefficients of the original equation as to reduce this number. If we speak of all surfaces of which any two are connected by a rational reversible transformation as belonging to the same class†, it becomes a question whether there is any limit to the reduction obtainable, by rational reversible transformation, in the number of constants in the equation of a surface of the class.

* See Weber, *Crelle*, 76, 345.

† So that surfaces of the same class will be of the same deficiency.

It will appear in the course of the book* that there is a limit, and that the various classes of surfaces of given deficiency are of essentially different character according to the least number of constants upon which they depend. Further it will appear, that the most general class of deficiency p is characterised by $3p-3$ constants when $p>1$—the number for $p=1$ being one, and for $p=0$ none.

For the explanatory purposes of the present Chapter we shall content ourselves with the proof of the following statement—When a surface is reversibly transformed as explained in this Chapter, we cannot, even though we choose the new independent variable ξ to contain a very large number of disposeable constants, prescribe the position of all the branch points of the new surface; there will be $3p-3$ of them whose position is settled by the position of the others. Since the correspondence is reversible we may regard the new surface as fundamental, equally with the original surface. We infer therefore that the original surface depends on $3p-3$ parameters—*or on less*, for the $3p-3$ undetermined branch points of the new surface may have mutually dependent positions.

In order to prove this statement we recall the fact that a function of order Q contains† $Q-p+1$ linearly entering constants when its poles are prescribed: it may contain more for values of $Q<2p-1$, but we shall not thereby obtain as many constants as if we suppose $Q>2p-2$ and large enough. Also the Q infinities are at our disposal. We can then presumably dispose of $2Q-p+1$ of the branch points of the new surface. But these are, in number, $2Q+2p-2$ when the correspondence is reversible. Hence we can dispose of all but $2Q+2p-2-(2Q-p+1)=3p-3$ of the branch points of the new surface‡.

Ex. 1. The surface associated with the equation

$$y^2=x(1-x)(1-k^2x)(1-\lambda^2x)(1-\mu^2x)(1-\nu^2x)(1-\rho^2x)$$

is of deficiency 3. It depends on $5=2p-1$ parameters, κ^2, λ^2, μ^2, ν^2, ρ^2.

Ex. 2. The surface associated with the equation

$$y^3+y^2(x,\ 1)_1+y(x,\ 1)_2+(x,\ 1)_4=0,$$

wherein the coefficients are integral polynomials of the orders specified by the suffixes, is of deficiency 3. Shew that it can be transformed to a form containing only $5=2p-1$ parametric constants.

* See the Chapters on the geometrical theory and on the inversion of Abelian Integrals. The reason for the exception in case $p=0$ or 1 will appear most clearly in the Chapter on the self-correspondence of a Riemann surface. But it is a familiar fact that the elliptic functions which can be constructed for a surface of deficiency 1 depend upon one parameter, commonly called the modulus: and the trigonometrical functions involve no such parameter.

† Forsyth, p. 459. The theorems here quoted are considered in detail in Chapter III. of the present book.

‡ Cf. Riemann, *Ges. Werke* (1876), p. 113. Klein, *Ueber Riemann's Theorie* (Leipzig, Teubner, 1882), p. 65.

8. But there is a case in which this argument fails. If it be possible to transform the original surface into itself by a rational reversible transformation involving r parameters, any r places on the surface are effectively equivalent with, as being transformable into, any other r places. Then the Q poles of the function ξ do not effectively supply Q but only $Q-r$ disposeable constants with which to fix the new surface. So that there are $3p-3+r$ branch points of the new surface which remain beyond our control. In this case we may say that all the surfaces of the class contain $3p-3$ disposeable parameters *beside* r parameters which remain indeterminate and serve to represent the possibility of the self-transformation of the surface. It will be shewn in the chapter on self-transformation that the possibility only arises for $p=0$ or $p=1$, and that the values of r are, in these cases, respectively 3 and 1. We remark as to the case $p=0$ that when the fundamental surface has only one sheet it can clearly be transformed into itself by a transformation involving three constants $x=\dfrac{a\xi+b}{c\xi+d}$: and in regard to $p=1$, the case of elliptic functions, that effectively a point represented by the elliptic argument u is equivalent to any other point represented by an argument $u+\gamma$. For instance a function of two poles is

$$F_{\alpha,\beta}=\frac{A}{\wp\left(u-\dfrac{\alpha+\beta}{2}\right)-\wp\dfrac{\alpha-\beta}{2}}+B,$$

and clearly $F_{\alpha,\beta}$ has the same value at u as has $F_{\alpha+\gamma,\beta+\gamma}$ at $u+\gamma$: so that the poles (α, β) are not, so far as absolute determinations are concerned, effective for the determination of more than one point.

9. The fundamental equation

$$a_0y^n+a_1y^{n-1}+\ldots+a_n=0,$$

so far considered as associated with a Riemann surface, may also be regarded as the equation of a plane curve: and it is possible to base our theory on the geometrical notions thus suggested. Without doing this we shall in the following pages make frequent use of them for purposes of illustration. It is therefore proper to remind the reader of some fundamental properties*.

The branch points of the surface correspond to those points of the curve where a line $x=$ constant meets the curve in two or more *consecutive* points: as for instance when it touches the curve, or passes through a cusp. On the other hand a double point of the curve corresponds to a point on the surface where two sheets just touch without further connexion. Thus the branch place of the surface which corresponds to a cusp is really a different singularity to that which corresponds to a place where the curve is touched by a

* Cf. Forsyth, *Theory of Functions*, p. 355 etc. Harkness and Morley, *Theory of Functions*, p. 273 etc.

line $x =$ constant, being obtained by the coincidence of an ordinary branch place with such a place of the Riemann surface as corresponds to a double point of the curve.

Properties of either the Riemann surface or a plane curve are, in the simpler cases, immediately transformed. For instance, by Plücker's formulae for a curve, since the number of tangents from any point is

$$t = (n-1)n - 2\delta - 3\kappa,$$

where n is the aggregate order in x and y, it follows that the number of branch places of the corresponding surface is

$$\begin{aligned} w = t + \kappa &= (n-1)n - 2(\delta + \kappa) \\ &= 2n - 2 + 2\{\tfrac{1}{2}(n-1)(n-2) - \delta - \kappa\}. \end{aligned}$$

Thus since $w = 2n - 2 + 2p$, the deficiency of the surface is

$$\tfrac{1}{2}(n-1)(n-2) - \delta - \kappa,$$

namely the number which is ordinarily called the deficiency of the curve.

To the theory of the birational transformation of the surface corresponds a theory of the birational transformation of plane curves. For example, the branch places of the new surface obtained from the surface $f(x, y) = 0$ by means of equations of the form $\phi(x, y) - \xi\psi(x, y) = 0$, $\vartheta(x, y) - \eta\chi(x, y) = 0$ will arise for those values of ξ for which the curve $\phi(x, y) - \xi\psi(x, y) = 0$ touches $f(x, y) = 0$. The condition this should be so, called the tact invariant, is known to involve the coefficients of $\phi(x, y) - \xi\psi(x, y) = 0$, and therefore in particular to involve ξ, to a degree* $n(n-3) - 2\delta - 3\kappa + 2nn'$, where n' is the order of $\phi(x, y) - \xi\psi(x, y) = 0$. Branch places of the new surface also arise corresponding to the cusps of the original curve. The total number is therefore $n(n-3) - 2\delta - 2\kappa + 2nn' = 2p - 2 + 2nn'$. Now nn' is the number of intersections of the curves $f(x, y) = 0$ and $\phi(x, y) - \xi\psi(x, y) = 0$, namely it is the number of values of η arising for any value of ξ, and is thus the number of sheets of the new surface, which we have previously denoted by ν: so that the result is as before.

In these remarks we have assumed that the dependent variable occurs to the order which is the highest aggregate order in x and y together—and we have spoken of this as the order of the curve. And in regarding two curves as intersecting in a number of points equal to the product of their orders we have allowed count of branches of the curve which are entirely at infinity. Some care is necessary in this regard. In speaking of the Riemann surface represented by a given equation it is intended, unless the contrary be stated, that such infinite branches are unrepresented. As an example the curve $y^2 = (x, 1)_6$ may be cited.

Ex. Prove that if from any point of a curve, ordinary or multiple, or from a point not on the curve, t be the number of tangents which can be drawn other than those touching

* See Salmon, *Higher Plane Curves* (1879), p. 81.

at the point, and κ be the number of cusps of the curve—and if ν be the number of points other than the point itself in which the curve is intersected by an arbitrary line through the point—then $t+\kappa-2\nu$ is independent of the position of the point. If the equation of the variable lines through the point be written $u-\xi v=0$, interpret the result by regarding the curve as giving rise to a Riemann surface whose independent variable is ξ*.

10. The geometrical considerations here referred to may however be stated with advantage in a very general manner.

In space of any (k) dimensions let there be a curve—(a one-dimensionality). Let points on this curve be given by the ratios of the $k+1$ homogeneous variables $x_1, \ldots, x_{k+1}$. Let u, v be any two rational integral homogeneous functions of these variables of the same order. The locus $u-\xi v=0$ will intersect the curve in a certain number, say ν, points—*we assume the curve to be such that this is the same for all values of* ξ, *and is finite.* Let all the possible values of ξ be represented by the real points of an infinite plane in the ordinary way. Let w, t be any two other integral functions of the coordinates of the same order. The values of $\eta=\dfrac{w}{t}$ at the points where $u-\xi v=0$ cuts the curve for any specified value of ξ will be ν in number. As before it follows thence that η satisfies an algebraic equation of order ν whose coefficients are one-valued functions of ξ. Since η can only be infinite to a finite order it follows that these coefficients are rational functions of ξ. Thence we can construct a Riemann surface, associated with this algebraic equation connecting ξ and η, such that every point of the curve gives rise to a place of the surface. In all cases in which the converse is true we may regard the curve as a representation of the surface, or conversely.

Thus such curves in space are divisible into sets according to their deficiency. And in connexion with such curves we can construct all the functions with which we deal upon a Riemann surface.

Of these principles sufficient account will be given below (Chapter VI.): familiar examples are the space cubic, of deficiency zero, and the most general space quartic of deficiency 1 which is representable by elliptic functions.

11. In this chapter we have spoken primarily of the algebraic equation —and of the curve or the Riemann surface as determined thereby. But this is by no means the necessary order. If the Riemann surface be given, the algebraic equation can be determined from it—and in many forms, according to the function selected as dependent variable (y). It is necessary to keep this in view in order fully to appreciate the generality of Riemann's methods. For instance, we may start with a surface in space whose shape is that of an

* The reader who desires to study the geometrical theory referred to may consult:— Cayley, *Quart. Journal*, VII.; H. J. S. Smith, *Proc. Lond. Math. Soc.* VI.; Noether, *Math. Annal.* 9; Brill, *Math. Annal.* 16; Brill u. Noether, *Math. Annal.* 7.

anchor ring*, and construct upon this surface a set of elliptic functions. Or we may start with the surface on a plane which is exterior to two circles drawn upon the plane, and construct for this surface a set of elliptic functions. Much light is thrown upon the functions occurring in the theory by thus considering them in terms of what are in fact different independent variables. And further gain arises by going a step further. The infinite plane upon which uniform functions of a single variable are represented may be regarded as an infinite sphere; and such surfaces as that of which the anchor ring above is an example may be regarded as generalizations of that simple case. Now we can treat of *branches* of a multiform function without the use of a Riemann surface, by supposing the branch points of the function marked on a single infinite plane and suitably connected by barriers, or cuts, across which the independent variable is supposed not to pass. In the same way, for any general Riemann surface, we may consider branches of functions which are not uniform upon that surface, the branches being separated by drawing barriers upon the surface. The properties obtained will obviously generalize the properties of the functions which are uniform upon the surface.

* Forsyth, p. 318; Riemann, *Ges. Werke* (1876), pp. 89, 415.