CHAPTER XVII

SPECIAL INFINITE DEVELOPMENTS

171. The trigonometric functions. If m is an odd integer, say $m = 2n + 1$, De Moivre's Theorem (§ 72) gives

$$\frac{\sin m\phi}{m \sin \phi} = \cos^{2n}\phi - \frac{(m-1)(m-2)}{3!}\cos^{2n-2}\phi \sin^2\phi + \cdots, \qquad (1)$$

where by virtue of the relation $\cos^2\phi = 1 - \sin^2\phi$ the right-hand member is a polynomial of degree n in $\sin^2\phi$. From the left-hand side it is seen that the value of the polynomial is 1 when $\sin\phi = 0$ and that the n roots of the polynomials are

$$\sin^2 \pi/m, \qquad \sin^2 2\pi/m, \qquad \cdots, \qquad \sin^2 n\pi/m.$$

Hence the polynomial may be factored in the form

$$\frac{\sin m\phi}{m \sin \phi} = \left(1 - \frac{\sin^2\phi}{\sin^2 \pi/m}\right)\left(1 - \frac{\sin^2\phi}{\sin^2 2\pi/m}\right)\cdots\left(1 - \frac{\sin^2\phi}{\sin^2 n\pi/m}\right). \qquad (2)$$

If the substitutions $\phi = x/m$ and $\phi = ix/m$ be made,

$$\frac{\sin x}{m \sin x/m} = \left(1 - \frac{\sin^2 x/m}{\sin^2 \pi/m}\right)\left(1 - \frac{\sin^2 x/m}{\sin^2 2\pi/m}\right)\cdots\left(1 - \frac{\sin^2 x/m}{\sin^2 n\pi/m}\right), \qquad (3)$$

$$\frac{\sinh x}{m \sinh x/m} = \left(1 + \frac{\sinh^2 x/m}{\sin^2 \pi/m}\right)\left(1 + \frac{\sinh^2 x/m}{\sin^2 2\pi/m}\right)\cdots\left(1 + \frac{\sinh^2 x/m}{\sin^2 n\pi/m}\right). \qquad (3')$$

Now if m be allowed to become infinite, passing through successive odd integers, these equations remain true and it would appear that the limiting relations would hold:

$$\frac{\sin x}{x} = \left(1 - \frac{x^2}{\pi^2}\right)\left(1 - \frac{x^2}{2^2\pi^2}\right)\cdots = \prod_1^\infty \left(1 - \frac{x^2}{k^2\pi^2}\right), \qquad (4)$$

$$\frac{\sinh x}{x} = \left(1 + \frac{x^2}{\pi^2}\right)\left(1 + \frac{x^2}{2^2\pi^2}\right)\cdots = \prod_1^\infty \left(1 + \frac{x^2}{k^2\pi^2}\right), \qquad (4')$$

since $$\lim_{m=\infty} \frac{\sin^2 \frac{x}{m}}{\sin^2 \frac{k\pi}{m}} = \lim_{m=\infty} \frac{\left(\frac{x}{m} - \frac{1}{6}\frac{x^3}{m^3} + \cdots\right)^2}{\left(\frac{k\pi}{m} - \frac{1}{6}\left(\frac{k\pi}{m}\right)^3 + \cdots\right)^2} = \frac{x^2}{k^2\pi^2}.$$

In this way *the expansions into infinite products*

$$\sin x = x \prod_1^\infty \left(1 - \frac{x^2}{k^2\pi^2}\right), \qquad \sinh x = x \prod_1^\infty \left(1 + \frac{x^2}{k^2\pi^2}\right) \tag{5}$$

would be found. As the theorem that the limit of a product is the product of the limits holds in general only for finite products, the process here followed must be justified in detail.

For the justification the consideration of $\sinh x$, which involves only positive quantities, is simpler. Take the logarithm and split the sum into two parts

$$\log \frac{\sinh x}{m \sinh \frac{x}{m}} = \sum_1^p \log \left(1 + \frac{\sinh^2 \frac{x}{m}}{\sin^2 \frac{k\pi}{m}}\right) + \sum_{p+1}^n \log \left(1 + \frac{\sinh^2 \frac{x}{m}}{\sin^2 \frac{k\pi}{m}}\right).$$

As $\log(1+\alpha) < \alpha$, the second sum may be further transformed to

$$R = \sum_{p+1}^n \log \left(1 + \frac{\sinh^2 \frac{x}{m}}{\sin^2 \frac{k\pi}{m}}\right) < \sum_{p+1}^n \frac{\sinh^2 \frac{x}{m}}{\sin^2 \frac{k\pi}{m}} = \sinh^2 \frac{x}{m} \sum_{p+1}^n \frac{1}{\sin^2 \frac{k\pi}{m}}.$$

Now as $n < \frac{1}{2} m$, the angle $k\pi/m$ is less than $\frac{1}{2}\pi$, and $\sin \xi > 2\xi/\pi$ for $\xi < \frac{1}{2}\pi$, by Ex. 28, p. 11. Hence

$$R < \sinh^2 \frac{x}{m} \sum_{p+1}^n \frac{m^2}{4k^2} = \frac{m^2}{4} \sinh^2 \frac{x}{m} \sum_{p+1}^n \frac{1}{k^2} < \frac{m^2}{4} \sinh^2 \frac{x}{m} \int_p^\infty \frac{dk}{k^2}.$$

Hence
$$\log \frac{\sinh x}{m \sinh \frac{x}{m}} - \sum_1^p \left(1 + \frac{\sinh^2 \frac{x}{m}}{\sin^2 \frac{k\pi}{m}}\right) < \frac{m^2}{4p} \sinh^2 \frac{x}{m}.$$

Now let m become infinite. As the sum on the left is a finite, the limit is simply

$$\log \frac{\sinh x}{x} - \sum_1^p \left(1 + \frac{x^2}{k^2\pi^2}\right) < \frac{x^2}{4p}; \text{ and } \log \frac{\sinh x}{x} = \sum_1^\infty \left(1 + \frac{x^2}{k^2\pi^2}\right)$$

then follows easily by letting p become infinite. Hence the justification of (4′).

By the differentiation of the series of logarithms of (5),

$$\log \frac{\sin x}{x} = \sum_1^\infty \log \left(1 - \frac{x^2}{k^2\pi^2}\right), \qquad \log \frac{\sinh x}{x} = \sum_1^\infty \log \left(1 + \frac{x^2}{k^2\pi^2}\right), \tag{6}$$

the expressions of $\cot x$ and $\coth x$ in series of fractions

$$\cot x = \frac{1}{x} - \sum_1^\infty \frac{2x}{k^2\pi^2 - x^2}, \qquad \coth x = \frac{1}{x} + \sum_1^\infty \frac{2x}{k^2\pi^2 + x^2} \tag{7}$$

are found. And the differentiation is legitimate if these series converge uniformly. For the hyperbolic function the uniformity of the convergence follows from the M-test

$$\frac{1}{k^2\pi^2 + x^2} < \frac{1}{k^2\pi^2}, \quad \text{and} \quad \sum \frac{1}{k^2\pi^2} \text{ converges.}$$

The accuracy of the series for $\cot x$ may then be inferred by the substitution of ix for x instead of by direct examination. As

$$\frac{-2x}{k^2\pi^2 - x^2} = \frac{1}{x - k\pi} + \frac{1}{x + k\pi}, \qquad \cot x = \sum_{-\infty}^{+\infty} \frac{1}{x - k\pi}. \tag{8}$$

In this expansion, however, it is necessary still to associate the terms for $k = +n$ and $k = -n$; for each of the series for $k > 0$ and for $k < 0$ diverges.

172. In the series for $\coth x$ replace x by $\frac{1}{2}x$. Then, by (22), p. 447,

$$\frac{x}{2}\coth\frac{x}{2} = 1 + \sum_1^\infty \frac{2x^2}{4k^2\pi^2 + x^2} = 1 + \sum_1^\infty B_{2n}\frac{x^{2n}}{2n!}. \tag{9}$$

If the first series can be arranged according to powers of x, an expression for B_{2n} will be found. Consider the identity

$$\frac{t}{1+t} = -\sum_{p=1}^{n-1}(-t)^p - \frac{(-t)^n}{1+t} = -\sum_1^{n-1}(-t)^p - \theta(-t)^n,$$

which is derived by division and in which θ is a proper fraction if t is positive. Substitute $t = x^2/4k^2\pi^2$; then

$$\frac{x^2}{4k^2\pi^2 + x^2} = -\sum_1^{n-1}\left(-\frac{x^2}{4k^2\pi^2}\right)^p - \theta_k\left(-\frac{x^2}{4k^2\pi^2}\right)^n,$$

$$\frac{x}{2}\coth\frac{x}{2} - 1 = -2\sum_{k=1}^{\infty}\left[\sum_{p=1}^{n-1}\left(\frac{-x^2}{4k^2p^2}\right)^p - \theta_k\left(\frac{-x^2}{4k^2\pi^2}\right)^n\right]$$

$$= -2\sum_{p=1}^{n-1}\left[\left(\frac{-x^2}{4\pi^2}\right)^p\sum_{k=1}^{\infty}\frac{1}{k^{2p}}\right] - 2\theta\left(\frac{-x^2}{4\pi^2}\right)^n\sum_{k=1}^{\infty}\frac{1}{k^{2n}}.*$$

Let $$\sum_1^\infty \frac{1}{k^{2p}} = 1 + \frac{1}{2^{2p}} + \frac{1}{3^{2p}} + \cdots = S_{2p}.$$

$$\frac{x}{2}\coth\frac{x}{2} - 1 = -2\sum_1^{n-1} S_{2p}\left(\frac{-x^2}{4\pi^2}\right)^p - 2\theta S_{2n}\left(\frac{-x^2}{4\pi^2}\right)^n.$$

* The θ is still a proper fraction since each θ_k is. The interchange of the order of summation is legitimate because the series would still converge if all signs were positive, since Σk^{-2p} is convergent.

As S_{2n} approaches 1 when n becomes infinite, the last term approaches 0 if $x < 2\pi$, and the identical expansions are

$$2\sum_1^\infty S_{2p}(-1)^{p-1}\frac{x^{2p}}{(2\pi)^{2p}} = \sum_1^\infty B_{2p}\frac{x^{2p}}{2p!} = \frac{x}{2}\coth\frac{x}{2} - 1. \tag{10}$$

Hence
$$B_{2p} = (-1)^{p-1}\frac{2(2p)!}{(2\pi)^{2p}}S_{2p} \tag{11}$$

and
$$\frac{x}{2}\coth\frac{x}{2} = 1 + \sum_1^{n-1} B_{2p}\frac{x^{2p}}{2p!} + \theta B_{2n}\frac{x^{2n}}{2n!}. \tag{12}$$

The desired expression for B_{2n} is thus found, and it is further seen that the expansion for $\frac{1}{2}x\coth\frac{1}{2}x$ can be broken off at any term with an error less than the first term omitted. This did not appear from the formal work of § 170. Further it may be noted that for large values of n the numbers B_{2n} are very large.

It was seen in treating the Γ-function that (Ex. 17, p. 385)

$$\log\Gamma(n) = (n - \tfrac{1}{2})\log n - n + \log\sqrt{2\pi} + \omega(n),$$

where
$$\omega(n) = \int_{-\infty}^0 \left(\frac{x}{2}\coth\frac{x}{2} - 1\right)e^{nx}\frac{dx}{x^2}.$$

As
$$\int_{-\infty}^0 x^{2p}e^{nx}dx = \int_0^\infty x^{2p}e^{-nx}dx = \frac{\Gamma(2p+1)}{n^{2p+1}} = \frac{2p!}{n^{2p+1}},$$

the substitution of (12), and the integration gives the result

$$\omega(n) = \frac{B_2 n^{-1}}{1\cdot 2} + \frac{B_4 n^{-3}}{3\cdot 4} + \cdots + \frac{B_{2p-2}n^{-2p+3}}{(2p-3)(2p-2)} + \frac{\theta B_{2p}n^{-2p+1}}{(2p-1)2p}. \tag{13}$$

For large values of n this development starts to converge very rapidly, and by taking a few terms a very good value of $\omega(n)$ can be obtained; but too many terms must not be taken. Compare §§ 151, 154.

EXERCISES

1. Prove $\cos x = \dfrac{\sin 2x}{2\sin x} = \prod_0^\infty\left(1 - \dfrac{4x^2}{(2k+1)^2\pi^2}\right)$.

2. On the assumption that the product for $\sinh x$ may be multiplied out and collected according to powers of x, show that

$$(\alpha)\ \sum_{k=1}^\infty \frac{1}{k^2} = \frac{\pi^2}{6}, \qquad (\beta)\ \sum_{k=1}^\infty\sum_{l=1}^\infty \frac{1}{k^2l^2} = \frac{\pi^4}{120},\ \text{where } k \neq l,$$

$$(\gamma)\ \sum_{k=1}^\infty \frac{1}{k^4} = \frac{\pi^4}{90}, \qquad (\delta)\ \sum_{k=1}^\infty\sum_{l=1}^\infty \frac{1}{k^2l^2} = \frac{\pi^4}{36},\ \text{if } k \text{ may equal } l.$$

3. By aid of Ex. 21 (∂), p. 452, show: (α) $1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\cdots=\frac{\pi^2}{6}$,

(β) $1+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\cdots=\frac{\pi^2}{8}$, ($\gamma$) $1-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\cdots=\frac{\pi^2}{12}$.

4. Prove: (α) $\int_0^1 \frac{\log x}{1-x}dx=-\frac{\pi^2}{6}$, ($\beta$) $\int_0^1 \frac{\log x}{1+x}dx=-\frac{\pi^2}{12}$,

(γ) $\int_0^1 \frac{\log x}{1-x^2}dx=-\frac{\pi^2}{8}$, ($\delta$) $\int_0^1 \log\frac{1+x}{1-x}\frac{dx}{x}=\frac{\pi^2}{4}$.

5. From $\tan x=-\cot\left(x-\frac{1}{2}\pi\right)=-\sum_{-\infty}^{+\infty}\frac{1}{x-(k+\frac{1}{2})\pi}$

show $\csc x=\frac{1}{2}\left(\cot\frac{x}{2}+\tan\frac{x}{2}\right)=\sum_{-\infty}^{+\infty}\frac{(-1)^k}{x-k\pi}=\frac{1}{x}+\sum_{1}^{\infty}\frac{(-1)^k 2x}{x^2-k^2\pi^2}$.

6. From $\frac{1}{1+x}=\sum_0^{n-1}(-x)^k+(-1)^n\frac{x^n}{1+x}=\sum_0^{n-1}(-x)^k+(-1)^n\theta x^n$

show $\int_0^1\frac{x^{a-1}}{1+x}dx=\sum_0^{\infty}\frac{(-1)^k}{a+k}$, and compute for $a=\frac{1}{4}$ by Ex. 21, p. 452.

7. If a is a proper fraction so that $1-a$ is a proper fraction, show

(α) $\int_0^1\frac{x^{-a}dx}{1+x}=\sum_1^{\infty}\frac{(-1)^k}{a-k}=\int_1^{\infty}\frac{x^{a-1}}{1+x}dx$, ($\beta$) $\int_0^{\infty}\frac{x^{a-1}}{1+x}dx=\frac{\pi}{\sin a\pi}$.

8. When n is large $B_{2n}=(-1)^{n-1}4\sqrt{\pi n}\left(\frac{n}{\pi e}\right)^{2n}$ approximately (Ex. 13).

9. Expand the terms of $\frac{x}{2}\coth\frac{x}{2}=1+\sum_1^{\infty}\frac{2x^2}{4k^2\pi^2+x^2}$ by division when $x<2\pi$ and rearrange according to powers of x. Is it easy to justify this derivation of (11)?

10. Find $\omega'(n)$ by differentiating under the sign and substituting. Hence get

$$\frac{\Gamma'(n)}{\Gamma(n)}=\log n-\frac{1}{2n}-\frac{B_2}{2n^2}-\frac{B_4}{4n^4}-\cdots-\frac{B_{2p-2}}{(2p-2)n^{2p-2}}-\frac{\theta B_{2p}}{2pn^{2p}}.$$

11. From $\frac{\Gamma'(n)}{\Gamma(n)}+\gamma=\int_0^1\frac{1-\alpha^{n-1}}{1-\alpha}d\alpha$ of § 149 show that, if n is integral,

$$\frac{\Gamma'(n)}{\Gamma(n)}+\gamma=1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n-1},\quad\text{and}\quad\gamma=-\frac{\Gamma'(1)}{\Gamma(1)}=0.5772156649\cdots$$

by taking $n=10$ and using the necessary number of terms of Ex. 10.

12. Prove $\log\Gamma(n+\frac{1}{2})=n(\log n-1)+\log\sqrt{2\pi}+\omega_1(n)$, where

$$\omega_1(n)=\int_{-\infty}^0\left(\frac{1}{x}-\frac{e^{\frac{x}{2}}}{e^x-1}\right)e^{nx}\frac{dx}{x},\qquad \omega_1(n)=\omega(n)-\omega(2n),$$

$$\omega_1(n)=\frac{B_2 n^{-1}}{1\cdot 2}\left(1-\frac{1}{2}\right)+\frac{B_4 n^{-3}}{3\cdot 4}\left(1-\frac{1}{2^3}\right)+\frac{B_6 n^{-5}}{5\cdot 6}\left(1-\frac{1}{2^5}\right)+\cdots.$$

13. Show $n! = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n e^{\frac{\theta}{12n}}$ or $\sqrt{2\pi}\left(\frac{n+\frac{1}{2}}{e}\right)^{n+\frac{1}{2}} e^{-\frac{\theta}{24n+12}}$. Note that the results of § 149 are now obtained rigorously.

14. From $\frac{1}{1-e^{-x}} = \sum_0^{n-1} e^{-kx} + \frac{e^{-nx}}{1-e^{-x}} = \sum_0^{n-1} e^{-kx} + \theta\frac{e^{-(n-1)x}}{x}$, and the formulas of § 149, prove the expansions

$$(\alpha)\ \frac{d^2}{dn^2}\log\Gamma(n) = \sum_0^\infty \frac{1}{(n+k)^2}, \qquad (\beta)\ \frac{d}{dn}\log\Gamma(n) + \gamma = \sum_0^\infty\left(\frac{1}{1+k} - \frac{1}{n+k}\right),$$

$$(\gamma)\ \log\Gamma(n+1) + \gamma n = \sum_1^\infty\left(\frac{n}{k} - \log\frac{n+k}{k}\right), \qquad (\delta)\ \frac{1}{\Gamma(n+1)} = e^{\gamma n}\prod_1^\infty\left(1+\frac{n}{k}\right)e^{-\frac{n}{k}}.$$

173. Trigonometric or Fourier series. If the series

$$\begin{aligned} f(x) &= \tfrac{1}{2}a_0 + \sum_1^\infty (a_k\cos kx + b_k\sin kx) \\ &= \tfrac{1}{2}a_0 + a_1\cos x + a_2\cos 2x + a_3\cos 3x + \cdots \\ &\qquad + b_1\sin x + b_2\sin 2x + b_3\sin 3x + \cdots \end{aligned} \tag{14}$$

converges over an interval of length 2π in x, say $0 \leqq x < 2\pi$ or $-\pi < x \leqq \pi$, the series will converge for all values of x and will define a periodic function $f(x+2\pi) = f(x)$ of period 2π. As

$$\int_0^{2\pi}\cos kx\sin lx\,dx = 0 \quad\text{and}\quad \int_0^{2\pi}\begin{matrix}\cos kx\cos lx\\ \sin kx\sin lx\end{matrix}dx = 0 \text{ or } \pi \tag{15}$$

according as $k \neq l$ or $k = l$, the coefficients in (14) may be determined formally by multiplying $f(x)$ and the series by

$$1 = \cos 0x, \qquad \cos x, \qquad \sin x, \qquad \cos 2x, \qquad \sin 2x, \cdots$$

successively and integrating from 0 to 2π. By virtue of (15) each of the integrals vanishes except one, and from that one

$$a_k = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos kx\,dx, \qquad b_k = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin kx\,dx. \tag{16}$$

Conversely if $f(x)$ be a function which is defined in an interval of length 2π, and which is continuous except at a finite number of points in the interval, the numbers a_k and b_k may be computed according to (16) and the series (14) may then be constructed. If this series converges to the value of $f(x)$, there has been found an expansion of $f(x)$ over the interval from 0 to 2π in a *trigonometric or Fourier series.** The question of whether the series thus found does really converge to

* By special devices some Fourier expansions were found in Ex. 10, p. 439.

the value of the function, and whether that series can be integrated or differentiated term by term to find the integral or derivative of the function will be left for special investigation. At present it will be assumed that the function may be represented by the series, that the series may be integrated, and that it may be differentiated if the differentiated series converges.

For example let e^x be developed in the interval from 0 to 2π. Here

$$a_k = \frac{1}{\pi}\int_0^{2\pi} e^x \cos kx dx = \frac{1}{k\pi}\int_0^{2\pi k} e^{\frac{y}{k}} \cos y dy = \left[\frac{e^{\frac{y}{k}}}{\pi}\left(\frac{k\sin y + \cos y}{k^2+1}\right)\right]_0^{2\pi k}$$

or
$$a_0 = \frac{1}{\pi}e^{2\pi} - \frac{1}{\pi}, \qquad a_k = \frac{1}{\pi}e^{2\pi}\frac{1}{k^2+1} - \frac{1}{\pi}\frac{1}{k^2+1},$$

and
$$b_k = \frac{1}{\pi}\int_0^{2\pi} e^x \sin kx dx = -\frac{1}{\pi}e^{2\pi}\frac{k}{k^2+1} + \frac{1}{\pi}\frac{k}{k^2+1}.$$

Hence
$$\frac{\pi e^x}{e^{2\pi}-1} = \frac{1}{2} + \frac{1}{1^2+1}\cos x + \frac{1}{2^2+1}\cos 2x + \frac{1}{3^2+1}\cos 3x + \cdots$$
$$- \frac{1}{1^2+1}\sin x - \frac{2}{2^2+1}\sin 2x - \frac{3}{3^2+1}\sin 3x + \cdots.$$

This expansion is valid only in the interval from 0 to 2π; outside that interval the series automatically repeats that portion of the function which lies in the interval. It may be remarked that the expansion does not hold for 0 or 2π but gives the point midway in the break. Note further that if the series were differentiated the coefficient of the cosine terms would be $1 + 1/k^2$ and would not approach 0 when k became infinite, so that the series would apparently oscillate. Integration from 0 to x would give

$$\frac{\pi(e^x-1)}{e^{2\pi}-1} = \frac{1}{2}x + \frac{1}{1^2+1}\sin x + \frac{1}{2^2+1}\frac{\sin 2x}{2} + \frac{1}{3^2+1}\frac{\sin 3x}{3} + \cdots$$
$$+ \frac{1}{1^2+1}\cos x + \frac{1}{2^2+1}\cos 2x + \frac{1}{3^2+1}\cos 3x + \cdots,$$

and the term $\frac{1}{2}x$ may be replaced by its Fourier series if desired.

As the relations (15) hold not only when the integration is from 0 to 2π but also when it is over any interval of 2π from α to $\alpha + 2\pi$, the function may be expanded into series in the interval from α to $\alpha + 2\pi$ by using these values instead of 0 and 2π as limits in the formulas (16) for the coefficients. It may be shown that a function may be expanded in only one way into a trigonometric series (14) valid for an interval of length 2π; but the proof is somewhat intricate and will not be given here. If, however, the expansion of the function is desired for an interval $\alpha < x < \beta$ less than 2π, there are an infinite number of developments (14) which will answer; for if $\phi(x)$ be a

function which coincides with $f(x)$ during the interval $\alpha < x < \beta$, over which the expansion of $f(x)$ is desired, and which has any value whatsoever over the remainder of the interval $\beta < x < \alpha + 2\pi$, the expansion of $\phi(x)$ from α to $\alpha + 2\pi$ will converge to $f(x)$ over the interval $\alpha < x < \beta$.

In practice it is frequently desirable to restrict the interval over which $f(x)$ is expanded to a length π, say from 0 to π, and to seek an expansion in terms of sines or cosines alone. Thus suppose that in the interval $0 < x < \pi$ the function $\phi(x)$ be identical with $f(x)$, and that in the interval $-\pi < x < 0$ it be equal to $f(-x)$; that is, the function $\phi(x)$ is an even function, $\phi(x) = \phi(-x)$, which is equal to $f(x)$ in the interval from 0 to π. Then

$$\int_{-\pi}^{+\pi} \phi(x) \cos kx dx = 2\int_0^{\pi} \phi(x) \cos kx dx = 2\int_0^{\pi} f(x) \cos kx dx,$$

$$\int_{-\pi}^{+\pi} \phi(x) \sin kx dx = \int_0^{\pi} \phi(x) \sin kx dx - \int_0^{\pi} \phi(x) \sin kx dx = 0.$$

Hence for the expansion of $\phi(x)$ from $-\pi$ to $+\pi$ the coefficients b_k all vanish and the expansion is in terms of cosines alone. As $f(x)$ coincides with $\phi(x)$ from 0 to π, the expansion

$$f(x) = \sum_0^{\infty} a_k \cos kx, \qquad a_k = \frac{2}{\pi}\int_0^{\pi} f(x) \cos kx dx \tag{17}$$

of $f(x)$ in terms of cosines alone, and valid over the interval from 0 to π, has been found. In like manner the expansion

$$f(x) = \sum_1^{\infty} b_k \sin kx, \qquad b_k = \frac{2}{\pi}\int_0^{\pi} f(x) \sin kx dx \tag{18}$$

in term of sines alone may be found by taking $\phi(x)$ equal to $f(x)$ from 0 to π and equal to $-f(-x)$ from 0 to $-\pi$.

Let $\frac{1}{2}x$ be developed into a series of sines and into a series of cosines valid over the interval from 0 to π. For the series of sines

$$b_k = \frac{2}{\pi}\int_0^{\pi} \frac{1}{2} x \sin kx dx = -\frac{(-1)^k}{k}, \qquad \frac{x}{2} = \sum_1^{\infty} \pm \frac{\sin kx}{k}$$

or

$$\tfrac{1}{2}x = \sin x - \tfrac{1}{2}\sin 2x + \tfrac{1}{3}\sin 3x - \tfrac{1}{4}\sin 4x + \cdots. \tag{A}$$

Also $$a_0 = \frac{2}{\pi}\int_0^{\pi} \frac{1}{2} x dx = \frac{\pi}{2}, \qquad a_k = \frac{2}{\pi}\int_0^{\pi} \frac{1}{2} x \cos kx dx = \begin{cases} 0, & k \text{ even} \\ -\dfrac{2}{\pi k}, & k \text{ odd.} \end{cases}$$

Hence $$\frac{1}{2}x = \frac{\pi}{4} - \frac{2}{\pi}\left[\cos x + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \frac{\cos 7x}{7^2} + \cdots\right]. \tag{B}$$

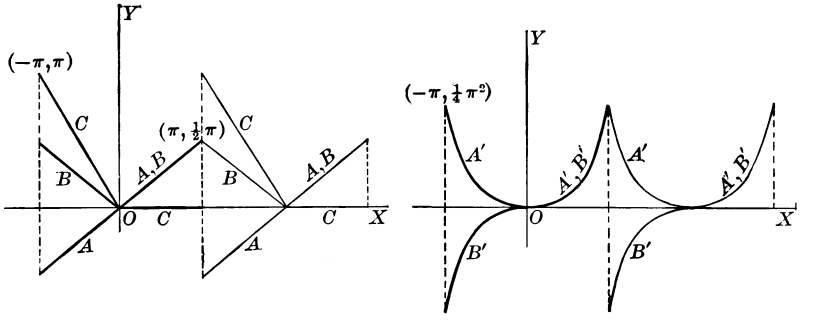
Although the two expansions define the same function $\frac{1}{2}x$ over the interval 0 to π, they will define different functions in the interval 0 to $-\pi$, as in the figure.

The development for $\frac{1}{4}x^2$ may be had by integrating either series (A) or (B).

$$\tfrac{1}{4}x^2 = 1 - \cos x - \tfrac{1}{4}(1 - \cos 2x) + \tfrac{1}{9}(1 - \cos 3x) - \tfrac{1}{16}(1 - \cos 4x) + \cdots$$

$$= \frac{\pi}{4}x - \frac{2}{\pi}\left[\sin x + \frac{\sin 3x}{3^3} + \frac{\cos 5x}{5^3} + \cdots\right].$$

These are not yet Fourier series because of the terms $\frac{1}{4}\pi x$ and the various 1's. For $\frac{1}{4}\pi x$ its sine series may be substituted and the terms $1 - \frac{1}{4} + \frac{1}{9} - \cdots$ may be collected by Ex. 3, p. 457. Hence



$$\frac{1}{4}x^2 = \frac{\pi^2}{12} - \cos x + \frac{1}{4}\cos 2x - \frac{1}{9}\cos 3x + \frac{1}{16}\cos 4x - \cdots \qquad (A')$$

$$\text{or } \frac{1}{4}x^2 = \frac{2}{\pi}\left[\left(\frac{\pi^2}{4} - 1\right)\sin x - \frac{\pi^2}{2}\sin 2x + \left(\frac{\pi^2}{12} - \frac{1}{3^2}\right)\sin 3x - \frac{\pi^2}{4}\sin 4x + \cdots\right]. \qquad (B')$$

The differentiation of the series (A) of sines will give a series in which the individual terms do not approach 0; the differentiation of the series (B) of cosines gives

$$\tfrac{1}{4}\pi = \sin x + \tfrac{1}{3}\sin 3x + \tfrac{1}{5}\sin 5x + \tfrac{1}{7}\sin 7x + \cdots$$

and that this is the series for $\pi/4$ may be verified by direct calculation. The difference of the two series (A) and (B) is a Fourier series

$$f(x) = \frac{\pi}{4} - \frac{2}{\pi}\left[\cos x + \frac{\cos 3x}{3^2} + \cdots\right] - \left[\sin x - \frac{\sin 2x}{2} + \cdots\right] \qquad (C)$$

which defines a function that vanishes when $0 < x < \pi$ but is equal to $-x$ when $0 > x > -\pi$.

174. For discussing the convergence of the trigonometric series as formally calculated, the sum of the first $2n+1$ terms may be written as

$$S_n = \frac{1}{\pi}\int_0^{2\pi}\left[\frac{1}{2} + \cos(t-x) + \cos 2(t-x) + \cdots + \cos n(t-x)\right] f(t)\,dt$$

$$= \frac{1}{\pi}\int_0^{2\pi} \frac{\sin(2n+1)\dfrac{t-x}{2}}{2\sin\dfrac{t-x}{2}} f(t)\,dt = \frac{1}{\pi}\int_{-\frac{x}{2}}^{\pi-\frac{x}{2}} f(x+2u)\frac{\sin(2n+1)u}{\sin u}\,du,$$

where the first step was to combine $a_k \cos kx$ and $b_k \sin kx$ after replacing x in the definite integrals (16) by t to avoid confusion, then summing by the formula of Ex. 9, p. 30, and finally changing the variable to $u = \frac{1}{2}(t - x)$. The sum S_n is therefore represented as a definite integral whose limit must be evaluated as n becomes infinite.

Let the restriction be imposed upon $f(x)$ that it shall be of limited variation in the interval $0 < x < 2\pi$. As the function $f(x)$ is of limited variation, it may be regarded as the difference $P(x) - N(x)$ of two positive limited functions which are constantly increasing and which will be continuous wherever $f(x)$ is continuous (§ 127). If $f(x)$ is discontinuous at $x = x_0$, it is still true that $f(x)$ approaches a limit, which will be denoted by $f(x_0 - 0)$ when x approaches x_0 from below; for each of the functions $P(x)$ and $N(x)$ is increasing and limited and hence each must approach a limit, and $f(x)$ will therefore approach the difference of the limits. In like manner $f(x)$ will approach a limit $f(x_0 + 0)$ as x approaches x_0 from above. Furthermore as $f(x)$ is of limited variation the integrals required for S_n, a_k, b_k will all exist and there will be no difficulty from that source. It will now be shown that

$$\lim_{n=\infty} S_n(x_0) = \lim_{n=\infty} \frac{1}{\pi} \int_{-\frac{x_0}{2}}^{\pi - \frac{x_0}{2}} f(x_0 + 2u) \frac{\sin(2n+1)u}{\sin u} du = \frac{1}{2}[f(x_0 + 0) - f(x_0 - 0)].$$

This will show that *the series converges to the function wherever the function is continuous and to the mid-point of the break wherever the function is discontinuous.*

$$\text{Let} \quad f(x_0 + 2u) \frac{\sin(2n+1)u}{\sin u} = f(x_0 + 2u) \frac{u}{\sin u} \frac{\sin(2n+1)u}{u} = F(u) \frac{\sin ku}{u},$$

$$\text{then} \quad S_n(x_0) = \frac{1}{\pi} \int_{-\frac{x_0}{2}}^{\pi - \frac{x_0}{2}} F(u) \frac{\sin ku}{u} du = \frac{1}{\pi} \int_a^b F(u) \frac{\sin ku}{u} du, \quad -\pi < a < 0 < b < \pi.$$

As $f(x)$ is of limited variation provided $-\pi < a \leqq u \leqq b < \pi$, so must $f(x_0 + 2u)$ be of limited variation and also $F(u) = uf/\sin u$. Then $F(u)$ may be regarded as the difference of two constantly increasing positive functions, or, if preferable, of two constantly decreasing positive functions; and it will be sufficient to investigate the integral of $F(u)\, u^{-1} \sin ku$ under the hypothesis that $F(u)$ is constantly decreasing. Let n be the number of times $2\pi/k$ is contained in b.

$$\int_0^b F(u) \frac{\sin ku}{u} du = \int_0^{\frac{2\pi}{k}} + \int_{\frac{2\pi}{k}}^{\frac{4\pi}{k}} + \cdots + \int_{\frac{2(n-1)\pi}{k}}^{\frac{2n\pi}{k}} + \int_{\frac{2n\pi}{k}}^b F(u) \frac{\sin ku}{u} du$$

$$= \int_0^{2\pi} + \int_{2\pi}^{4\pi} + \cdots + \int_{2(n-1)\pi}^{2n\pi} F\left(\frac{u}{k}\right) \frac{\sin u}{u} du + \int_{\frac{2n\pi}{k}}^b F(u) \frac{\sin ku}{u} du.$$

As $F(u)$ is a decreasing function, so is $u^{-1}F(u/k)$, and hence each of the integrals which extends over a complete period 2π will be positive because the negative elements are smaller than the corresponding positive elements. The integral from $2n\pi/k$ to b approaches zero as k becomes infinite. Hence for large values of k,

$$\int_0^b F(u) \frac{\sin ku}{u} du < \int_0^{2p\pi} F\left(\frac{u}{k}\right) \frac{\sin u}{u} du, \qquad p \text{ fixed and less than } n.$$

Again, $\int_0^b F(u)\frac{\sin ku}{u}du = \int_0^{\pi} + \int_{\pi}^{3\pi} + \int_{3\pi}^{5\pi}$

$$+ \cdots + \int_{(2n-3)\pi}^{(2n-1)\pi} F\left(\frac{u}{k}\right)\frac{\sin u}{u}du + \int_{\frac{(2n-1)\pi}{k}}^{b} F(u)\frac{\sin ku}{u}du.$$

Here all the terms except the first and last are negative because the negative elements of the integrals are larger than the positive elements. Hence for k large,

$$\int_0^b F(u)\frac{\sin ku}{u}du > \int_0^{(2p-1)\pi} F\left(\frac{u}{k}\right)\frac{\sin u}{u}du, \qquad p \text{ fixed and less than } n.$$

In the inequalities thus established let k become infinite. Then $u/k \doteq 0$ from above and $F(u/k) \doteq F(+0)$. It therefore follows that

$$F(+0)\int_0^{(2p-1)\pi}\frac{\sin u}{u}du < \lim_{k=\infty}\int_0^b F(u)\frac{\sin ku}{u}du < F(+0)\int_0^{2p\pi}\frac{\sin u}{u}du.$$

Although p is fixed, there is no limit to the size of the number at which it is fixed. Hence the inequality may be transformed into an equality

$$\lim_{k=\infty}\int_0^b F(u)\frac{\sin ku}{u}du = F(+0)\int_0^{\infty}\frac{\sin u}{u}du = \frac{\pi}{2}F(+0).$$

Likewise $$\lim_{k=\infty}\int_a^0 F(u)\frac{\sin ku}{u}du = F(-0)\int_0^{\infty}\frac{\sin u}{u}du = \frac{\pi}{2}F(-0).$$

Hence $$\lim_{k=\infty}\int_a^b F(u)\frac{\sin ku}{u}du = \frac{\pi}{2}[F(+0)+F(-0)]$$

or $$\lim_{n=\infty}\frac{1}{\pi}\int_{-\frac{x_0}{2}}^{\pi-\frac{x_0}{2}} f(x_0+2u)\frac{\sin(2n+1)u}{\sin u}du = \frac{1}{2}[f(x_0+0)+f(x_0-0)].$$

Hence for every point x_0 in the interval $0 < x < 2\pi$ the series converges to the function where continuous, and to the mid-point of the break where discontinuous.

As the function $f(x)$ has the period 2π, it is natural to suppose that the convergence at $x = 0$ and $x = 2\pi$ will not differ materially from that at any other value, namely, that it will be to the value $\frac{1}{2}[f(+0)+f(2\pi-0)]$. This may be shown by a transformation. If k is an odd integer, $2n+1$,

$$\sin(2n+1)u = \sin(2n+1)(\pi-u) = \sin(2n+1)u',$$

$$\lim_{n=\infty}\int_b^{\pi} F(u)\frac{\sin(2n+1)u}{u}du = \lim_{n=\infty}\int_0^{\pi-b} F(u')\frac{\sin(2n+1)u'}{u'}du' = \frac{\pi}{2}F(u'=+0).$$

Hence $$\lim_{n=\infty}\int_0^{\pi} F(u)\frac{\sin(2n+1)u}{u}du = \lim_{n=\infty}\int_0^b + \int_b^{\pi} = \frac{\pi}{2}[F(+0)+F(\pi-0)].$$

Now for $x = 0$ or $x = 2\pi$ the sum $S_n = \frac{1}{\pi}\int_0^{\pi} f(2u)\frac{\sin(2n+1)u}{\sin u}du$, and the limit will therefore be $\frac{1}{2}[f(+0)+f(2\pi-0)]$ as predicted above.

The convergence may be examined more closely. In fact

$$S_n(x) = \frac{1}{\pi}\int_{-\frac{x}{2}}^{\pi-\frac{x}{2}} f(x+2u)\frac{u}{\sin u}\frac{\sin ku}{u}du = \frac{1}{\pi}\int_{a(x)}^{b(x)} F(x,u)\frac{\sin ku}{u}du.$$

Suppose $0 < \alpha \leqq x \leqq \beta < 2\pi$ so that the least possible upper limit $b(x)$ is $\pi - \frac{1}{2}\beta$ and the greatest possible lower limit $a(x)$ is $-\frac{1}{2}\alpha$. Let n be the number of times $2\pi/k$ is contained in $\pi - \frac{1}{2}\beta$. Then for all values of x in $\alpha \leqq x \leqq \beta$,

$$\int_0^{(2p-1)\pi} F\left(x, \frac{u}{k}\right)\frac{\sin u}{u}du + \epsilon < \int_0^{b(x)} F(x, u)\frac{\sin ku}{u}du$$

$$< \int_0^{2p\pi} F\left(x, \frac{u}{k}\right)\frac{\sin u}{u}du + \eta, \qquad p < n,$$

where ϵ and η are the integrals over partial periods neglected above and are uniformly small for all x's of $\alpha \leqq x \leqq \beta$ since $F(x, u)$ is everywhere finite. This shows that the number p may be chosen uniformly for all x's in the interval and yet ultimately may be allowed to become infinite. If it be now assumed that $f(x)$ is continuous for $\alpha \leqq x \leqq \beta$, then $F(x, u)$ will be continuous and hence uniformly continuous in (x, u) for the region defined by $\alpha \leqq x \leqq \beta$ and $-\frac{1}{2}x \leqq u \leqq \pi - \frac{1}{2}x$. Hence $F(x, u/k)$ will converge uniformly to $F(x, +0)$ as k becomes infinite. Hence

$$F(x, +0)\int_0^{\infty}\frac{\sin u}{u}du + \epsilon' < \int_0^{b(x)} F(x, u)\frac{\sin ku}{u}du < F(x, +0)\int_0^{\infty}\frac{\sin u}{u}du + \eta'$$

where, if $\delta > 0$ is given, K may be taken so large that $|\epsilon'| < \delta$ and $|\eta'| < \delta$ for $k > K$; with a similar relation for the integration from $a(x)$ to 0. Hence in any interval $0 < \alpha \leqq x \leqq \beta < 2\pi$ over which $f(x)$ is continuous $S_n(x)$ converges uniformly toward its limit $f(x)$. Over such an interval the series may be integrated term by term. If $f(x)$ has a finite number of discontinuities, the series may still be integrated term by term throughout the interval $0 \leqq x \leqq 2\pi$ because $S_n(x)$ remains always finite and limited and such discontinuities may be disregarded in integration.

EXERCISES

1. Obtain the expansions over the indicated intervals. Integrate the series. Also discuss the differentiated series. Make graphs.

$$(\alpha)\ \frac{\pi e^x}{2\sinh\pi} = \frac{1}{2} - \frac{1}{2}\cos x + \frac{1}{5}\cos 2x - \frac{1}{10}\cos 3x + \frac{1}{17}\cos 4x - \cdots$$

$$+ \frac{1}{2}\sin x - \frac{2}{5}\sin 2x + \frac{3}{10}\sin 3x - \frac{4}{17}\sin 4x + \cdots, \qquad -\pi \text{ to } +\pi,$$

(β) $\frac{1}{4}\pi$, as sine series, 0 to π, (γ) $\frac{1}{4}\pi$, as cosine series, 0 to π,

$$(\delta)\ \sin x = \frac{4}{\pi}\left[\frac{1}{2} - \frac{\cos 2x}{1\cdot 3} - \frac{\cos 4x}{3\cdot 5} - \frac{\cos 6x}{5\cdot 7} - \cdots\right], \ 0 \text{ to } \pi,$$

(ϵ) $\cos x$, as sine series, 0 to π, (ζ) e^x, as cosine series, 0 to π,

(η) $x\sin x$, $-\pi$ to π, (θ) $x\cos x$, $-\pi$ to π, (ι) $\pi + x$, $-\pi$ to π,

(κ) $\sin\theta x$, $-\pi$ to π, θ fractional, (λ) $\cos\theta x$, $-\pi$ to π, θ fractional,

$$(\mu)\ f(x) = \begin{cases} \frac{1}{4}\pi, & 0 < x < \pi, \\ 0, & \pi < x < 2\pi, \end{cases} \quad (\nu)\ f(x) = \begin{cases} \frac{1}{4}\pi, & 0 < x < \frac{1}{2}\pi, \\ -\frac{1}{4}\pi, & \frac{1}{2}\pi < x < \pi, \end{cases} \text{ as a sine series, 0 to } \pi,$$

$$(o)\ -\log\left(2\sin\frac{x}{2}\right) = \cos x + \frac{1}{2}\cos 2x + \frac{1}{3}\cos 3x + \frac{1}{4}\cos 4x + \cdots, \ 0 \text{ to } \pi,$$

(π) $x, -\frac{1}{2}\pi$ to $\frac{3}{2}\pi$, (ρ) $\sin\frac{1}{2}x, -\frac{1}{2}\pi$ to $\frac{3}{2}\pi$, (σ) $\cos\frac{1}{2}x, -\frac{3}{2}\pi$ to $\frac{1}{2}\pi$,

(τ) from (o) find expansions for $\log\cos\frac{1}{2}x$, $\log\operatorname{vers}x$, $\log\tan\frac{1}{2}x$. Note that in these cases, as in (o), the function does not remain finite, but its integral does.

2. What peculiarities occur in the trigonometric development from $-\pi$ to π for an odd function for which $f(x)=f(\pi-x)$? for an even function for which $f(x)=f(\pi-x)$?

3. Show that $f(x)=\sum_1^\infty b_k\sin\frac{k\pi x}{c}$ with $b_k=\frac{2}{c}\int_0^c f(x)\sin\frac{k\pi x}{c}dx$ is the trigonometric sine series for $f(x)$ over the interval $0<x<c$ and that the function thus defined is odd and of period $2c$. Write the corresponding results for the cosine series and for the general Fourier series.

4. Obtain Nos. 808–812 of Peirce's Tables. Graph the sum of Nos. 809 and 810.

5. Let $e(x)=f(x)-\frac{1}{2}a_0-a_1\cos x-\cdots-a_n\cos nx-b_1\sin x-\cdots-b_n\sin nx$ be the error made by taking for $f(x)$ the first $2n+1$ terms of a trigonometric series. The mean value of the square of $e(x)$ is $\frac{1}{2\pi}\int_{-\pi}^{+\pi}[e(x)]^2dx$ and is a function $F(a_0, a_1, \cdots, a_n, b_1, \cdots, b_n)$ of the coefficients. Show that if this mean square error is to be as small as possible, the constants $a_0, a_1, \cdots, a_n, b_1, \cdots, b_n$ must be precisely those given by (16); that is, show that (16) is equivalent to

$$\frac{\partial F}{\partial a_0}=\frac{\partial F}{\partial a_1}=\cdots=\frac{\partial F}{\partial a_n}=\frac{\partial F}{\partial b_1}=\cdots=\frac{\partial F}{\partial b_n}=0.$$

6. By using the variable λ in place of x in (16) deduce the equations

$$f(x)=\frac{1}{2\pi}\int_0^{2\pi}f(\lambda)\cos 0(\lambda-x)d\lambda+\frac{1}{\pi}\sum_1^\infty\int_0^{2\pi}f(\lambda)\cos k(\lambda-x)d\lambda$$

$$=\frac{1}{2\pi}\sum_{-\infty}^{\infty}\int_0^{2\pi}f(\lambda)e^{\pm k(\lambda-x)i}d\lambda=\frac{1}{2\pi}\sum_{-\infty}^{\infty}e^{\mp kxi}\int_0^{2\pi}f(x)e^{\pm kxi}dx;$$

and hence infer $\quad f(x)=\sum_{-\infty}^{\infty}\alpha_k e^{\mp kxi}, \qquad \alpha_k=\frac{1}{2\pi}\int_0^{2\pi}f(x)e^{\pm kxi}dx.$

7. Without attempting rigorous analysis show formally that

$$\int_{-\infty}^{\infty}\phi(\alpha)d\alpha=\lim_{\Delta\alpha\doteq 0}[\cdots+\phi(-n\cdot\Delta\alpha)\Delta\alpha+\phi(-n+1\cdot\Delta\alpha)\Delta\alpha+\cdots+\phi(-1\cdot\Delta\alpha)\Delta\alpha$$
$$+\phi(0\cdot\Delta\alpha)\Delta\alpha+\phi(1\cdot\Delta\alpha)\Delta\alpha+\cdots+\phi(n\cdot\Delta\alpha)\Delta\alpha+\cdots]$$

$$=\lim_{\Delta\alpha\doteq 0}\sum_{-\infty}^{\infty}\phi(k\cdot\Delta\alpha)\Delta\alpha=\lim_{c=\infty}\sum_{-\infty}^{\infty}\phi\left(k\frac{a}{c}\right)\frac{a}{c}.$$

Show $\quad f(x)=\frac{1}{2c}\sum_{-\infty}^{\infty}\int_{-c}^{c}f(\lambda)e^{\pm\frac{k\pi}{c}(\lambda-x)i}d\lambda=\frac{1}{2\pi}\sum_{-\infty}^{\infty}\int_{-c}^{c}f(\lambda)e^{\pm\frac{k\pi}{c}(\lambda-x)i}\frac{\pi}{c}d\lambda$

is the expansion of $f(x)$ by Fourier series from $-c$ to c. Hence infer that

$$f(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f(\lambda)e^{\pm\alpha(\lambda-x)i}d\lambda d\alpha=\lim_{c=\infty}\frac{1}{2\pi}\sum_{-\infty}^{\infty}\int_{-c}^{c}f(\lambda)e^{\pm\frac{k\pi}{c}(\lambda-x)i}d\lambda\frac{\pi}{c}$$

is an expression for $f(x)$ as a double integral, which may be expected to hold for all values of x. Reduce this to the form of a Fourier Integral (Ex. 15, p. 377)

$$f(x) = \frac{1}{\pi}\int_0^\infty \int_{-\infty}^{\infty} f(\lambda)\cos\alpha(\lambda - x)\,d\lambda d\alpha.$$

8. Assume the possibility of expanding $f(x)$ between -1 and $+1$ as a series of Legendre polynomials (Exs. 13–20, p. 252, Ex. 16, p.440) in the form

$$f(x) = a_0P_0(x) + a_1P_1(x) + a_2P_2(x) + \cdots + a_nP_n(x) + \cdots.$$

By the aid of Ex. 19, p. 253, determine the coefficients as $a_k = \frac{2k+1}{2}\int_{-1}^{1} f(x)\,P_k(x)\,dx$. For this expansion, form $e(x)$ as in Ex. 5 and show that the determination of the coefficients a_i so as to give a least mean square error agrees with the determination here found.

9. Note that the expansion of Ex. 8 represents a function $f(x)$ between the limits ± 1 as a polynomial of the nth degree in x, plus a remainder. It may be shown that precisely this polynomial of degree n gives a smaller mean square error over the interval than any other polynomial of degree n. For suppose

$$g_n(x) = c_0 + c_1x + \cdots + c_nx^n = b_0 + b_1P_1 + \cdots + b_nP_n$$

be any polynomial of degree n and its equivalent expansion in terms of Legendre polynomials. Now if the c's are so determined that the mean value of $[f(x) - g_n(x)]^2$ is a minimum, so are the b's, which are linear homogeneous functions of the c's. Hence the b's must be identical with the a's above. Note that whereas the Maclaurin expansion replaces $f(x)$ by a polynomial in x which is a very good approximation near $x = 0$, the Legendre expansion replaces $f(x)$ by a polynomial which is the best expansion when the whole interval from -1 to $+1$ is considered.

10. Compute (cf. Ex. 17, p. 252) the polynomials $P_1 = x$, $P_2 = -\frac{1}{2} + \frac{3}{2}x^2$,

$$P_3 = -\tfrac{3}{2}x + \tfrac{5}{2}x^3, \qquad P_4 = \tfrac{3}{8} - \tfrac{15}{4}x^2 + \tfrac{35}{8}x^4, \qquad P_5 = \tfrac{15}{8}x - \tfrac{35}{4}x^3 + \tfrac{63}{8}x^5.$$

Compute $\int_{-1}^{1} x^i \sin\pi x\,dx = 0, \frac{2}{\pi}\left(1 - \frac{6}{\pi^2}\right), 0, \frac{2}{\pi}, 0$ when $i = 4, 3, 2, 1, 0$. Hence show that the polynomial of the fourth degree which best represents $\sin\pi x$ from -1 to $+1$ reduces to degree three, and is

$$\sin\pi x = \frac{3}{\pi}x - \frac{7}{\pi}\left(\frac{15}{\pi^2} = 1\right)\left(\frac{5}{2}x^3 - \frac{3}{2}x\right) = 2.69x - 2.89x^3.$$

Show that the mean square error is 0.004 and compare with that due to Maclaurin's expansion if the term in x^4 is retained or if the term in x^3 is retained.

11. Expand $\sin\frac{1}{2}\pi x = \frac{12}{\pi^2}P_1 - \frac{168}{\pi^2}\left(\frac{10}{\pi^2} - 1\right)P_3 = 1.553x - 0.562x^3$.

12. Expand from -1 to $+1$, as far as indicated, these functions:

(α) $\cos\pi x$ to P_4, (β) e^x to P_5, (γ) $\log(1+x)$ to P_4,

(δ) $\sqrt{1-x^2}$ to P_4, (ϵ) $\cos^{-1}x$ to P_4, (ζ) $\tan^{-1}x$ to P_5,

(η) $\frac{1}{\sqrt{1+x}}$, to P_3, (θ) $\frac{1}{\sqrt{1-x^2}}$ to P_3, (ι) $\frac{1}{\sqrt{1+x^2}}$ to P_3.

What simplifications occur if $f(x)$ is odd or if it is even?

175. The Theta functions. It has been seen that a function with the period 2π may be expanded into a trigonometric series; that if the function is odd, the series contains only sines; and if, furthermore, the function is symmetric with respect to $x = \frac{1}{2}\pi$, the odd multiples of the angle will alone occur. In this case let

$$f(x) = 2\,[a_0 \sin x - a_1 \sin 3x + \cdots + (-1)^n a_n \sin (2n+1)x + \cdots].$$

As $2 \sin nx = -i\,(e^{nxi} - e^{-nxi})$, the series may be written

$$f(x) = 2\sum_0^\infty (-1)^n a_n \sin (2n+1)x = -i\sum_{-\infty}^{\infty} (-1)^n a_n e^{(2n+1)xi},\ a_{-n} = a_{n-1}.$$

This exponential form is very convenient for many purposes. Let $i\rho$ be added to x. The general term of the series is then

$$a_{n-1}e^{(2n-1)(x+i\rho)i} = a_{n-1}e^{-(2n-1)\rho}e^{-2xi}e^{(2n+1)xi}.$$

Hence if the coefficients of the series satisfy $a_{n-1}e^{-2n\rho} = a_n$, the new general term is identical with the succeeding term in the given series multiplied by $-e^{\rho}e^{-2xi}$. Hence

$$f(x + i\rho) = -e^{\rho}e^{-2xi}f(x) \quad \text{if} \quad a_{n-1} = a_n e^{2n\rho}.$$

The recurrent relation between the coefficients will determine them in terms of a_0. For let $q = e^{-\rho}$. Then

$$a_n = a_{n-1}q^{2n} = a_{n-2}q^{2n}q^{2n-2} = \cdots = a_0 q^{2n}q^{2n-2}\cdots q^2 = a_0 q^{n^2+n},$$

$$a_0 = a_{-1} = a_{-2}q^{-2} = a_{-3}q^{-2}q^{-4} = \cdots = a_{-n-1}q^{-n^2-n}.$$

The new relation on the coefficients is thus compatible with the original relation $a_{-n} = a_{n-1}$. If $a_0 = q^{\frac{1}{4}}$, the series thus becomes

$$f(x) = 2q^{\frac{1}{4}}\sin x - 2q^{\frac{9}{4}}\sin 3x + \cdots + (-1)^n 2q^{\frac{1}{4}(2n+1)^2}\sin(2n+1)x + \cdots,$$

$$f(x + 2\pi) = f(x), \quad f(x+\pi) = -f(x), \quad f(x + i\rho) = -q^{-1}e^{-2xi}f(x).$$

The function thus defined formally has important properties.

In the first place it is important to discuss the convergence of the series. Apply the test ratio to the exponential form.

$$u_{n+1}/u_n = q^{2n}e^{2xi}, \qquad u_{-n-1}/u_{-n} = q^{2n}e^{-2xi}.$$

For any x this ratio will approach the limit 0 if q is numerically less than 1. Hence the series converges for all values of x provided $|q| < 1$. Moreover if $|x| < \frac{1}{2}G$, the absolute value of the ratio is less than $|q|^{2n}e^{G}$, which approaches 0 as n becomes infinite. The terms of the series therefore ultimately become less than those of any assigned geometric

series. This establishes the uniform convergence and consequently the continuity of $f(x)$ for all real or complex values of x. As the series for $f'(x)$ may be treated similarly, the function has a continuous derivative and is everywhere analytic.

By a change of variable and notation let

$$H(u) = f\left(\frac{\pi u}{2K}\right), \qquad q = e^{-\pi\frac{K'}{K}}, \tag{19}$$

$$H(u) = 2q^{\frac{1}{4}}\sin\frac{\pi u}{2K} - 2q^{\frac{9}{4}}\sin\frac{3\pi u}{2K} + 2q^{\frac{25}{4}}\sin\frac{5\pi u}{2K} - \cdots. \tag{20}$$

The function $H(u)$, called eta of u, has therefore the properties

$$H(u + 2K) = -H(u), \qquad H(u + 2iK') = -q^{-1}e^{-\frac{i\pi}{K}u}H(u), \tag{21}$$

$$H(u + 2mK + 2inK') = (-1)^{m+n}q^{-n}e^{-\frac{in\pi}{K}u}H(u), \quad m, n \text{ integers.}$$

The quantities $2K$ and $2iK'$ are called the *periods* of the function. They are not true periods in the sense that 2π is a period of $f(x)$; for when $2K$ is added to u, the function does not return to its original value, but is changed in sign; and when $2iK'$ is added to u, the function takes the multiplier written above.

Three new functions will be formed by adding to u the quantity K or iK' or $K + iK'$, that is, the *half periods*, and making slight changes suggested by the results. First let $H_1(u) = H(u + K)$. By substitution in the series (20),

$$H_1(u) = 2q^{\frac{1}{4}}\cos\frac{\pi u}{2K} + 2q^{\frac{9}{4}}\cos\frac{3\pi u}{2K} + 2q^{\frac{25}{4}}\cos\frac{5\pi u}{2K} + \cdots. \tag{22}$$

By using the properties of H, corresponding properties of H_1,

$$H_1(u + 2K) = -H_1(u), \qquad H_1(u + 2iK') = +q^{-1}e^{-\frac{i\pi}{K}u}H_1(u), \tag{23}$$

are found. Second let iK' be added to u in $H(u)$. Then

$$q^{\frac{1}{4}(2n+1)^2}e^{(2n+1)\frac{\pi i}{2K}(u+iK')} = q^{n^2+n+\frac{1}{4}}e^{-\pi(n+\frac{1}{2})\frac{K'}{K}}e^{(2n+1)\frac{\pi i}{2K}u}$$

is the general term in the exponential development of $H(u + iK')$ apart from the coefficient $\pm i$. Hence

$$H(u + iK') = i\sum_{-\infty}^{\infty}(-1)^n q^{n^2-\frac{1}{4}}e^{-\frac{\pi i}{2K}u}e^{2n\frac{\pi i}{2K}u}$$

$$= iq^{-\frac{1}{4}}e^{-\frac{\pi i}{2K}u}\sum_{-\infty}^{\infty}(-1)^n q^{n^2}e^{2n\frac{\pi i}{2K}u}.$$

Let $$\Theta(u) = -iq^{\frac{1}{4}} e^{\frac{i\pi}{2K}u} H(u + iK') = \sum_{-\infty}^{\infty} (-1)^n q^{n^2} e^{2n\frac{\pi i}{2K}u}.$$

The development of $\Theta(u)$ and further properties are evidently

$$\Theta(u) = 1 - 2q\cos\frac{2\pi u}{2K} + 2q^4\cos\frac{4\pi u}{2K} - 2q^9\cos\frac{6\pi u}{2K} + \cdots, \quad (24)$$

$$\Theta(u + 2K) = \Theta(u), \qquad \Theta(u + 2iK') = -q^{-1}e^{-\frac{i\pi}{K}u}\Theta(u). \quad (25)$$

Finally instead of adding $K + iK'$ to u in $H(u)$, add K in $\Theta(u)$.

$$\Theta_1(u) = 1 + 2q\cos\frac{2\pi u}{2K} + 2q^4\cos\frac{4\pi u}{2K} + 2q^9\cos\frac{6\pi u}{2K} + \cdots, \quad (26)$$

$$\Theta_1(u + 2K) = \Theta_1(u), \qquad \Theta_1(u + 2iK') = +q^{-1}e^{-\frac{i\pi}{K}u}\Theta_1(u). \quad (27)$$

For a tabulation of properties of the four functions see Ex. 1 below.

176. As $H(u)$ vanishes for $u = 0$ and is reproduced except for a finite multiplier when $2mK + 2niK'$ is added to u, the table

$$\begin{aligned} H(u) &= 0 \quad \text{for} \quad u = 2mK + 2niK', \\ H_1(u) &= 0 \quad \text{for} \quad u = (2m+1)K + 2niK', \\ \Theta(u) &= 0 \quad \text{for} \quad u = 2mK + (2n+1)iK', \\ \Theta_1(u) &= 0 \quad \text{for} \quad u = (2m+1)K + (2n+1)iK', \end{aligned}$$

contains the known vanishing points of the four functions. Now it is possible to form infinite products which vanish for these values. From such products it may be seen that the functions have no other vanishing points. Moreover the products themselves are useful.

It will be most convenient to use the function $\Theta_1(u)$. Now

$$e^{\frac{i\pi}{K}(2mK + K + 2niK' + iK')} = -q^{(2n+1)}, \qquad -\infty < n < \infty.$$

Hence $e^{\frac{i\pi}{K}u} + q^{-(2n+1)}$ and $e^{-\frac{i\pi}{K}u} + q^{-(2n+1)}$, $n \geqq 0$,

are two expressions of which the second vanishes for all the roots of $\Theta_1(u)$ for which $n \geqq 0$, and the first for all roots with $n < 0$. Hence

$$\Pi = C\prod_0^{\infty}\left(1 + q^{2n+1}e^{\frac{i\pi}{K}u}\right)\left(1 + q^{2n+1}e^{-\frac{i\pi u}{K}}\right)$$

is an infinite product which vanishes for all the roots of $\Theta_1(u)$. The product is readily seen to converge absolutely and uniformly. In particular it does not diverge to 0 and consequently has no other roots than those of $\Theta_1(u)$ above given. It remains to show that the product is identical with $\Theta_1(u)$ with a proper determination of C.

Let $\Theta_1(u)$ be written in exponential form as follows, with $z = e^{\frac{i\pi}{K}u}$:

$$\phi(z) = \Theta_1(u) = 1 + q\left(z + \frac{1}{z}\right) + q^4\left(z^2 + \frac{1}{z^2}\right) + \cdots + q^{n^2}\left(z^n + \frac{1}{z^n}\right) + \cdots,$$

$$\psi(z) = C^{-1}\prod(u) = (1 + qz)(1 + q^3z)(1 + q^5z)\cdots(1 + q^{2n-1}z)\cdots$$
$$\times\left(1 + \frac{q}{z}\right)\left(1 + \frac{q^3}{z}\right)\left(1 + \frac{q^5}{z}\right)\cdots\left(1 + \frac{q^{2n-1}}{z}\right)\cdots.$$

A direct substitution will show that $\phi(q^2z) = q^{-1}z^{-1}\phi(z)$ and $\psi(q^2z) = q^{-1}z^{-1}\psi(z)$. In fact this substitution is equivalent to replacing u by $u + 2iK'$ in Θ_1. Next consider the first $2n$ terms of $\psi(z)$ written above, and let this finite product be $\psi_n(z)$. Then by substitution

$$(q^{2n} + qz)\psi_n(q^2z) = (1 + q^{2n+1}z)\psi_n(z).$$

Now $\psi_n(z)$ is reciprocal in z in such a way that, if multiplied out,

$$\psi_n(z) = a_0 + a_1\left(z + \frac{1}{z}\right) + a_2\left(z^2 + \frac{1}{z^2}\right) + \cdots + a_n\left(z^n + \frac{1}{z^n}\right), \qquad a_n = q^{n^2}.$$

Then $$(q^{2n} + qz)\sum_0^n a_i(q^{2i}z^i + q^{-2i}z^{-i}) = (1 + q^{2n+1}z)\sum_0^n a_i(z^i + z^{-i}),$$

and the expansion and equation of coefficients of z^i gives the relation

$$a_i = a_{i-1}\frac{q^{2i-1}(1 - q^{2n-2i+2})}{1 - q^{2n+2i}} \quad \text{or} \quad a_i = a_0\frac{q^{i^2}\prod_{k=1}^{i}(1 - q^{2n-2k+2})}{\prod_{k=0}^{i-1}(1 - q^{2n+2k+2})}.$$

From $a_n = q^{n^2}$, $$a_0 = \frac{\prod_{k=0}^{n-1}(1 - q^{2n+2k+2})}{\prod_{k=1}^{n}(1 - q^{2k})}, \qquad a_i = \frac{q^{i^2}\prod_{k=1}^{n-i}(1 - q^{2n+2i+2k})}{\prod_{k=1}^{n-i}(1 - q^{2k})}.$$

Now if n be allowed to become infinite, each coefficient a_i approaches the limit

$$\lim a_i = \frac{q^{i^2}}{C}, \qquad C = \prod_1^\infty(1 - q^{2n}) = (1 - q^2)(1 - q^4)(1 - q^6)\cdots.$$

Hence $$\Theta_1(u) = \prod_1^\infty(1 - q^{2n})\cdot\prod_0^\infty\left(1 + q^{2n+1}e^{\frac{i\pi}{k}u}\right)\left(1 + q^{2n+1}e^{\frac{-i\pi}{k}u}\right),$$

provided the limit of $\psi_n(z)$ may be found by taking the series of the limits of the terms. The justification of this process would be similar to that of § 171.

The products for Θ, H_1, H may be obtained from that for Θ_1 by subtracting K, iK', $K + iK'$ from u and making the needful slight alterations to conform with the definitions. The products may be converted into trigonometric form by multiplying. Then

$$H(u) = C\,2\,q^{\frac{1}{4}}\sin\frac{\pi u}{2K}\prod_1^\infty\left(1 - 2q^{2n}\cos\frac{2\pi u}{2K} + q^{4n}\right), \qquad (28)$$

$$H_1(u) = C\, 2\, q^{\frac{1}{4}} \cos\frac{\pi u}{2K} \prod_1^\infty \left(1 + 2\, q^{2n} \cos\frac{2\pi u}{2K} + q^{4n}\right), \tag{29}$$

$$\Theta(u) = C \prod_0^\infty \left(1 - 2\, q^{2n+1} \cos\frac{2\pi u}{2K} + q^{4n+2}\right), \tag{30}$$

$$\Theta_1(u) = C \prod_0^\infty \left(1 + 2\, q^{2n+1} \cos\frac{2\pi u}{2K} + q^{4n+2}\right), \tag{31}$$

$$C = \prod_1^\infty (1 - q^{2n}) = (1 - q^2)(1 - q^4)(1 - q^6)\cdots, \tag{32}$$

$$H_1(0) = C\, 2\, q^{\frac{1}{4}} \prod_1^\infty (1 + q^{2n})^2, \qquad \Theta(0) = C \prod_0^\infty (1 - q^{2n+1})^2,$$

$$H'(0) = C\, 2\, q^{\frac{1}{4}} \frac{\pi}{2K} \prod_1^\infty (1 - q^{2n})^2, \qquad \Theta_1(0) = C \prod_0^\infty (1 + q^{2n+1})^2.$$

The value of $H'(0)$ is found by dividing $H(u)$ by u and letting $u \doteq 0$. Then

$$H'(0) = \frac{\pi}{2K} H_1(0)\Theta(0)\Theta_1(0) \tag{33}$$

follows by direct substitution and cancellation or combination.

177. Other functions may be built from the theta functions. Let

$$\sqrt{k} = \frac{H(K)}{\Theta(K)} = \frac{H_1(0)}{\Theta_1(0)}, \qquad \sqrt{k'} = \frac{\Theta(0)}{\Theta_1(0)}, \qquad \sqrt{\frac{k'}{k}} = \frac{\Theta(0)}{H_1(0)}, \tag{34}$$

$$\operatorname{sn} u = \frac{1}{\sqrt{k}} \frac{H(u)}{\Theta(u)}, \qquad \operatorname{cn} u = \sqrt{\frac{k'}{k}} \frac{H_1(u)}{\Theta(u)}, \qquad \operatorname{dn} u = \sqrt{k'} \frac{\Theta_1(u)}{\Theta(u)}. \tag{35}$$

The functions $\operatorname{sn} u$, $\operatorname{cn} u$, $\operatorname{dn} u$ are called elliptic functions* of u. As H is the only odd theta function, $\operatorname{sn} u$ is odd but $\operatorname{cn} u$ and $\operatorname{dn} u$ are even. *All three functions have two actual periods* in the same sense that $\sin x$ and $\cos x$ have the period 2π. Thus $\operatorname{dn} u$ has the periods $2K$ and $4iK'$ by (25), (27); and $\operatorname{sn} u$ has the periods $4K$ and $2iK'$ by (25), (21). That $\operatorname{cn} u$ has $4K$ and $2K + 2iK'$ as periods is also easily verified. The values of u which make the functions vanish are known; they are those which make the numerators vanish. In like manner the values of u for which the three functions become infinite are the known roots of $\Theta(u)$.

If q is known, the values of $\sqrt{k}$ and $\sqrt{k'}$ may be found from their definitions. Conversely the expression for $\sqrt{k'}$,

$$\sqrt{k'} = \frac{\Theta(0)}{\Theta_1(0)} = \frac{1 - 2\,q + 2\,q^4 - 2\,q^9 + \cdots}{1 + 2\,q + 2\,q^4 + 2\,q^9 + \cdots}, \tag{36}$$

* The study of the elliptic functions is continued in Chapter XIX.

is readily solved for q by reversion. If powers of q higher than the first are neglected, the approximate value of q is found by solution, as

$$\frac{1}{2}\frac{1-\sqrt{k'}}{1+\sqrt{k'}} = \frac{q+q^9+\cdots}{1-2q^4+\cdots} = q - 2q^5 + 5q^9 + \cdots.$$

$$\text{Hence}\quad q = \frac{1}{2}\frac{1-\sqrt{k'}}{1+\sqrt{k'}} + \frac{2}{2^5}\left(\frac{1-\sqrt{k'}}{1+\sqrt{k'}}\right)^5 + \frac{15}{2^9}\left(\frac{1-\sqrt{k'}}{1+\sqrt{k'}}\right)^9 + \cdots \tag{37}$$

is the series for q. For values of k' near 1 this series converges with great rapidity; in fact if $k'^2 \geqq \frac{1}{2}$, $k' > 0.7$, $\sqrt{k'} > 0.82$, the second term of the expansion amounts to less than $1/10^6$ and may be disregarded in work involving four or five figures. The first two terms here given are sufficient for eleven figures.

Let ϑ denote any one of the four theta series H, H_1, Θ, Θ_1. Then

$$\vartheta^2(u) = \phi(z) = \sum_{-\infty}^{\infty} b_n z^n, \qquad z = e^{-\frac{i\pi}{K}u} \tag{38}$$

may be taken as the form of development of ϑ^2; this is merely the Fourier series for a function with period $2K$. But all the theta functions take the same multiplier, except for sign, when $2iK'$ is added to u; hence the squares of the functions take the same multiplier, and in particular $\phi(q^2 z) = q^{-2}z^{-2}\phi(z)$. Apply this relation.

$$\sum b_n q^{2n} z^n = q^{-2}z^{-2}\sum b_n z^n, \qquad b_n q^{2n+2} = b_{n-2}.$$

It then is seen that a recurrent relation between the coefficients is found which will determine all the even coefficients in terms of b_0 and all the odd in terms of b_1. Hence

$$\vartheta^2(u) = b_0\Phi(z) + b_1\Psi(z), \qquad b_0, b_1, \text{ constants}, \tag{38'}$$

is the expansion of any ϑ^2 or of any function which may be developed as (38) and satisfies $\phi(q^2 z) = q^{-2}z^{-2}\phi(z)$. Moreover Φ and Ψ are identical for all such functions, and the only difference is in the values of the constants b_0, b_1.

As any three theta functions satisfy (38') with different values of the constants, the functions Φ and Ψ may be eliminated and

$$\alpha\vartheta_1^2(u) + \beta\vartheta_2^2(u) + \gamma\vartheta_3^2(u) = 0,$$

where α, β, γ are constants. In words, the squares of any three theta functions satisfy a linear homogeneous equation with constant coefficients. The constants may be determined by assigning particular values to the argument u. For example, take H, H_1, Θ. Then*

*For brevity the parenthesis about the arguments of a function will frequently be omitted.

$$\alpha H^2(u) + \beta H_1^2(u) = \gamma\Theta^2(u), \qquad \beta H_1^2 0 = \gamma\Theta^2 0, \qquad \alpha H^2 K = \gamma\Theta^2 K,$$

$$\frac{\Theta^2 K}{H^2 K}\frac{H^2(u)}{\Theta^2(u)} + \frac{\Theta^2 0}{H_1^2 0}\frac{H_1^2(u)}{\Theta^2(u)} = 1, \text{ or } \operatorname{sn}^2 u + \operatorname{cn}^2 u = 1. \tag{39}$$

By treating H, Θ_1, Θ in a similar manner may be proved

$$k^2 \operatorname{sn}^2 u + \operatorname{dn}^2 u = 1 \quad \text{and} \quad k^2 + k'^2 = 1. \tag{40}$$

The function $\vartheta(u)\vartheta(u-a)$, where a is a constant, satisfies the relation $\phi(q^2 z) = q^{-2}z^{-2}C\phi(z)$ if $\log C = i\pi a/K$. Reasoning like that used for treating ϑ^2 then shows that between any three such expressions there is a linear relation. Hence

$$\alpha H(u)H(u-a) + \beta H_1(u)H_1(u-a) = \gamma\Theta(u)\Theta(u-a),$$

$$u = 0, \qquad \beta H_1(0)H_1(a) = \gamma\Theta(0)\Theta(a),$$

$$u = K, \qquad \alpha H_1(0)H_1(a) = \gamma\Theta_1(0)\Theta_1(a),$$

$$\frac{\Theta 0\Theta_1 0\Theta_1 a\, H(u)H(u-a)}{H_1^2 0\Theta a\,\Theta(u)\Theta(u-a)} + \frac{\Theta^2 0}{H_1^2 0}\frac{H_1(u)H_1(u-a)}{\Theta(u)\Theta(u-a)} = \frac{\Theta 0}{H_1 0}\frac{H_1 a}{\Theta a},$$

or
$$\operatorname{dn} a \operatorname{sn} u \operatorname{sn}(u-a) + \operatorname{cn} u \operatorname{cn}(u-a) = \operatorname{cn} a. \tag{41}$$

In this relation replace a by $-v$. Then there results

$$\operatorname{cn} u \operatorname{cn}(u+v) + \operatorname{sn} u \operatorname{dn} v \operatorname{sn}(u+v) = \operatorname{cn} v,$$

or
$$\operatorname{cn} v \operatorname{cn}(u+v) + \operatorname{sn} v \operatorname{dn} u \operatorname{sn}(u+v) = \operatorname{cn} u,$$

and
$$\operatorname{sn}(u+v) = \frac{\operatorname{cn}^2 u - \operatorname{cn}^2 v = \operatorname{sn}^2 v - \operatorname{sn}^2 u}{\operatorname{sn} v \operatorname{cn} u \operatorname{dn} u - \operatorname{sn} u \operatorname{cn} v \operatorname{dn} v}, \tag{42}$$

by symmetry and by solution. The fraction may be reduced by multiplying numerator and denominator by the denominator with the middle sign changed, and by noting that

$$\operatorname{sn}^2 v \operatorname{cn}^2 u \operatorname{dn}^2 u - \operatorname{sn}^2 u \operatorname{cn}^2 v \operatorname{dn}^2 v = (\operatorname{sn}^2 v - \operatorname{sn}^2 u)(1 - k^2 \operatorname{sn}^2 u \operatorname{sn}^2 v).$$

Then
$$\operatorname{sn}(u+v) = \frac{\operatorname{sn} u \operatorname{cn} v \operatorname{dn} v + \operatorname{sn} v \operatorname{cn} u \operatorname{dn} u}{1 - k^2 \operatorname{sn}^2 u \operatorname{sn}^2 v}, \tag{43}$$

and
$$\operatorname{sn}(u-v) = \frac{\operatorname{sn} u \operatorname{cn} v \operatorname{dn} v - \operatorname{sn} v \operatorname{cn} u \operatorname{dn} u}{1 - k^2 \operatorname{sn}^2 u \operatorname{sn}^2 v},$$

and
$$\operatorname{sn}(u+v) - \operatorname{sn}(u-v) = \frac{2 \operatorname{sn} v \operatorname{cn} u \operatorname{dn} u}{1 - k^2 \operatorname{sn}^2 u \operatorname{sn}^2 v}. \tag{44}$$

The last result may be used to differentiate $\operatorname{sn} u$. For

$$\frac{\operatorname{sn}(u+\Delta u) - \operatorname{sn} u}{\Delta u} = \frac{\operatorname{sn} \tfrac{1}{2}\Delta u}{\tfrac{1}{2}\Delta u}\frac{\operatorname{cn}(u+\tfrac{1}{2}\Delta u)\operatorname{dn}(u+\tfrac{1}{2}\Delta u)}{1 - k^2 \operatorname{sn}^2 \tfrac{1}{2}\Delta u \operatorname{sn}^2(u+\tfrac{1}{2}\Delta u)},$$

$$\frac{d}{du}\operatorname{sn} u = g \operatorname{cn} u \operatorname{dn} u, \qquad g = \lim_{u \doteq 0}\frac{\operatorname{sn} u}{u}. \tag{45}$$

Here g is called the *multiplier*. By definition of $\operatorname{sn} u$ and by (33)

$$g = \frac{\Theta_1(0)}{H_1(0)}\,\frac{H'(0)}{\Theta(0)} = \frac{\pi}{2K}\,\Theta_1^2(0). \tag{45'}$$

The periods $2K$, $2iK'$ have been independent up to this point. It will, however, be a convenience to have $g = 1$ and thus simplify the formula for differentiating $\operatorname{sn} u$. Hence let

$$g = 1, \qquad \sqrt{\frac{2K}{\pi}} = \Theta_1(0) = 1 + 2q + 2q^4 + \cdots. \tag{46}$$

Now of the five quantities K, K', k, k', q only one is independent. If q is known, then k' and K may be computed by (36), (46); k is determined by $k^2 + k'^2 = 1$, and K' by $\pi K'/K = -\log q$ of (19). If, on the other hand, k' is given, q may be computed by (37) and then the other quantities may be determined as before.

EXERCISES

1. With the notations $\lambda = q^{-\frac{1}{4}} e^{-\frac{i\pi}{2K}u}$, $\mu = q^{-1} e^{-\frac{i\pi}{K}u}$ establish:

$$\begin{array}{lll}
H(-u) = -H(u), & H(u+2K) = -H(u), & H(u+2iK') = -\mu H(u),\\
H_1(-u) = +H_1(u), & H_1(u+2K) = -H_1(u). & H_1(u+2iK') = +\mu H_1(u),\\
\Theta(-u) = +\Theta(u), & \Theta(u+2K) = +\Theta(u), & \Theta(u+2iK') = -\mu\Theta(u),\\
\Theta_1(-u) = +\Theta_1(u). & \Theta_1(u+2K) = +\Theta_1(u). & \Theta_1(u+2iK') = +\mu\Theta_1(u),\\
H(u+K) = +H_1(u). & H(u+iK') = i\lambda\Theta(u). & H(u+K+iK') = +\lambda\Theta_1(u),\\
H_1(u+K) = -H(u), & H_1(u+iK') = +\lambda\Theta_1(u). & H_1(u+K+iK') = -i\lambda\Theta(u),\\
\Theta(u+K) = +\Theta_1(u). & \Theta(u+iK') = i\lambda H(u), & \Theta(u+K+iK') = +\lambda H_1(u),\\
\Theta_1(u+K) = +\Theta(u), & \Theta_1(u+iK') = +\lambda H_1(u), & \Theta_1(u+K+iK') = +i\lambda H(u).
\end{array}$$

2. Show that if u is real and $q \leqq \frac{1}{8}$, the first two trigonometric terms in the series for H, H_1, Θ, Θ_1, give four-place accuracy. Show that with $q \leqq 0.1$ these terms give about six-place accuracy.

3. Use $\dfrac{q\sin\alpha}{1 - 2q\cos\alpha + q^2} = q\sin\alpha + q^2\sin 2\alpha + q^3\sin 3\alpha + \cdots$ to prove

$$\frac{d}{du}\log\Theta(u) = \frac{\Theta'(u)}{\Theta(u)} = \frac{2\pi}{K}\left(\frac{q\sin\dfrac{\pi u}{K}}{1-q^2} + \frac{q^2\sin\dfrac{2\pi u}{K}}{1-q^4} + \frac{q^3\sin\dfrac{3\pi u}{K}}{1-q^6} + \cdots\right).$$

4. Prove the double periodicity of $\operatorname{cn} u$ and show that:

$$\operatorname{sn}(u+K) = \frac{\operatorname{cn} u}{\operatorname{dn} u}, \qquad \operatorname{sn}(u+iK') = \frac{1}{k\operatorname{sn} u}, \qquad \operatorname{sn}(u+K+iK') = \frac{\operatorname{dn} u}{k\operatorname{cn} u},$$

$$\operatorname{cn}(u+K) = \frac{-k'\operatorname{sn} u}{\operatorname{dn} u}, \qquad \operatorname{cn}(u+iK') = \frac{-i\operatorname{dn} u}{k\operatorname{sn} u}, \qquad \operatorname{cn}(u+K+iK') = \frac{-ik'}{k\operatorname{cn} u},$$

$$\operatorname{dn}(u+K) = \frac{k'}{\operatorname{dn} u}, \qquad \operatorname{dn}(u+iK') = -i\frac{\operatorname{cn} u}{\operatorname{sn} u}, \qquad \operatorname{dn}(u+K+iK') = ik'\frac{\operatorname{sn} u}{\operatorname{cn} u}.$$

5. Tabulate the values of $\operatorname{sn} u$, $\operatorname{cn} u$, $\operatorname{dn} u$ at 0, K, iK', $K + iK'$.

6. Compute k' and k^2 for $q = \frac{1}{6}$ and $q = 0.1$. Hence show that two trigonometric terms in the theta series give four-place accuracy if $k' \geqq \frac{1}{4}$.

7. Prove $$\operatorname{cn}(u+v) = \frac{\operatorname{cn} u \operatorname{cn} v - \operatorname{sn} u \operatorname{sn} v \operatorname{dn} u \operatorname{dn} v}{1 - k^2 \operatorname{sn}^2 u \operatorname{sn}^2 v},$$

and $$\operatorname{dn}(u+v) = \frac{\operatorname{dn} u \operatorname{dn} v - k^2 \operatorname{sn} u \operatorname{sn} v \operatorname{cn} u \operatorname{cn} v}{1 - k^2 \operatorname{sn}^2 u \operatorname{sn}^2 v}.$$

8. Prove $$\frac{d}{du}\operatorname{cn} u = -\operatorname{sn} u \operatorname{dn} u, \qquad \frac{d}{du}\operatorname{dn} u = -k^2 \operatorname{sn} u \operatorname{cn} u, \qquad g = 1.$$

9. Prove $$\operatorname{sn}^{-1} u = \int_0^u \frac{du}{\sqrt{(1-u^2)(1-k^2u^2)}}$$ from (45) with $g = 1$.

10. If $g = 1$, compute k, k', K, K', for $q = 0.1$ and $q = 0.01$.

11. If $g = 1$, compute k', q, K, K', for $k^2 = \frac{1}{2}, \frac{3}{4}, \frac{1}{4}$.

12. In Exs. 10, 11 write the trigonometric expressions which give $\operatorname{sn} u$, $\operatorname{cn} u$, $\operatorname{dn} u$ with four-place accuracy.

13. Find $\operatorname{sn} 2u$, $\operatorname{cn} 2u$, $\operatorname{dn} 2u$, and hence $\operatorname{sn} \frac{1}{2}u$, $\operatorname{cn} \frac{1}{2}u$, $\operatorname{dn} \frac{1}{2}u$, and show

$$\operatorname{sn} \tfrac{1}{2}K = (1+k')^{-\frac{1}{2}}, \qquad \operatorname{cn} \tfrac{1}{2}K = \sqrt{k'}(1+k')^{-\frac{1}{2}}, \qquad \operatorname{dn} \tfrac{1}{2}K = \sqrt{k'}.$$

14. Prove $-k\int \operatorname{sn} u \operatorname{dn} = \log(\operatorname{dn} u + k \operatorname{cn} u)$; also

$$\Theta^2(0) H(u+a) H(u-a) = \Theta^2(a) H^2(u) - H^2(a) \Theta^2(u),$$
$$\Theta^2(0) \Theta(u+a) \Theta(u-a) = \Theta^2(u) \Theta^2(a) - H^2(u) H^2(a).$$