PART IV. THEORY OF FUNCTIONS

CHAPTER XVI

INFINITE SERIES

162. Convergence or divergence of series.* Let a series

$$\sum_0^\infty u = u_0 + u_1 + u_2 + \cdots + u_{n-1} + u_n + \cdots, \tag{1}$$

the terms of which are constant but infinite in number, be given. Let the sum of the first n terms of the series be written

$$S_n = u_0 + u_1 + u_2 + \cdots + u_{n-1} = \sum_0^{n-1} u. \tag{2}$$

Then $$S_1, S_2, S_3, \cdots, S_n, S_{n+1}, \cdots$$

form a definite suite of numbers which *may approach a definite limit* $\lim S_n = S$ when n becomes infinite. In this case the series is said to *converge to the value* S, and S, which is the limit of the sum of the first n terms, is called the *sum* of the series. Or S_n *may not approach a limit* when n becomes infinite, either because the values of S_n become infinite or because, though remaining finite, they oscillate about and fail to settle down and remain in the vicinity of a definite value. In these cases the series is said to *diverge.*

The necessary and sufficient condition that a series converge is that a value of n may be found so large that the numerical value of $S_{n+p} - S_n$ *shall be less than any assigned value for every value of p.* (See § 21, Theorem 3, and compare p. 356.) A sufficient condition that a series diverge is that the terms u_n do not approach the limit 0 when n becomes infinite. For if there are always terms numerically as great as some number r no matter how far one goes out in the series, there must always be successive values of S_n which differ by as much as r no matter how large n, and hence the values of S_n cannot possibly settle down and remain in the vicinity of some definite limiting value S.

* It will be useful to read over Chap. II, §§ 18–22, and Exercises. It is also advisable to compare many of the results for infinite series with the corresponding results for infinite integrals (Chap. XIII).

A series in which the terms are alternately positive and negative is called an *alternating series. An alternating series in which the terms approach* 0 *as a limit when n becomes infinite, each term being less than its predecessor, will converge and the difference between the sum S of the series and the sum S_n of the first n terms is less than the next term u_n.* This follows (p. 39, Ex. 3) from the fact that $|S_{n+p} - S_n| < u_n$ and $u_n \doteq 0$.

For example, consider the alternating series

$$1 - x^2 + 2x^4 - 3x^6 + \cdots + (-1)^n nx^{2n} + \cdots.$$

If $|x| \geqq 1$, the individual terms in the series do not approach 0 as n becomes infinite and the series diverges. If $|x| < 1$, the individual terms do approach 0; for

$$\lim_{n=\infty} nx^{2n} = \lim_{n=\infty} \frac{n}{x^{-2n}} = \lim_{n=\infty} \frac{1}{-2x^{-2n}\log x} = 0.$$

And for sufficiently large* values of n the successive terms decrease in magnitude since

$$nx^{2n} < (n-1)x^{2n-2} \quad \text{gives} \quad \frac{n-1}{n} > x^2 \quad \text{or} \quad n > \frac{1}{1-x^2}.$$

Hence the series is seen to converge for any value of x numerically less than unity and to diverge for all other values.

The Comparison Test. *If the terms of a series are all positive (or all negative) and each term is numerically less than the corresponding term of a series of positive terms which is known to converge, the series converges and the difference $S - S_n$ is less than the corresponding difference for the series known to converge.* (Cf. p. 355.) Let

$$u_0 + u_1 + u_2 + \cdots + u_{n-1} + u_n + \cdots$$

and

$$u_0' + u_1' + u_2' + \cdots + u_{n-1}' + u_n' + \cdots$$

be respectively the given series and the series known to converge. Since the terms of the first are less than those of the second,

$$S_{n+p} - S_n = u_n + \cdots + u_{n+p-1} < u_n' + \cdots + u_{n+p-1}' = S_{n+p}' - S_n'.$$

Now as the second quantity $S_{n+p}' - S_n'$ can be made as small as desired, so can the first quantity $S_{n+p} - S_n$, which is less; and the series must converge. The remainders

$$R_n = S - S_n = u_n + u_{n+1} + \cdots = \sum_n^\infty u,$$

$$R_n' = S' - S_n' = u_n' + u_{n+1}' + \cdots = \sum_n^\infty u'$$

* It should be remarked that the behavior of a series near its beginning is of no consequence in regard to its convergence or divergence; the first N terms may be added and considered as a finite sum S_N and the series may be written as $S_N + u_N + u_{N+1} + \cdots$; it is the properties of $u_N + u_{N+1} + \cdots$ which are important, that is, the ultimate behavior of the series.

clearly satisfy the stated relation $R_n < R'_n$. The series which is most frequently used for comparison with a given series is the geometric,

$$a + ar + ar^2 + ar^3 + \cdots, \qquad R_n = \frac{ar^n}{1-r}, \qquad 0 < r < 1, \tag{3}$$

which is known to converge for all values of r less than 1.

For example, consider the series

$$1 + 1 + \frac{1}{2} + \frac{1}{2\cdot 3} + \frac{1}{2\cdot 3\cdot 4} + \cdots + \frac{1}{n!} + \cdots$$

and
$$1 + \frac{1}{2} + \frac{1}{2\cdot 2} + \frac{1}{2\cdot 2\cdot 2} + \cdots + \frac{1}{2^{n-1}} + \cdots.$$

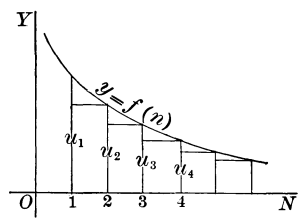
Here, after the first two terms of the first and the first term of the second, each term of the second is greater than the corresponding term of the first. Hence the first series converges and the remainder after the term $1/n!$ is less than

$$R_n < \frac{1}{2^n} + \frac{1}{2^{n+1}} + \cdots = \frac{1}{2^n}\frac{1}{1-\frac{1}{2}} = \frac{1}{2^{n-1}}.$$

A better estimate of the remainder after the term $1/n!$ may be had by comparing

$$R_n = \frac{1}{(n+1)!} + \frac{1}{(n+2)!} + \cdots \quad \text{with} \quad \frac{1}{(n+1)!} + \frac{1}{(n+1)!\,(n+1)} + \cdots = \frac{1}{n!\,n}.$$

163. As the convergence and divergence of a series are of vital importance, it is advisable to have a number of tests for the convergence or divergence of a given series. The test by comparison with a series known to converge requires that at least a few types of convergent series be known. For the establishment of such types and for the test of many series, the terms of which are positive, *Cauchy's integral test* is useful. Suppose that the terms of the series are decreasing and that a function $f(n)$ which decreases can be found such that $u_n = f(n)$. Now if the terms u_n be plotted at unit intervals along the n-axis, the value of the terms may be interpreted as the area of certain rectangles. The curve $y = f(n)$ lies above the rectangles and the area under the curve is



$$\int_1^n f(n)\,dn > u_2 + u_3 + \cdots + u_n. \tag{4}$$

Hence if the integral converges (which in practice means that if

$$\int f(n)\,dn = F(n), \quad \text{then} \quad \int_1^\infty f(n) = F(\infty) - F(1) \text{ is finite}),$$

it follows that the series must converge. For instance, if

$$\frac{1}{1^p}+\frac{1}{2^p}+\frac{1}{3^p}+\cdots+\frac{1}{n^p}+\cdots \tag{5}$$

be given, then $u_n = f(n) = 1/n^p$, and from the integral test

$$\frac{1}{2^p}+\frac{1}{3^p}+\cdots<\int_1^\infty \frac{dn}{n^p}=\frac{-1}{(p-1)n^{p-1}}\Bigg]_1^\infty=\frac{1}{p-1}$$

provided $p > 1$. Hence the series converges if $p > 1$. This series is also very useful for comparison with others; it diverges if $p \leqq 1$ (see Ex. 8).

THE RATIO TEST. *If the ratio of two successive terms in a series of positive terms approaches a limit which is less than* 1, *the series converges;* if the ratio approaches a limit which is greater than one or if the ratio becomes infinite, the series diverges. That is

if $\qquad \lim\limits_{n=\infty}\dfrac{u_{n+1}}{u_n}=\gamma<1$, the series converges,

if $\qquad \lim\limits_{n=\infty}\dfrac{u_{n+1}}{u_n}=\gamma'>1$, the series diverges.

For in the first case, as the ratio approaches a limit less than 1, it must be possible to go so far in the series that the ratio shall be as near to $\gamma < 1$ as desired, and hence shall be less than r if r is an assigned number between γ and 1. Then

$$u_{n+1} < ru_n, \qquad u_{n+2} < ru_{n+1} < r^2u_n, \cdots$$

and $\qquad u_n + u_{n+1} + u_{n+2} + \cdots < u_n(1 + r + r^2 + \cdots) = u_n\dfrac{1}{1-r}.$

The proof of the divergence when u_{n+1}/u_n becomes infinite or approaches a limit greater than 1 consists in noting that the individual terms cannot approach 0. Note that if the limit of the ratio is 1, *no information* relative to the convergence or divergence is furnished by this test.

If the series of numerical or absolute values

$$|u_0| + |u_1| + |u_2| + \cdots + |u_n| + \cdots$$

of the terms of a series which contains positive and negative terms converges, the series converges and is said to *converge absolutely*. For consider the two sums

$$S_{n+p} - S_n = u_n + \cdots + u_{n+p-1} \quad \text{and} \quad |u_n| + \cdots + |u_{n+p-1}|.$$

The first is surely not numerically greater than the second; as the second can be made as small as desired, so can the first. It follows therefore that the given series must converge. The converse proposition that if a series of positive and negative terms converges, then the series of absolute values converges, is not true.

As an example on convergence consider the binomial series

$$1 + mx + \frac{m(m-1)}{1 \cdot 2}x^2 + \frac{m(m-1)(m-2)}{1 \cdot 2 \cdot 3}x^3 + \cdots + \frac{m(m-1)\cdots(m-n+1)}{1 \cdot 2 \cdots n}x^n + \cdots,$$

where $$\frac{|u_{n+1}|}{|u_n|} = \frac{|m-n|}{n+1}|x|, \qquad \lim_{n=\infty}\frac{|u_{n+1}|}{|u_n|} = |x|.$$

It is therefore seen that the limit of the quotient of two successive terms in the series of absolute values is $|x|$. This is less than 1 for values of x numerically less than 1, and hence for such values the series converges and converges absolutely. (That the series converges for *positive* values of x less than 1 follows from the fact that for values of n greater than $m + 1$ the series alternates and the terms approach 0; the proof above holds equally for negative values.) For values of x numerically greater than 1 the series does not converge absolutely. As a matter of fact when $|x| > 1$, the series does not converge at all; for as the ratio of successive terms approaches a limit greater than unity, the individual terms cannot approach 0. For the values $x = \pm 1$ the test fails to give information. The conclusions are therefore that for values of $|x| < 1$ the binomial series converges absolutely, for values of $|x| > 1$ it diverges, and for $|x| = 1$ the question remains doubtful.

A word about series with *complex terms.* Let

$$\begin{aligned} &u_0 + u_1 + u_2 + \cdots + u_{n-1} + u_n + \cdots \\ &= u_0' + u_1' + u_2' + \cdots + u_{n-1}' + u_n' + \cdots \\ &+ i(u_0'' + u_1'' + u_2'' + \cdots + u_{n-1}'' + u_n'' + \cdots) \end{aligned}$$

be a series of complex terms. The sum to n terms is $S_n = S_n' + iS_n''$. The series is said to converge if S_n approaches a limit when n becomes infinite. If the complex number S_n is to approach a limit, both its real part S_n' and the coefficient S_n'' of its imaginary part must approach limits, and hence the series of real parts and the series of imaginary parts must converge. It will then be possible to take n so large that for any value of p the simultaneous inequalities

$$|S_{n+p}' - S_n'| < \tfrac{1}{2}\epsilon \quad \text{and} \quad |S_{n+p}'' - S_n''| < \tfrac{1}{2}\epsilon,$$

where ϵ is any assigned number, hold. Therefore

$$|S_{n+p} - S_n| \leqq |S_{n+p}' - S_n'| + |iS_{n+p}'' - iS_n''| < \epsilon.$$

Hence if the series converges, the same condition holds as for a series of real terms. Now conversely the condition

$$|S_{n+p} - S_n| < \epsilon \quad \text{implies} \quad |S_{n+p}' - S_n'| < \epsilon, \qquad |S_{n+p}'' - S_n''| < \epsilon.$$

Hence if the condition holds, the two real series converge and the complex series will then converge.

164. As Cauchy's integral test is not easy to apply except in simple cases and the ratio test fails when the limit of the ratio is 1, other sharper tests for convergence or divergence are sometimes needed, as in the case of the binomial series when $x = \pm 1$. Let there be given two series of positive terms

$$u_0 + u_1 + \cdots + u_n + \cdots \quad \text{and} \quad v_0 + v_1 + \cdots + v_n + \cdots$$

of which the first is to be tested and the second is known to converge (or diverge). *If the ratio of two successive terms u_{n+1}/u_n ultimately becomes and remains less (or greater) than the ratio v_{n+1}/v_n, the first series is also convergent (or divergent).* For if

$$\frac{u_{n+1}}{u_n} < \frac{v_{n+1}}{v_n}, \quad \frac{u_{n+2}}{u_{n+1}} < \frac{v_{n+2}}{v_{n+1}}, \quad \cdots, \quad \text{then} \quad \frac{u_n}{v_n} > \frac{u_{n+1}}{v_{n+1}} > \frac{u_{n+2}}{v_{n+2}} > \cdots.$$

Hence if $\quad u_n = \rho v_n, \qquad$ then $\quad u_{n+1} < \rho v_{n+1}, \qquad u_{n+2} < \rho v_{n+2}, \qquad \cdots,$

and
$$u_n + u_{n+1} + u_{n+2} + \cdots < \rho\,(v_n + v_{n+1} + v_{n+2} + \cdots).$$

As the v-series is known to converge, the ρv-series serves as a comparison series for the u-series which must then converge. If $u_{n+1}/u_n > v_{n+1}/v_n$ and the v-series diverges, similar reasoning would show that the u-series diverges.

This theorem serves to establish the useful *test due to Raabe*, which is

$$\text{if} \lim_{n=\infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) > 1,\ S_n \text{ converges}; \qquad \text{if} \lim_{n=\infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) < 1,\ S_n \text{ diverges.}$$

Again, if the limit is 1, no information is given. This test need never be tried except when the ratio test gives a limit 1 and fails. The proof is simple. For

$$\int^{\infty} \frac{dn}{n(\log n)^{1+\alpha}} = -\frac{1}{\alpha}\frac{1}{(\log n)^{\alpha}}\bigg]^{\infty} \text{is finite}$$

and
$$\int^{\infty} \frac{dn}{n \log n} = \log \log n\bigg]^{\infty} \text{is infinite,}$$

hence $\displaystyle \frac{1}{2(\log 2)^{1+\alpha}} + \cdots + \frac{1}{n(\log n)^{1+\alpha}} + \cdots$ and $\displaystyle \frac{1}{2(\log 2)} + \cdots + \frac{1}{n(\log n)} + \cdots$

are respectively convergent and divergent by Cauchy's integral test. Let these be taken as the v-series with which to compare the u-series. Then

$$\frac{v_n}{v_{n+1}} = \frac{n+1}{n}\left(\frac{\log(n+1)}{\log n}\right)^{1+\alpha} = \left(1 + \frac{1}{n}\right)\left(\frac{\log(1+n)}{\log n}\right)^{1+\alpha}$$

and
$$\frac{v_n}{v_{n+1}} = \left(1 + \frac{1}{n}\right)\frac{\log(1+n)}{\log n}$$

in the two respective cases. Next consider Raabe's expression. If first

$$\lim_{n=\infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) > 1, \quad \text{then ultimately} \quad n\left(\frac{u_n}{u_{n+1}} - 1\right) > \gamma > 1 \quad \text{and} \quad \frac{u_n}{u_{n+1}} > 1 + \frac{\gamma}{n}.$$

Now $\displaystyle \lim_{n=\infty}\left(\frac{\log(1+n)}{\log n}\right)^{1+\alpha} = 1$ and ultimately $\displaystyle \left(\frac{\log(1+n)}{\log n}\right)^{1+\alpha} < 1 + \epsilon,$

where ϵ is arbitrarily small. Hence ultimately if $\gamma > 1$,

$$\left(1+\frac{1}{n}\right)\left(\frac{\log(1+n)}{\log n}\right)^{1+\alpha} < 1 + \frac{1+\epsilon}{n} + \frac{\epsilon}{n^2} < 1 + \frac{\gamma}{n},$$

or $$v_n/v_{n+1} < u_n/u_{n+1} \quad \text{or} \quad u_{n+1}/u_n < v_{n+1}/v_n,$$

and the u-series converges. In like manner, secondly, if

$$\lim_{n=\infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) < 1, \quad \text{then ultimately} \quad \frac{u_n}{u_{n+1}} < 1 + \frac{\gamma}{n}, \qquad \gamma < 1;$$

and $$1 + \frac{\gamma}{n} < \left(1 + \frac{1}{n}\right)\frac{\log(1+n)}{\log n} \quad \text{or} \quad \frac{u_n}{u_{n+1}} < \frac{v_n}{v_{n+1}} \quad \text{or} \quad \frac{u_{n+1}}{u_n} > \frac{v_{n+1}}{v_n}.$$

Hence as the v-series now diverges, the u-series must diverge.

Suppose this test applied to the binomial series for $x = -1$. Then

$$\frac{u_n}{u_{n+1}} = \frac{n+1}{n-m}, \qquad \lim_{n=\infty} n\left(\frac{n+1}{n-m} - 1\right) = \lim_{n=\infty} \frac{m+1}{1 - \frac{m}{n}} = m + 1.$$

It follows that the series will converge if $m > 0$, but diverge if $m < 0$. If $x = +1$, the binomial series becomes alternating for $n > m + 1$. If the series of absolute values be considered, the ratio of successive terms $|u_n/u_{n+1}|$ is still $(n+1)/(n-m)$ and the binomial series converges absolutely if $m > 0$; but when $m < 0$ the series of absolute values diverges and it remains an open question whether the alternating series diverges or converges. Consider therefore the alternating series

$$1 + m + \frac{m(m-1)}{1 \cdot 2} + \frac{m(m-1)(m-2)}{1 \cdot 2 \cdot 3} + \cdots + \frac{m(m-1)\cdots(m-n+1)}{1 \cdot 2 \cdots n} + \cdots, \; m < 0.$$

This will converge if the limit of u_n is 0, but otherwise it will diverge. Now if $m \leqq -1$, the successive terms are multiplied by a factor $|m - n + 1|/n \geqq 1$ and they cannot approach 0. When $-1 < m < 0$, let $1 + m = \theta$, a fraction. Then the nth term in the series is

$$|u_n| = (1-\theta)\left(1 - \frac{\theta}{2}\right)\cdots\left(1 - \frac{\theta}{n}\right)$$

and $$-\log|u_n| = -\log(1-\theta) - \log\left(1 - \frac{\theta}{2}\right) - \cdots - \log\left(1 - \frac{\theta}{n}\right).$$

Each successive factor diminishes the term but diminishes it by so little that it may not approach 0. The logarithm of the term is a series. Now apply Cauchy's test.

$$\int^{\infty} -\log\left(1 - \frac{\theta}{n}\right) dn = \left[-n\log\left(1 - \frac{\theta}{n}\right) + \theta\log(n - \theta)\right]^{\infty} = \infty.$$

The series of logarithms therefore diverges and $\lim|u_n| = e^{-\infty} = 0$. Hence the terms approach 0 as a limit. The final results are therefore that when $x = -1$ the binomial series converges if $m > 0$ but diverges if $m < 0$; and when $x = +1$ it converges (absolutely) if $m > 0$, diverges if $m < -1$, and converges (not absolutely) if $-1 < m < 0$.

EXERCISES

1. State the number of terms which must be taken in these alternating series to obtain the sum accurate to three decimals. If the number is not greater than 8, compute the value of the series to three decimals, carrying four figures in the work:

(α) $\frac{1}{3}-\frac{1}{2\cdot 3^2}+\frac{1}{3\cdot 3^3}-\frac{1}{4\cdot 3^4}+\cdots$, ($\beta$) $\frac{1}{2}-\frac{1}{2\cdot 2^2}+\frac{1}{3\cdot 2^3}-\frac{1}{4\cdot 2^4}+\cdots$,

(γ) $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots$, ($\delta$) $\frac{1}{\log 2}-\frac{1}{\log 3}+\frac{1}{\log 4}-\cdots$,

(ϵ) $1-\frac{1}{3^2}+\frac{1}{5^2}-\frac{1}{7^2}+\cdots$, ($\zeta$) $e^{-1}-2e^{-2}+3e^{-3}-4e^{-4}+\cdots$.

2. Find the values of x for which these alternating series converge or diverge:

(α) $1-x^2+\frac{1}{2}x^4-\frac{1}{3}x^6+\cdots$, ($\beta$) $1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\cdots$,

(γ) $x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\cdots$, ($\delta$) $x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\cdots$,

(ϵ) $1-\frac{x^2}{1^p}+\frac{x^4}{2^p}-\frac{x^6}{3^p}+\cdots$, ($\zeta$) $2x-\frac{2^3x^3}{3}+\frac{2^5x^5}{5}-\frac{2^7x^7}{7}+\cdots$,

(η) $\frac{1}{x}-\frac{1}{x+1}+\frac{1}{x+2}-\frac{1}{x+3}+\cdots$, ($\theta$) $\frac{1}{x}-\frac{2}{x+1}+\frac{2^2}{x+2}-\frac{2^3}{x+3}+\cdots$.

3. Show that these series converge and estimate the error after n terms:

(α) $1+\frac{1}{2^2}+\frac{1}{3^3}+\frac{1}{4^4}+\cdots$, ($\beta$) $\frac{1}{3}+\frac{1\cdot 2}{3\cdot 5}+\frac{1\cdot 2\cdot 3}{3\cdot 5\cdot 7}+\cdots$,

(γ) $\frac{1}{2}+\frac{1}{2\cdot 2^2}+\frac{1}{3\cdot 2^3}+\frac{1}{4\cdot 2^4}+\cdots$, ($\delta$) $\left(\frac{1}{3}\right)^2+\left(\frac{1\cdot 2}{3\cdot 5}\right)^2+\left(\frac{1\cdot 2\cdot 3}{3\cdot 5\cdot 7}\right)^2+\cdots$.

From the estimate of error state how many terms are required to compute the series accurate to two decimals and make the computation, carrying three figures. Test for convergence or divergence:

(ϵ) $\sin 1+\sin\frac{1}{2}+\sin\frac{1}{3}+\cdots$, ($\zeta$) $\sin^2 1+\sin^2\frac{1}{2}+\sin^2\frac{1}{3}+\cdots$,

(η) $\tan^{-1}1+\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{3}+\cdots$, ($\theta$) $\tan 1+\frac{1}{\sqrt{2}}\tan\frac{1}{2}+\frac{1}{\sqrt{3}}\tan\frac{1}{3}+\cdots$,

(ι) $\frac{1}{1+1}+\frac{1}{2+\sqrt{2}}+\frac{1}{3+\sqrt{3}}+\cdots$, ($\kappa$) $\frac{1}{2^2-1^2}+\frac{1}{3^2-2^2}+\frac{1}{4^2-3^2}+\cdots$,

(λ) $\frac{1}{x}+\frac{2}{x^2}+\frac{2\cdot 3}{x^3}+\frac{2\cdot 3\cdot 4}{x^4}+\cdots$, ($\mu$) $\frac{1}{x}+\frac{\sqrt{2}}{x^2}+\frac{\sqrt[3]{3}}{x^3}+\frac{\sqrt[4]{4}}{x^4}+\cdots$.

4. Apply Cauchy's integral to determine the convergence or divergence:

(α) $1+\frac{\log 2}{2^p}+\frac{\log 3}{3^p}+\frac{\log 4}{4^p}+\cdots$, ($\beta$) $1+\frac{1}{2(\log 2)^p}+\frac{1}{3(\log 3)^p}+\frac{1}{4(\log 4)^p}+\cdots$,

(γ) $1+\sum_{2}^{\infty}\frac{1}{n\log n\log\log n}$, ($\delta$) $1+\sum_{2}^{\infty}\frac{1}{n\log n(\log\log n)^p}$,

(ϵ) $\cot^{-1}1+\cot^{-1}2+\cdots$, ($\zeta$) $1+\frac{2}{2^2+1}+\frac{3}{3^2+2}+\frac{4}{4^2+3}+\cdots$.

5. Apply the ratio test to determine convergence or divergence:

(α) $\frac{1}{2}+\frac{2}{2^2}+\frac{3}{2^3}+\frac{4}{2^4}+\cdots$, ($\beta$) $\frac{2^2}{2^{10}}+\frac{2^3}{3^{10}}+\frac{2^4}{4^{10}}+\cdots$,

(γ) $\frac{2\,!}{2^5}+\frac{3\,!}{3^5}+\frac{4\,!}{4^5}+\frac{5\,!}{5^5}+\cdots$, ($\delta$) $\frac{2^2}{2\,!}+\frac{3^3}{3\,!}+\frac{4^4}{4\,!}+\cdots$,

(ϵ) Ex. 3(α), (β), (γ), (δ); Ex. 4(α), (ζ), (ζ) $\frac{2^{10}}{10^2}+\frac{3^{10}}{10^3}+\frac{4^{10}}{10^4}+\cdots$,

(η) $1+\frac{x^2}{2\,!}+\frac{x^4}{4\,!}+\frac{x^6}{6\,!}+\cdots$, ($\theta$) $1+\frac{x^2}{2^p}+\frac{x^4}{4^p}+\cdots$,

(ι) $x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\cdots$, ($\kappa$) $\frac{1}{a}+\frac{bx}{a^2}+\frac{b^2x^3}{a^3}+\cdots$.

6. Where the ratio test fails, discuss the above exercises by any method.

7. Prove that if a series of decreasing positive terms converges, $\lim nu_n = 0$.

8. Formulate the Cauchy integral test for divergence and check the statement on page 422. The test has been used in the text and in Ex. 4. *Prove* the test.

9. Show that if the ratio test indicates the divergence of the series of absolute values, the series diverges no matter what the distribution of signs may be.

10. Show that if $\sqrt[n]{u_n}$ approaches a limit less than 1, the series (of positive terms) converges; but if $\sqrt[n]{u_n}$ approaches a limit greater than 1, it diverges.

11. If the terms of a convergent series $u_0+u_1+u_2+\cdots$ of positive terms be multiplied respectively by a set of positive numbers $a_0, a_1, a_2, \cdots$ all of which are less than some number G, the resulting series $a_0u_0+a_1u_1+a_2u_2+\cdots$ converges. State the corresponding theorem for divergent series. What if the given series has terms of opposite signs, but converges absolutely?

12. Show that the series $\frac{\sin x}{1^2}-\frac{\sin 2x}{2^2}+\frac{\sin 3x}{3^2}-\frac{\sin 4x}{4^2}+\cdots$ converges absolutely for any value of x, and that the series $1+x\sin\theta+x^2\sin 2\theta+x^3\sin 3\theta+\cdots$ converges absolutely for any x numerically less than 1, no matter what θ may be.

13. If $a_0, a_1, a_2, \cdots$ are any suite of numbers such that $\sqrt[n]{|a_n|}$ approaches a limit less than or equal to 1, show that the series $a_0+a_1x+a_2x^2+\cdots$ converges absolutely for any value of x numerically less than 1. Apply this to show that the following series converge absolutely when $|x|<1$;

(α) $1+\frac{1}{2}x^2+\frac{1\cdot 3}{2\cdot 4}x^4+\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}x^6+\cdots$, ($\beta$) $1-2x+3x^2-4x^3+\cdots$,

(γ) $1+x+2^px^2+3^px^3+4^px^4+\cdots$, ($\delta$) $1-x\log 1+x^2\log 4-x^3\log 9+\cdots$.

14. Show that in Ex. 10 it will be sufficient for convergence if $\sqrt[n]{u_n}$ becomes and remains less than $\gamma < 1$ without approaching a limit, and sufficient for divergence if there are an infinity of values for n such that $\sqrt[n]{u_n} > 1$. Note a similar generalization in Ex. 13 and state it.

15. If a power series $a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots$ converges for $x = X > 0$, it converges absolutely for any x such that $|x| < X$, and the series

$$a_0x + \tfrac{1}{2}a_1x^2 + \tfrac{1}{3}a_2x^3 + \cdots \quad \text{and} \quad a_1 + 2a_2x + 3a_3x^2 + \cdots,$$

obtained by integrating and differentiating term by term, also converge absolutely for any value of x such that $|x| < X$. The same result, by the same proof, holds if the terms $a_0, a_1X, a_2X^2, \cdots$ remain less than a fixed value G.

16. If the ratio of the successive terms in a series of positive terms be regarded as a function of $1/n$ and may be expanded by Maclaurin's Formula to give

$$\frac{u_n}{u_{n+1}} = \alpha + \beta\frac{1}{n} + \frac{\mu}{2}\left(\frac{1}{n}\right)^2, \qquad \mu \text{ remaining finite as } \frac{1}{n} \doteq 0,$$

the series converges if $\alpha > 1$ or $\alpha = 1$, $\beta > 1$, but diverges if $\alpha < 1$ or $\alpha = 1$, $\beta \leqq 1$. This test covers most of the series of positive terms which arise in practice. Apply it to various instances in the text and previous exercises. Why are there series to which this test is inapplicable?

17. If $\rho_0, \rho_1, \rho_2, \cdots$ is a decreasing suite of positive numbers approaching a limit λ and $S_0, S_1, S_2, \cdots$ is any limited suite of numbers, that is, numbers such that $|S_n| \leqq G$, show that the series

$$(\rho_0 - \rho_1)S_0 + (\rho_1 - \rho_2)S_1 + (\rho_2 - \rho_3)S_2 + \cdots \text{ converges absolutely,}$$

and
$$\left|\sum_0^\infty (\rho_n - \rho_{n+1})S_n\right| \leqq G(\rho_0 - \lambda).$$

18. Apply Ex. 17 to show that, $\rho_0, \rho_1, \rho_2, \cdots$ being a decreasing suite, if

$$u_0 + u_1 + u_2 + \cdots \text{ converges,} \qquad \rho_0u_0 + \rho_1u_1 + \rho_2u_2 + \cdots \text{ will converge also.}$$

N.B.
$$\begin{aligned}\rho_0u_0 + \rho_1u_1 + \cdots + \rho_nu_n &= \rho_0S_1 + \rho_1(S_2 - S_1) + \cdots + \rho_n(S_{n+1} - S_n)\\ &= S_1(\rho_0 - \rho_1) + \cdots + S_n(\rho_{n-1} - \rho_n) + \rho_nS_{n+1}.\end{aligned}$$

19. Apply Ex. 18 to prove Ex. 15 after showing that $\rho_0u_0 + \rho_1u_1 + \cdots$ must converge absolutely if $\rho_0 + \rho_1 + \cdots$ converges.

20. If $a_1, a_2, a_3, \cdots, a_n$ are n positive numbers less than 1, show that

$$(1 + a_1)(1 + a_2)\cdots(1 + a_n) > 1 + a_1 + a_2 + \cdots + a_n$$

and
$$(1 - a_1)(1 - a_2)\cdots(1 - a_n) > 1 - a_1 - a_2 - \cdots - a_n$$

by induction or any other method. Then since $1 + a_1 < 1/(1 - a_1)$ show that

$$\frac{1}{1 - (a_1 + a_2 + \cdots + a_n)} > (1 + a_1)(1 + a_2)\cdots(1 + a_n) > 1 + (a_1 + a_2 + \cdots + a_n),$$

$$\frac{1}{1 + (a_1 + a_2 + \cdots + a_n)} > (1 - a_1)(1 - a_2)\cdots(1 - a_n) > 1 - (a_1 + a_2 + \cdots + a_n),$$

if $a_1 + a_2 + \cdots + a_n < 1$. Or if $\prod$ be the symbol for a *product*,

$$\left(1-\sum_1^n a\right)^{-1} > \prod_1^n (1+a) > 1 + \sum_1^n a, \qquad \left(1+\sum_1^n a\right)^{-1} > \prod_1^n (1-a) > 1 - \sum_1^n a.$$

21. Let $\prod_1^\infty (1+u_1)(1+u_2)\cdots(1+u_n)(1+u_{n+1})\cdots$ be an infinite product and let P_n be the product of the first n factors. Show that $|P_{n+p} - P_n| < \epsilon$ is the necessary and sufficient condition that P_n approach a limit when n becomes infinite. Show that u_n must approach 0 as a limit if P_n approaches a limit.

22. In case P_n approaches a limit different from 0, show that if ϵ be assigned, a value of n can be found so large that for any value of p

$$\left|\frac{P_{n+p}}{P_n} - 1\right| = \left|\prod_{n+1}^{n+p} (1+u_i) - 1\right| < \epsilon \quad \text{or} \quad \prod_{n+1}^{n+p} (1+u_i) = 1 + \eta, \qquad |\eta| < \epsilon.$$

Conversely show that if this relation holds, P_n must approach a limit other than 0. The *infinite product* is said to *converge* when P_n approaches a limit other than 0; in all other cases it is said to diverge, including the case where $\lim P_n = 0$.

23. By combining Exs. 20 and 22 show that the necessary and sufficient condition that

$$P_n = (1+a_1)(1+a_2)\cdots(1+a_n) \quad \text{and} \quad Q_n = (1-a_1)(1-a_2)\cdots(1-a_n)$$

converge as n becomes infinite is that the series $a_1 + a_2 + \cdots + a_n + \cdots$ shall converge. Note that P_n is increasing and Q_n decreasing. Show that in case Σa diverges, P_n diverges to ∞ and Q_n to 0 (provided ultimately $a_i < 1$).

24. Define absolute convergence for infinite products and show that if a product converges absolutely it converges in its original form.

25. Test these products for convergence, divergence, or absolute convergence:

$$(\alpha)\ \left(1-\frac{1}{2}\right)\left(1-\frac{1}{4}\right)\left(1-\frac{1}{8}\right)\cdots, \qquad (\beta)\ \left(1+\frac{1}{2^2}\right)\left(1+\frac{1}{3^2}\right)\left(1+\frac{1}{4^2}\right)\cdots,$$

$$(\gamma)\ \prod_1^\infty\left[1-\left(\frac{nx}{n+1}\right)^n\right], \qquad (\delta)\ (1+x)(1+x^2)(1+x^4)(1+x^8)\cdots,$$

$$(\epsilon)\ \left(1-\frac{1}{\log 2}\right)\left(1-\frac{1}{(\log 4)^2}\right)\left(1-\frac{1}{(\log 8)^3}\right)\cdots, \qquad (\zeta)\ \prod_1^\infty\left[\left(1-\frac{x}{c+n}\right)e^{\frac{x}{n}}\right].$$

26. Given $\dfrac{\frac{1}{2}u^2}{1+u}$ or $\dfrac{1}{2}u^2 < u - \log(1+u) < \dfrac{1}{2}u^2$ or $\dfrac{\frac{1}{2}u^2}{1+u}$ according as u is a positive or negative fraction (see Ex. 29, p. 11). Prove that if Σu_n^2 converges, then

$$u_{n+1} + u_{n+2} + \cdots + u_{n+p} - \log(1+u_{n+1})(1+u_{n+2})\cdots(1+u_{n+p})$$
$$= (S_{n+p} - S_n) - (\log P_{n+p} - \log P_n)$$

can be made as small as desired by taking n large enough regardless of p. Hence prove that if Σu_n^2 converges, $\prod(1+u_n)$ converges if Σu_n does, but diverges to ∞ if Σu_n diverges to $+\infty$, and diverges to 0 if Σu_n diverges to $-\infty$; whereas if Σu_n^2 diverges while Σu_n converges, the product diverges to 0.

27. Apply Ex. 26 to: (α) $\left(1+\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1+\frac{1}{4}\right)\left(1-\frac{1}{5}\right)\cdots$,

(β) $\left(1-\frac{1}{\sqrt{2}}\right)\left(1+\frac{1}{\sqrt{3}}\right)\left(1-\frac{1}{\sqrt{4}}\right)\cdots$, ($\gamma$) $\left(1+\frac{x}{1}\right)\left(1-\frac{x^2}{2}\right)\left(1+\frac{x^3}{3}\right)\left(1-\frac{x^4}{4}\right)\cdots$

28. Suppose the integrand $f(x)$ of an infinite integral oscillates as x becomes infinite. What test might be applicable from the construction of an alternating series?

165. Series of functions. If the terms of a series

$$S(x) = u_0(x) + u_1(x) + \cdots + u_n(x) + \cdots \tag{6}$$

are functions of x, the series defines a function $S(x)$ of x for every value of x for which it converges. If the individual terms of the series are continuous functions of x over some interval $a \leqq x \leqq b$, the sum $S_n(x)$ of n terms will of course be a continuous function over that interval. Suppose that the series converges for all points of the interval. Will it then be true that $S(x)$, the limit of $S_n(x)$, is also a continuous function over the interval? Will it be true that the integral term by term,

$$\int_a^b u_0(x)\,dx + \int_a^b u_1(x)\,dx + \cdots, \quad \text{converges to} \quad \int_a^b S(x)\,dx?$$

Will it be true that the derivative term by term,

$$u_0'(x) + u_1'(x) + \cdots, \quad \text{converges to} \quad S'(x)?$$

There is no *a priori* reason why any of these things should be true; for the proofs which were given in the case of finite sums will not apply to the case of a limit of a sum of an infinite number of terms (cf. § 144).

These questions may readily be thrown into the form of questions concerning the possibility of inverting the order of two limits (see § 44).

For integration: Is $\displaystyle\int_a^b \lim_{n=\infty} S_n(x)\,dx = \lim_{n=\infty}\int_a^b S_n(x)\,dx$?

For differentiation: Is $\displaystyle\frac{d}{dx}\lim_{n=\infty} S_n(x) = \lim_{n=\infty}\frac{d}{dx}S_n(x)$?

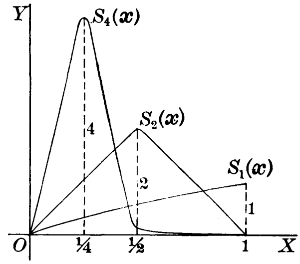
For continuity: Is $\displaystyle\lim_{x\doteq x_0}\lim_{n=\infty} S_n(x) = \lim_{n=\infty}\lim_{x\doteq x_0} S_n(x)$?

As derivatives and definite integrals are themselves defined as limits, the existence of a double limit is clear. That all three of the questions must be answered in the negative unless some restriction is placed on the way in which $S_n(x)$ converges to $S(x)$ is clear from some examples. Let $0 \leqq x \leqq 1$ and

$$S_n(x) = xn^2e^{-nx}, \quad \text{then} \quad \lim_{n=\infty} S_n(x) = 0, \quad \text{or} \quad S(x) = 0.$$

No matter what the value of x, the limit of $S_n(x)$ is 0. The limiting function is therefore continuous in this case; but from the manner in which $S_n(x)$ converges to $S(x)$ it is apparent that under suitable conditions the limit would not be continuous. The area under the limit $S(x) = 0$ from 0 to 1 is of course 0; but the limit of the area under $S_n(x)$ is

$$\lim_{n=\infty} \int_0^1 xn^2 e^{-nx} dx = \lim_{n=\infty} \left[e^{-nx}(-nx - 1) \right]_0^1 = 1.$$



The derivative of the limit at the point $x = 0$ is of course 0; but the limit,

$$\lim_{n=\infty} \left[\frac{d}{dx}(xn^2 e^{-nx}) \right]_{x=0}$$

$$= \lim_{n=\infty} \left[n^2 e^{-nx}(1 - nx) \right]_{x=0} = \lim_{n=\infty} n^2 = \infty,$$

of the derivative is infinite. Hence in this case two of the questions have negative answers and one of them a positive answer.

If a suite of functions such as $S_1(x), S_2(x), \cdots, S_n(x), \cdots$ converge to a limit $S(x)$ over an interval $a \leqq x \leqq b$, the conception of a limit requires that when ϵ is assigned and x_0 is assumed it must be possible to take n so large that $|R_n(x_0)| = |S(x_0) - S_n(x_0)| < \epsilon$ for this and any larger n. The suite is said to *converge uniformly* toward its limit, if this condition can be satisfied simultaneously for all values of x in the interval, that is, if when ϵ is assigned it is possible to take n so large that $|R_n(x)| < \epsilon$ for every value of x in the interval and for this and any larger n. In the above example the convergence was not uniform; the figure shows that no matter how great n, there are always values of x between 0 and 1 for which $S_n(x)$ departs by a large amount from its limit 0.

The uniform convergence of a continuous function $S_n(x)$ to its limit is sufficient to insure the continuity of the limit $S(x)$. To show that $S(x)$ is continuous it is merely necessary to show that when ϵ is assigned it is possible to find a Δx so small that $|S(x + \Delta x) - S(x)| < \epsilon$. But $|S(x + \Delta x) - S(x)| = |S_n(x + \Delta x) - S_n(x) + R_n(x + \Delta x) - R_n(x)|$; and as by hypothesis R_n converges uniformly to 0, it is possible to take n so large that $|R_n(x + \Delta x)|$ and $|R_n(x)|$ are less than $\frac{1}{3}\epsilon$ irrespective of x. Moreover, as $S_n(x)$ is continuous it is possible to take Δx so small that $|S_n(x + \Delta x) - S_n(x)| < \frac{1}{3}\epsilon$ irrespective of x. Hence $|S(x + \Delta x) - S(x)| < \epsilon$, and the theorem is proved. Although the uniform convergence of S_n to S is a sufficient condition for the continuity of S, it is not a necessary condition, as the above example shows.

The uniform convergence of $S_n(x)$ to its limit insures that

$$\lim_{n=\infty} \int_a^b S_n(x)\,dx = \int_a^b S(x)\,dx.$$

For in the first place $S(x)$ must be continuous and therefore integrable. And in the second place when ϵ is assigned, n may be taken so large that $|R_n(x)| < \epsilon/(b-a)$. Hence

$$\left|\int_a^b S(x)\,dx - \int_a^b S_n(x)\,dx\right| = \left|\int_a^b R_n(x)\,dx\right| < \int_a^b \frac{\epsilon}{b-a}\,dx = \epsilon,$$

and the result is proved. Similarly *if $S_n'(x)$ is continuous and converges uniformly to a limit $T(x)$, then $T(x) = S'(x)$.* For by the above result on integrals,

$$\int_a^x T(x)\,dx = \lim_{n=\infty}\int_a^x S_n'(x)\,dx = \lim_{n=\infty}\Big[S_n(x) - S_n(a)\Big] = S(x) - S(a).$$

Hence $T(x) = S'(x)$. It should be noted that this proves incidentally that if $S_n'(x)$ is continuous and converges uniformly to a limit, then $S(x)$ actually has a derivative, namely $T(x)$.

In order to apply these results to a series, it is necessary to have a *test for the uniformity of the convergence of the series;* that is, for the uniform convergence of $S_n(x)$ to $S(x)$. One such test is *Weierstrass's M-test: The series*

$$u_0(x) + u_1(x) + \cdots + u_n(x) + \cdots \tag{7}$$

will converge uniformly provided a convergent series

$$M_0 + M_1 + \cdots + M_n + \cdots \tag{8}$$

of positive terms may be found such that ultimately $|u_i(x)| \leqq M_i$. The proof is immediate. For

$$|R_n(x)| = |u_n(x) + u_{n+1}(x) + \cdots| \leqq M_n + M_{n+1} + \cdots$$

and as the M-series converges, its remainder can be made as small as desired by taking n sufficiently large. Hence any series of continuous functions defines a continuous function and may be integrated term by term to find the integral of that function provided an M-test series may be found; and the derivative of that function is the derivative of the series term by term if this derivative series admits an M-test.

To apply the work to an example consider whether the series

$$S(x) = \frac{\cos x}{1^2} + \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} + \cdots + \frac{\cos nx}{n^2} + \cdots \tag{7'}$$

defines a continuous function and may be integrated and differentiated term by term as

$$\int_0^x S(x) = \frac{\sin x}{1^3} + \frac{\sin 2x}{2^3} + \frac{\sin 3x}{3^3} + \cdots + \frac{\sin nx}{n^3} + \cdots \tag{7''}$$

and

$$\frac{d}{dx}S(x) = -\frac{\sin x}{1} - \frac{\sin 2x}{2} - \frac{\sin 3x}{3} - \cdots - \frac{\sin nx}{n} - \cdots. \tag{7'''}$$

As $|\cos x| \leqq 1$, the convergent series $1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} + \cdots$ may be taken as an M-series for $S(x)$. Hence $S(x)$ is a continuous function of x for all real values of x, and the integral of $S(x)$ may be taken as the limit of the integral of $S_n(x)$, that is, as the integral of the series term by term as written. On the other hand, an M-series for (7‴) cannot be found, for the series $1 + \frac{1}{2} + \frac{1}{3} + \cdots$ is not convergent. It therefore appears that $S'(x)$ may not be identical with the term-by-term derivative of $S(x)$; it does not follow that it will not be, — merely that it may not be.

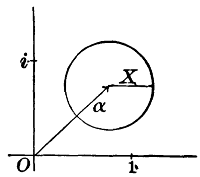
166. Of series with variable terms, the *power series*

$$f(z) = a_0 + a_1(z - \alpha) + a_2(z - \alpha)^2 + \cdots + a_n(z - \alpha)^n + \cdots \tag{9}$$

is perhaps the most important. Here z, α, and the coefficients a_i may be either real or complex numbers. This series may be written more simply by setting $x = z - \alpha$; then

$$f(x + \alpha) = \phi(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n + \cdots \tag{9'}$$

is a series which surely converges for $x = 0$. It may or may not converge for other values of x, but from Ex. 15 or 19 above it is seen that if the series converges for X, it converges absolutely for any x of smaller absolute value; that is, if a circle of radius X be drawn around the origin in the complex plane for x or about the point α in the complex plane for z, the series (9) and (9′) respectively will converge absolutely for all complex numbers which lie within these circles.



Three cases should be distinguished. First the series may converge for any value x no matter how great its absolute value. The circle may then have an indefinitely large radius; the series converge for all values of x or z and the function defined by them is finite (whether real or complex) for all values of the argument. Such a function is called an *integral function* of the complex variable z or x. Secondly, the series may converge for no other value than $x = 0$ or $z = \alpha$ and therefore cannot define any function. Thirdly, there may be a definite largest value for the radius, say R, such that for any point within the respective circles of radius R the series converge and define a function, whereas for any point outside the circles the series diverge. The circle of radius R is called the *circle of convergence* of the series.

As the matter of the radius and circle of convergence is important, it will be well to go over the whole matter in detail. Consider the suite of numbers

$$|a_1|, \quad \sqrt[2]{|a_2|}, \quad \sqrt[3]{|a_3|}, \quad \cdots, \quad \sqrt[n]{|a_n|}, \quad \cdots.$$

Let them be imagined to be located as points with coördinates between 0 and $+\infty$ on a line. Three possibilities as to the distribution of the points arise. First they may be unlimited above, that is, it may be possible to pick out from the suite a set of numbers which increase without limit. Secondly, the numbers may converge to the limit 0. Thirdly, neither of these suppositions is true and the numbers from 0 to $+\infty$ may be divided into two classes such that every number in the first class is less than an infinity of numbers of the suite, whereas any number of the second class is surpassed by only a finite number of the numbers in the suite. The two classes will then have a frontier number which will be represented by $1/R$ (see §§ 19 ff.).

In the first case no matter what x may be it is possible to pick out members from the suite such that the set $\sqrt[i]{|a_i|}$, $\sqrt[j]{|a_j|}$, $\sqrt[k]{|a_k|}, \cdots$, with $i<j<k \cdots$, increases without limit. Hence the set $\sqrt[i]{|a_i|}|x|$, $\sqrt[j]{|a_j|}\,|x|, \cdots$ will increase without limit; the terms $a_i x^i$ $a_j x^j, \cdots$ of the series (9′) do not approach 0 as their limit, and the series diverges for all values of x other than 0. In the second case the series converges for any value of x. For let ϵ be any number less than $1/|x|$. It is possible to go so far in the suite that all subsequent numbers of it shall be less than this assigned ϵ. Then

$$|a_{n+p}x^{n+p}| < \epsilon^{n+p}|x|^{n+p} \quad \text{and} \quad \epsilon^n|x|^n + \epsilon^{n+1}|x|^{n+1} + \cdots, \qquad \epsilon|x| < 1,$$

serves as a comparison series to insure the absolute convergence of (9′). In the third case the series converges for any x such that $|x| < R$ but diverges for any x such that $|x| > R$. For if $|x| < R$, take $\epsilon < R - |x|$ so that $|x| < R - \epsilon$. Now proceed in the suite so far that all the subsequent numbers shall be less than $1/(R-\epsilon)$, which is greater than $1/R$. Then

$$|a_{n+p}x^{n+p}| < \frac{|x|^{n+p}}{(R-\epsilon)^{n+p}} < 1, \quad \text{and} \quad \sum_0^\infty \frac{|x|^{n+p}}{(R-\epsilon)^{n+p}}$$

will do as a comparison series. If $|x| > R$, it is easy to show the terms of (9′) do not approach the limit 0.

Let a circle of radius r less than R be drawn concentric with the circle of convergence. Then *within the circle of radius $r < R$ the power series* (9′) *converges uniformly and defines a continuous function; the integral of the function may be had by integrating the series term by term,*

$$\Phi(x) = \int_0^x \phi(x)\,dx = a_0 x + \frac{1}{2}a_1 x^2 + \frac{1}{3}a_2 x^3 + \cdots + \frac{1}{n}a_{n-1}x^n + \cdots;$$

and the series of derivatives converges uniformly and represents the derivative of the function,

$$\phi'(x) = a_1 + 2a_2 x + 3a_3 x^2 + \cdots + na_n x^{n-1} + \cdots.$$

To prove these theorems it is merely necessary to set up an M-series for the series itself and for the series of derivatives. Let X be any number between r and R. Then

$$|a_0| + |a_1|X + |a_2|X^2 + \cdots + |a_n|X^n + \cdots \tag{10}$$

converges because $X < R$; and furthermore $|a_n x^n| < |a_n| X^n$ holds for any x such that $|x| < X$, that is, for all points within and on the circle of radius r. Moreover as $|x| < X$,

$$|na_n x^{n-1}| = |a_n| \frac{n}{X}\left(\frac{|x|}{X}\right)^{n-1} X^n < |a_n| X^n$$

holds for sufficiently large values of n and for any x such that $|x| \leqq r$. Hence (10) serves as an M-series for the given series and the series of derivatives; and the theorems are proved. It should be noticed that it is incorrect to say that the convergence is uniform over the circle of radius R, although the statement is true of any circle within that circle no matter how small $R - r$. For an apparently slight but none the less important extension to include, in some cases, some points upon the circle of convergence see Ex. 5.

An immediate corollary of the above theorems is that *any power series* (9) *in the complex variable which converges for other values than* $z = a$, *and hence has a finite circle of convergence or converges all over the complex plane, defines an analytic function* $f(z)$ *of* z *in the sense of* §§ 73, 126; for the series is differentiable within any circle within the circle of convergence and thus the function has a definite finite and continuous derivative.

167. It is now possible to extend Taylor's and Maclaurin's Formulas, which developed a function of a real variable x into a polynomial plus a remainder, to *infinite series* known as Taylor's and Maclaurin's Series, which express the function as a power series, provided the remainder after n terms converges uniformly toward 0 as n becomes infinite. It will be sufficient to treat one case. Let

$$f(x) = f(0) + f'(0)x + \frac{1}{2!}f''(0)x^2 + \cdots + \frac{1}{(n-1)!}f^{(n-1)}(0)x^{n-1} + R_n,$$

$$R_n = \frac{x^n}{n!}f^{(n)}(\theta x) = \frac{x^n(1-\theta)^{n-1}}{(n-1)!}f^{(n)}(\theta x) = \frac{1}{(n-1)!}\int_0^x t^{n-1}f^{(n)}(x-t)\,dt,$$

$$\lim_{n=\infty} R_n(x) = 0 \text{ uniformly in some interval } -h \leqq x \leqq h,$$

where the first line is Maclaurin's Formula, the second gives differnet forms of the remainder, and the third expresses the condition that the remainder converges to 0. Then the series

$$f(0) + f'(0)x + \frac{1}{2!}f''(0)x^2 + \cdots + \frac{1}{(n-1)!}f^{(n-1)}(0)x^{n-1} + \frac{1}{n!}f^{(n)}(0)x^n + \cdots \qquad (11)$$

converges to the value $f(x)$ for any x in the interval. The proof consists merely in noting that $f(x) - R_n(x) = S_n(x)$ is the sum of the first n terms of the series and that $|R_n(x)| < \epsilon$.

In the case of the exponential function e^x the nth derivative is e^x, and the remainder, taken in the first form, becomes

$$R_n(x) = \frac{1}{n!} e^{\theta x} x^n, \qquad |R_n(x)| < \frac{1}{n!} e^h h^n, \qquad |x| \leqq h.$$

As n becomes infinite, R_n clearly approaches zero no matter what the value of h; and

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots$$

is the infinite series for the exponential function. The series converges for all values of x real or complex and may be taken as the definition of e^x for complex values. This definition may be shown to coincide with that obtained otherwise (§ 74).

For the expansion of $(1 + x)^m$ the remainder may be taken in the second form.

$$R_n(x) = \frac{m(m-1)\cdots(m-n+1)}{1\cdot 2\cdots(n-1)} x^n \left(\frac{1-\theta}{1+\theta x}\right)^{n-1} (1+\theta x)^{m-1},$$

$$|R_n(x)| < \left|\frac{m(m-1)\cdots(m-n+1)}{1\cdot 2\cdots(n-1)}\right| h^n (1+h)^{m-1}, \qquad h < 1.$$

Hence when $h < 1$ the limit of $R_n(x)$ is zero and the infinite expansion

$$(1+x)^m = 1 + mx + \frac{m(m-1)}{2!} x^2 + \frac{m(m-1)(m-2)}{3!} x^3 + \cdots$$

is valid for $(1 + x)^m$ for all values of x numerically less than unity.

If in the binomial expansion x be replaced by $-x^2$ and m by $-\frac{1}{2}$,

$$\frac{1}{\sqrt{1-x^2}} = 1 + \frac{1}{2}x^2 + \frac{1\cdot 3}{2\cdot 4}x^4 + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}x^6 + \frac{1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8}x^8 + \cdots.$$

This series converges for all values of x numerically less than 1, and hence converges uniformly whenever $|x| \leqq h < 1$. It may therefore be integrated term by term.

$$\sin^{-1}x = x + \frac{1}{2}\frac{x^3}{3} + \frac{1\cdot 3}{2\cdot 4}\frac{x^5}{5} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\frac{x^7}{7} + \frac{1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8}\frac{x^9}{9} + \cdots.$$

This series is valid for all values of x numerically less than unity. The series also converges for $x = \pm 1$, and hence by Ex. 5 is uniformly convergent when $-1 \leqq x \leqq 1$.

But Taylor's and Maclaurin's series may also be extended directly to functions $f(z)$ of a complex variable. If $f(z)$ is single valued and has a definite continuous derivative $f'(z)$ at every point of a region and on the boundary, the expansion

$$f(z) = f(a) + f'(a)(z-a) + \cdots + f^{(n-1)}(a)\frac{(z-a)^{n-1}}{(n-1)!} + R_n$$

has been established (§ 126) with the remainder in the form

$$|R_n(z)| = \left|\frac{(z-a)^n}{2\pi}\int_O \frac{f(t)\,dt}{(t-a)^n(t-z)}\right| \leqq \frac{1}{2\pi}\frac{r^n}{\rho^n}\frac{ML}{\rho - r}$$

for all points z within the circle of radius r (Ex. 7, p. 306). As n becomes infinite, R_n approaches zero uniformly, and hence the infinite series

$$f(z) = f(a) + f'(a)(z-a) + \cdots + f^{(n)}(a)\frac{(z-a)^n}{n!} + \cdots \qquad (12)$$

is valid at all points within the circle of radius r and upon its circumference. The expansion is therefore convergent and valid for any z actually within the circle of radius ρ.

Even for real expansions (11) the significance of this result is great because, except in the simplest cases, it is impossible to compute $f^{(n)}(x)$ and establish the convergence of Taylor's series for real variables. The result just found shows that if the values of the function be considered for complex values z in addition to real values x, the circle of convergence will extend out to the nearest point where the conditions imposed on $f(z)$ break down, that is, to the nearest point at which $f(z)$ becomes infinite or otherwise ceases to have a definite continuous derivative $f'(z)$. For example, there is nothing in the behavior of the function

$$(1+x^2)^{-1} = 1 - x^2 + x^4 - x^6 + x^8 - \cdots,$$

as far as real values are concerned, which should indicate why the expansion holds only when $|x| < 1$; but in the complex domain the function $(1+z^2)^{-1}$ becomes infinite at $z = \pm i$, and hence the greatest circle about $z = 0$ in which the series could be expected to converge has a unit radius. Hence by considering $(1+z^2)^{-1}$ for complex values, it can be predicted without the examination of the nth derivative that the Maclaurin development of $(1+x^2)^{-1}$ will converge when and only when x is a proper fraction.

EXERCISES

1. (α) Does $x + x(1-x) + x(1-x)^2 + \cdots$ converge uniformly when $0 \leqq x \leqq 1$?

(β) Does the series $(1+k)^{\frac{1}{k}} = 1 + 1 + \frac{1-k}{2!} + \frac{(1-k)(1-2k)}{3!} + \cdots$ converge uniformly for small values of k? Can the derivation of the limit e of § 4 thus be made rigorous and the value be found by setting $k = 0$ in the series?

2. Test these series for uniform convergence; also the series of derivatives:

(α) $1 + x\sin\theta + x^2\sin 2\theta + x^3 \sin 3\theta + \cdots, \qquad |x| \leqq X < 1,$

(β) $1 + \frac{\sin x}{1^{2\frac{1}{2}}} + \frac{\sin^2 x}{2^{2\frac{1}{2}}} + \frac{\sin^3 x}{3^{2\frac{1}{2}}} + \frac{\sin^4 x}{4^{2\frac{1}{2}}} + \cdots, \qquad |x| \leqq X < \infty,$

(γ) $\frac{x-1}{x} + \frac{1}{2}\left(\frac{x-1}{x}\right)^2 + \frac{1}{3}\left(\frac{x-1}{x}\right)^3 + \cdots, \qquad \frac{1}{2} < \gamma \leqq x \leqq X < \infty,$

(δ) $\frac{x-1}{x+1} + \frac{1}{3}\left(\frac{x-1}{x+1}\right)^3 + \frac{1}{5}\left(\frac{x-1}{x+1}\right)^5 + \cdots, \qquad 0 < \gamma \leqq x \leqq X < \infty.$

(ϵ) Consider complex as well as real values of the variable.

3. Determine the radius of convergence and draw the circle. Note that in practice the test ratio is more convenient than the theoretical method of the text:

(α) $x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \cdots$, ($\beta$) $x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \cdots$,

(γ) $\frac{1}{a}\left[1 + \frac{bx}{a} + \frac{b^2x^2}{a^2} + \frac{b^3x^3}{a^3} + \cdots\right]$, ($\delta$) $1 - x^2 + \frac{x^4}{2\,!} - \frac{x^6}{3\,!} + \frac{x^8}{4\,!} - \cdots$,

(ϵ) $\frac{1}{1}x - (\frac{1}{1} + \frac{1}{2})x^2 + (\frac{1}{1} + \frac{1}{2} + \frac{1}{3})x^3 - (\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4})x^4 + \cdots$,

(ζ) $1 - \frac{3^2 + 3}{4 \cdot 2\,!}x^2 + \frac{3^4 + 3}{4 \cdot 4\,!} - \frac{3^6 + 3}{4 \cdot 6\,!}x^6 + \cdots$,

(η) $1 - x + x^4 - x^5 + x^8 - x^9 + x^{12} - x^{13} + \cdots$,

(θ) $(x-1)^1 - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4 + \cdots$,

(ι) $x - \frac{(m-1)(m+2)}{3\,!}x^3 + \frac{(m-1)(m-3)(m+2)(m+4)}{5\,!}x^5 - \cdots$,

(κ) $1 - \frac{x^2}{2^2(m+1)} + \frac{x^4}{2^4 \cdot 2\,!(m+1)(m+2)} - \frac{x^6}{2^6 \cdot 3\,!(m+1)(m+2)(m+3)} + \cdots$,

(λ) $\frac{x^2}{2^2} - \frac{x^4}{2^4(2\,!)^2}\left(\frac{1}{1} + \frac{1}{2}\right) + \frac{x^6}{2^6(3\,!)^2}\left(\frac{1}{1} + \frac{1}{2} + \frac{1}{3}\right) - \frac{x^8}{2^8(4\,!)^2}\left(\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}\right) + \cdots$,

(μ) $1 + \frac{\alpha\beta}{1 \cdot \gamma}x + \frac{\alpha(\alpha+1)\beta(\beta+1)}{1 \cdot 2 \cdot \gamma(\gamma+1)}x^2 + \frac{\alpha(\alpha+1)(\alpha+2)\beta(\beta+1)(\beta+2)}{1 \cdot 2 \cdot 3 \cdot \gamma(\gamma+1)(\gamma+2)}x^3 + \cdots$.

4. Establish the Maclaurin expansions for the elementary functions:

(α) $\log(1-x)$, (β) $\sin x$, (γ) $\cos x$, (δ) $\cosh x$,
(ϵ) a^x, (ζ) $\tan^{-1}x$, (η) $\sinh^{-1}x$, (θ) $\tanh^{-1}x$.

5. *Abel's Theorem.* If the infinite series $a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots$ converges for the value X, it converges uniformly in the interval $0 \leqq x \leqq X$. Prove this by showing that (see Exs. 17–19, p. 428)

$$|R_n(x)| = |a_nx^n + a_{n+1}x^{n+1} + \cdots| < \left(\frac{x}{X}\right)^n |a_nX^n + \cdots + a_{n+p}X^{n+p}|,$$

when p is rightly chosen. Apply this to extending the interval over which the series is uniformly convergent to extreme values of the interval of convergence wherever possible in Exs. 4 (α), (ζ), (θ).

6. Examine sundry of the series of Ex. 3 in regard to their convergence at extreme points of the interval of convergence or at various other points of the circumference of their circle of convergence. Note the significance in view of Ex. 5.

7. Show that $f(x) = e^{-\frac{1}{x^2}}$, $f(0) = 0$, cannot be expanded into an infinite Maclaurin series by showing that $R_n = e^{-\frac{1}{x^2}}$, and hence that R_n does not converge uniformly toward 0 (see Ex. 9, p. 66). Show this also from the consideration of complex values of x.

8. From the consideration of complex values determine the interval of convergence of the Maclaurin series for

(α) $\tan x = \frac{\sin x}{\cos x}$, ($\beta$) $\frac{x}{e^x - 1}$, (γ) $\tanh x$, (δ) $\log(1 + e^x)$.

9. Show that if two similar infinite power series represent the same function in any interval the coefficients in the series must be equal (cf. § 32).

10. From $1+2r\cos x+r^2=(1+re^{ix})(1+re^{-ix})=r^2\left(1+\frac{e^{ix}}{r}\right)\left(1+\frac{e^{-ix}}{r}\right)$

prove
$$\log(1+2r\cos x+r^2)=2\left(r\cos x-\frac{r^2}{2}\cos 2x+\frac{r^3}{3}\cos 3x-\cdots\right),$$
$$\int_0^x\log(1+2r\cos x+r^2)\,dx=2\left(r\sin x-\frac{r^2}{2^2}\sin 2x+\frac{r^3}{3^2}\sin 3x-\cdots\right);\qquad r<1$$

and
$$\log(1+2r\cos x+r^2)=2\log r+2\left(\frac{\cos x}{r}-\frac{\cos 2x}{2r^2}+\frac{\cos 3x}{3r^2}-\cdots\right),$$
$$\int_0^x\log(1+2r\cos x+r^2)\,dx=2x\log r+2\left(\frac{\sin x}{r}-\frac{\sin 2x}{2^2r^2}+\frac{\sin 3x}{3^2r^2}-\cdots\right);\qquad r>1$$
$$\int_0^x\log(1+\sin\alpha\cos x)\,dx=2x\log\cos\frac{\alpha}{2}+2\left(\tan\frac{\alpha}{2}\sin x-\tan^2\frac{\alpha}{2}\frac{\sin 2x}{2^2}+\cdots\right).$$

11. Prove $\displaystyle\int_0^1\frac{dx}{\sqrt{1+x^4}}=1-\frac{1}{2\cdot 5}+\frac{1\cdot 3}{2\cdot 4\cdot 9}-\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 13}+\cdots=\int_1^\infty\frac{dx}{\sqrt{1+x^4}}.$

12. Evaluate these integrals by expansion into series (see Ex. 23, p. 452)

$$(\alpha)\ \int_0^\infty\frac{e^{-qx}\sin rx}{x}\,dx=\frac{r}{q}-\frac{1}{3}\left(\frac{r}{q}\right)^3+\frac{1}{5}\left(\frac{r}{q}\right)^5-\cdots=\tan^{-1}\frac{r}{q},$$
$$(\beta)\ \int_0^\pi\frac{\log(1+k\cos x)}{\cos x}\,dx=\pi\sin^{-1}k,\qquad(\gamma)\ \int_0^\pi\frac{x\sin x}{1+\cos^2x}\,dx=\frac{\pi^2}{4},$$
$$(\delta)\ \int_0^\infty e^{-\alpha^2x^2}\cos 2\beta x\,dx=\frac{\sqrt{\pi}}{2\alpha}e^{-\left(\frac{\beta}{\alpha}\right)^2},\qquad(\epsilon)\ \int_0^\pi\log(1+2r\cos x+r^2)\,dx.$$

13. By formal multiplication (§ 168) show that
$$\frac{1-\alpha^2}{1-2\alpha\cos x+\alpha^2}=1+2\alpha\cos x+2\alpha^2\cos 2x+\cdots,$$
$$\frac{\alpha\sin x}{1-2\alpha\cos x+\alpha^2}=\alpha\sin x+\alpha^2\sin 2x+\cdots.$$

14. Evaluate, by use of Ex. 13, these definite integrals, m an integer:

$$(\alpha)\ \int_0^\pi\frac{\cos mx\,dx}{1-2\alpha\cos x+\alpha^2}=\frac{\pi\alpha^m}{1-\alpha^2},\qquad(\beta)\ \int_0^\pi\frac{x\sin x\,dx}{1-2\alpha\cos x+\alpha^2}=\frac{\pi}{\alpha}\log(1+\alpha),$$
$$(\gamma)\ \int_0^\pi\frac{\sin x\sin mx\,dx}{1-2\alpha\cos x+\alpha^2}=\frac{\pi}{2}\alpha^{m-1},$$
$$(\delta)\ \int_0^\pi\frac{\sin^2x\,dx}{(1-2\alpha\cos x+\alpha^2)(1-2\beta\cos x+\beta^2)}.$$

15. In Ex. 14 (γ) let $\alpha=1-h/m$ and $x=z/m$. Obtain by a limiting process, and by a similar method exercised upon Ex. 14 (α):
$$\int_0^\infty\frac{z\sin z\,dz}{h^2+z^2}=\frac{\pi}{2}e^{-h},\qquad\int_0^\infty\frac{\cos z\,dz}{h^2+z^2}=\frac{\pi}{2}e^{-h}.$$

Can the use of these limiting processes be readily justified ?

16. Let h and x be less than 1. Assume the expansion

$$f(x, h) = \frac{1}{\sqrt{1 - 2xh + h^2}} = 1 + hP_1(x) + h^2P_2(x) + \cdots + h^nP_n(x) + \cdots.$$

Obtain therefrom the following expansions by differentiation:

$$\frac{1}{h}f'_x = \frac{1}{(1-2xh+h^2)^{\frac{3}{2}}} = P'_1 + hP'_2 + h^2P'_3 + \cdots + h^{n-1}P'_n + \cdots,$$

$$f'_h = \frac{x-h}{(1-2xh+h^2)^{\frac{3}{2}}} = P_1 + 2hP_2 + 3h^2P_3 + \cdots + nh^{n-1}P_n + \cdots.$$

Hence establish the given identities and consequent relations:

$$\begin{aligned}
\frac{x-h}{h}f'_x &= xP'_1 + h(xP'_2 - P'_1) + \cdots + h^{n-1}(xP'_n - P'_{n-1}) + \cdots = \\
f'_h &= P_1 + h(2P_2) + \cdots + h^{n-1}(nP_n) + \cdots, \\
\frac{(1+h^2)}{h}f'_x - f &= -1 + P'_1 + h(P'_2 - P_1) + \cdots + h^n(P'_{n+1} + P'_{n-1} - P_n) + \cdots = \\
2xhf &= h(2x) + \cdots + h^n(2xP_{n-1}).
\end{aligned}$$

Or $\quad nP_n = xP'_n - P'_{n-1} \quad$ and $\quad P'_{n+1} + P'_{n-1} - P_n = 2xP'_n.$

Hence $\quad xP'_n = P'_{n+1} - (n+1)P_n \quad$ and $\quad (x^2-1)P'_n = n(xP_n - P_{n-1}).$

Compare the results with Exs. 13 and 17, p. 252, to identify the functions with the Legendre polynomials. Write

$$\frac{1}{(1-2xh+h^2)^{\frac{1}{2}}} = \frac{1}{(1-2h\cos\theta+h^2)^{\frac{1}{2}}} = \frac{1}{(1-he^{i\theta})^{\frac{1}{2}}(1-he^{-i\theta})^{\frac{1}{2}}}$$

$$= \left(1 + \frac{1}{2}he^{i\theta} + \frac{1\cdot 3}{2\cdot 4}h^2e^{2i\theta} + \cdots\right)\left(1 + \frac{1}{2}he^{-i\theta} + \frac{1\cdot 3}{2\cdot 4}h^2e^{-2i\theta} + \cdots\right),$$

and show $P_n(\cos\theta) = 2\dfrac{1\cdot 3\cdots(2n-1)}{2\cdot 4\cdots 2n}\left\{\cos n\theta + \dfrac{1\cdot n}{1\cdot(2n-1)}\cos(n-2)\theta + \cdots\right\}.$

168. Manipulation of series. *If an infinite series*

$$S = u_0 + u_1 + u_2 + \cdots + u_{n-1} + u_n + \cdots \tag{13}$$

converges, the series obtained by grouping the terms in parentheses without altering their order will also converge. Let

$$S' = U_0 + U_1 + \cdots + U_{n'-1} + U_{n'} + \cdots \tag{13'}$$

and $$S'_1, S'_2, \cdots, S'_{n'}, \cdots$$

be the new series and the sums of its first n' terms. These sums are merely particular ones of the set $S_1, S_2, \cdots, S_n, \cdots$, and as $n' < n$ it follows that n becomes infinite when n' does if n be so chosen that $S_n = S'_{n'}$. As S_n approaches a limit, $S'_{n'}$ must approach the same limit. As a corollary it appears that if the series obtained by removing parentheses in a given series converges, the value of the series is not affected by removing the parentheses.

If two convergent infinite series be given as

$$S = u_0 + u_1 + \cdots, \quad \textit{and} \quad T = v_0 + v_1 + \cdots,$$

then $$(\lambda u_0 + \mu v_0) + (\lambda u_1 + \mu v_1) + \cdots$$

will converge to the limit $\lambda S + \mu T$, *and will converge absolutely provided both the given series converge absolutely.* The proof is left to the reader.

If a given series converges absolutely, the series formed by rearranging the terms in any order without omitting any terms will converge to the same value. Let the two arrangements be

$$S = u_0 + u_1 + u_2 + \cdots + u_{n-1} + u_n + \cdots$$

and $$S = u_{0'} + u_{1'} + u_{2'} + \cdots + u_{n'-1} + u_{n'} + \cdots.$$

As S converges absolutely, n may be taken so large that

$$|u_n| + |u_{n+1}| + \cdots < \epsilon;$$

and as the terms in S' are identical with those in S except for their order, n' may be taken so large that $S'_{n'}$ shall contain all the terms in S_n. The other terms in $S'_{n'}$ will be found among the terms $u_n, u_{n+1}, \cdots$. Hence

$$|S'_{n'} - S_n| < |u_n| + |u_{n+1}| + \cdots < \epsilon.$$

As $|S - S_n| < \epsilon$, it follows that $|S - S'_{n'}| < 2\epsilon$. Hence $S'_{n'}$ approaches S as a limit when n' becomes infinite. It may easily be shown that S' also converges absolutely.

The theorem is still true if the rearrangement of S is into a series some of whose terms are themselves infinite series of terms selected from S. Thus let

$$S' = U_0 + U_1 + U_2 + \cdots + U_{n'-1} + U_{n'} + \cdots,$$

where U_i may be any aggregate of terms selected from S. If U_i be an infinite series of terms selected from S, as

$$U_i = u_{i0} + u_{i1} + u_{i2} + \cdots + u_{in} + \cdots,$$

the absolute convergence of U_i follows from that of S (cf. Ex. 22 below). It is possible to take n' so large that every term in S_n shall occur in one of the terms $U_0, U_1, \cdots, U_{n'-1}$. Then if from

$$S - U_0 - U_1 - \cdots - U_{n'-1} \tag{14}$$

there be canceled all the terms of S_n, the terms which remain will be found among $u_n, u_{n+1}, \cdots$, and (14) will be less than ϵ. Hence as n' becomes infinite, the difference (14) approaches zero as a limit and the theorem is proved that

$$S = U_0 + U_1 + \cdots + U_{n'-1} + U_{n'} + \cdots = S'.$$

If a series of real terms is convergent, but not absolutely, the number of positive and the number of negative terms is infinite, the series of positive terms and the series of negative terms diverge, and the given series may be so rearranged as to comport itself in any desired manner. That the number of terms of each sign cannot be finite follows from the fact that if it were, it would be possible to go so far in the series that all subsequent terms would have the same sign and the series would therefore converge absolutely if at all. Consider next the sum $S_n = P_l - N_m$, $l + m = n$, of n terms of the series, where P_l is the sum of the positive terms and N_m that of the negative terms. If both P_l and N_m converged, then $P_l + N_m$ would also converge and the series would converge absolutely; if only one of the sums P_l or N_m diverged, then S would diverge. Hence both sums must diverge. The series may now be rearranged to approach any desired limit, to become positively or negatively infinite, or to oscillate as desired. For suppose an arrangement to approach L as a limit were desired. First take enough positive terms to make the sum exceed L, then enough negative terms to make it less than L, then enough positive terms to bring it again in excess of L, and so on. But as the given series converges, its terms approach 0 as a limit; and as the new arrangement gives a sum which never differs from L by more than the last term in it, the difference between the sum and L is approaching 0 and L is the limit of the sum. In a similar way it could be shown that an arrangement which would comport itself in any of the other ways mentioned would be possible.

If two absolutely convergent series be multiplied, as

$$S = u_0 + u_1 + u_2 + \cdots + u_n + \cdots,$$
$$T = v_0 + v_1 + v_2 + \cdots + v_n + \cdots,$$

and

$$\begin{aligned} W = {} & u_0v_0 + u_1v_0 + u_2v_0 + \cdots + u_nv_0 + \cdots \\ & + u_0v_1 + u_1v_1 + u_2v_1 + \cdots + u_nv_1 + \cdots \\ & + \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ & + u_0v_n + u_1v_n + u_2v_n + \cdots + u_nv_n + \cdots \\ & + \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \end{aligned}$$

and if the terms in W be arranged in a simple series as

$$u_0v_0 + (u_1v_0 + u_1v_1 + u_0v_1) + (u_2v_0 + u_2v_1 + u_2v_2 + u_1v_2 + u_0v_2) + \cdots$$

or in any other manner whatsoever, the series is absolutely convergent and converges to the value of the product ST.

In the particular arrangement above, S_1T_1, S_2T_2, S_nT_n is the sum of the first, the first two, the first n terms of the series of parentheses. As $\lim S_nT_n = ST$, the series of parentheses converges to ST. As S and T are absolutely convergent the same reasoning could be applied to the series of absolute values and

$$|u_0||v_0| + |u_1||v_0| + |u_1||v_1| + |u_0||v_1| + |u_2||v_0| + \cdots$$

would be seen to converge. Hence the convergence of the series

$$u_0v_0 + u_1v_0 + u_1v_1 + u_0v_1 + u_2v_0 + u_2v_1 + u_2v_2 + u_1v_2 + u_0v_2 + \cdots$$

is absolute and to the value ST when the parentheses are omitted. Moreover, any other arrangement, such in particular as

$$u_0v_0 + (u_1v_0 + u_0v_1) + (u_2v_0 + u_1v_1 + u_0v_2) + \cdots,$$

would give a series converging absolutely to ST.

The equivalence of a function and its Taylor or Maclaurin infinite series (wherever the series converges) lends importance to the operations of multiplication, division, and so on, which may be performed on the series. Thus if

$$f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots, \qquad |x| < R_1,$$
$$g(x) = b_0 + b_1x + b_2x^2 + b_3x^3 + \cdots, \qquad |x| < R_2,$$

the multiplication may be performed and the series arranged as

$$f(x)g(x) = a_0b_0 + (a_0b_1 + a_1b_0)x + (a_0b_2 + a_1b_1 + a_2b_0)x^2 + \cdots$$

according to ascending powers of x whenever x is numerically less than the smaller of the two radii of convergence R_1, R_2, because both series will then converge absolutely. Moreover, Ex. 5 above shows that this form of the product may still be applied at the extremities of its interval of convergence for real values of x provided the series converges for those values.

As an example in the multiplication of series let the product $\sin x \cos x$ be found.

$$\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \cdots, \qquad \cos x = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \cdots.$$

The product will contain only odd powers of x. The first few terms are

$$1x - \left(\frac{1}{3!} + \frac{1}{2!}\right)x^3 + \left(\frac{1}{5!} + \frac{1}{3!2!} + \frac{1}{4!}\right)x^5 - \left(\frac{1}{7!} + \frac{1}{5!2!} + \frac{1}{3!4!} + \frac{1}{6!}\right)x^7 + \cdots.$$

The law of formation of the coefficients gives as the coefficient of x^{2k+1}

$$(-1)^k\left[\frac{1}{(2k+1)!} + \frac{1}{(2k-1)!2!} + \frac{1}{(2k-3)!4!} + \cdots + \frac{1}{3!(2k-2)!} + \frac{1}{(2k)!}\right] =$$
$$\frac{(-1)^k}{(2k+1)!}\left[1 + \frac{(2k+1)2k}{2!} + \frac{(2k+1)(2k)(2k-1)(2k-2)}{4!} + \cdots + \frac{(2k+1)}{1!}\right].$$

But $\quad 2^{2k+1} = (1+1)^{2k+1} = 1 + (2k+1) + \dfrac{(2k+1)2k}{2!} + \cdots + (2k+1) + 1.$

Hence it is seen that the coefficient of x^{2k+1} takes every other term in this symmetrical sum of an even number of terms and must therefore be equal to half the sum. The product may then be written as the series

$$\sin x \cos x = \frac{1}{2}\left[2x - \frac{(2x)^3}{3!} + \frac{(2x)^5}{5!} - \cdots\right] = \frac{1}{2}\sin 2x.$$

169. If a function $f(x)$ be expanded into a power series

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots, \qquad |x| < R, \tag{15}$$

and if $x = \alpha$ is any point within the circle of convergence, *it may be desired to transform the series into one which proceeds according to powers of $(x - \alpha)$ and converges in a circle about the point $x = \alpha$.* Let $t = x - \alpha$. Then $x = \alpha + t$ and hence

$$x^2 = \alpha^2 + 2\,\alpha t + t^2, \qquad x^3 = \alpha^3 + 3\,\alpha^2 t + 3\,\alpha t^2 + t^3, \qquad \cdots,$$

$$f(x) = a_0 + a_1(\alpha + t) + a^2(\alpha^2 + 2\,\alpha t + t^2) + \cdots. \tag{15'}$$

Since $|\alpha| < R$, the relation $|\alpha| + |t| < R$ will hold for small values of t, and the series (15′) will converge for $x = |\alpha| + |t|$. Since

$$a_0 + a_1(|\alpha| + |t|) + a_2(|\alpha|^2 + 2\,|\alpha||t| + |t|^2) + \cdots$$

is absolutely convergent for small values of t, the parentheses in (15′) may be removed and the terms collected as

$$f(x) = \phi(t) = (a_0 + a_1\alpha + a_2\alpha^2 + a_3\alpha^3 + \cdots) + (a_1 + 2\,a_2\alpha + 3\,a_3\alpha^2 + \cdots)t$$
$$+ (a_2 + 3\,a_3\alpha + \cdots)t^2 + (a_3 + \cdots)t^3 + \cdots,$$

or
$$f(x) = \phi(x - \alpha) = A_0 + A_1(x - \alpha) + A_2(x - \alpha)^2$$
$$+ A_3(x - \alpha)^3 + \cdots, \tag{16}$$

where $A_0, A_1, A_2, \cdots$ are infinite series; in fact

$$A_0 = f(\alpha), \qquad A_1 = f'(\alpha), \qquad A_2 = \frac{1}{2!}f''(\alpha), \qquad A_3 = \frac{1}{3!}f'''(\alpha), \cdots.$$

The series (16) in $x - \alpha$ will surely converge within a circle of radius $R - |\alpha|$ about $x = \alpha$; but it may converge in a larger circle. As a matter of fact it will converge within the largest circle whose center is at α and within which the function has a definite continuous derivative. Thus Maclaurin's expansion for $(1 + x^2)^{-1}$ has a unit radius of convergence; but the expansion about $x = \frac{1}{2}$ into powers of $x - \frac{1}{2}$ will have a radius of convergence equal to $\frac{1}{2}\sqrt{5}$, which is the distance from $x = \frac{1}{2}$ to either of the points $x = \pm\, i$. If the function had originally been defined by its development about $x = 0$, the definition would have been valid only over the unit circle. The new development about $x = \frac{1}{2}$ will therefore extend the definition to a considerable region outside the original domain, and by repeating the process the region of definition may be extended further. As the function is at each step defined by a power series, it remains analytic. This process of extending the definition of a function is called *analytic continuation.*

Consider *the expansion of a function of a function.* Let

$$f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots \qquad |x| < R_1,$$
$$x = \phi(y) = b_0 + b_1y + b_2y^2 + b_3y^3 + \cdots, \qquad |y| < R_2,$$

and let $|b_0| < R_1$ so that, for sufficiently small values of y, the point x will still lie within the circle R_1. By the theorem on multiplication, the series for x may be squared, cubed, $\cdots$, and the series for x^2, x^3, $\cdots$ may be arranged according to powers of y. These results may then be substituted in the series for $f(x)$ and the result may be ordered according to powers of y. Hence the expansion for $f[\phi(y)]$ is obtained. That the expansion is valid at least for small values of y may be seen by considering

$$|a_0| + |a_1|\xi + |a_2|\xi^2 + |a_3|\xi^3 + \cdots, \qquad \xi < R_1,$$
$$\xi = |b_0| + |b_1||y| + |b_2||y|^2 + \cdots, \qquad |y| \text{ small},$$

which are series of positive terms. The radius of convergence of the series for $f[\phi(y)]$ may be found by discussing that function.

For example consider the problem of expanding $e^{\cos x}$ to five terms.

$$e^y = 1 + y + \tfrac{1}{2}y^2 + \tfrac{1}{6}y^3 + \tfrac{1}{24}y^4 + \cdots, \qquad y = \cos x = 1 - \tfrac{1}{2}x^2 + \tfrac{1}{24}x^4 + \cdots,$$
$$y^2 = 1 - x^2 + \tfrac{1}{3}x^4 - \cdots, \qquad y^3 = 1 - \tfrac{3}{2}x^2 + \tfrac{7}{8}x^4 - \cdots, \qquad y^4 = 1 - 2x^2 + 1\tfrac{2}{3}x^4 - \cdots,$$
$$e^y = 1 + (1 - \tfrac{1}{2}x^2 + \tfrac{1}{24}x^4 - \cdots) + \tfrac{1}{2}(1 - x^2 + \tfrac{1}{3}x^4 - \cdots) + \tfrac{1}{6}(1 - \tfrac{3}{2}x^2 + \tfrac{7}{8}x^4 - \cdots)$$
$$+ \tfrac{1}{24}(1 - 2x^2 + 1\tfrac{2}{3}x^4 - \cdots) + \cdots$$
$$= (1 + 1 + \tfrac{1}{2} + \tfrac{1}{6} + \tfrac{1}{24} + \cdots) - (\tfrac{1}{2} + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{12} + \cdots)x^2$$
$$+ (\tfrac{1}{2} + \tfrac{1}{6} + \tfrac{7}{48} + \tfrac{1}{9} + \cdots)x^4 + \cdots,$$
$$e^y = e^{\cos x} = 2\tfrac{17}{24} - 1\tfrac{1}{3}x^2 + \tfrac{22}{27}x^4 - \cdots.$$

It should be noted that the coefficients in this series for $e^{\cos x}$ are really infinite series and the final values here given are only the approximate values found by taking the first few terms of each series. This will always be the case when $y = b_0 + b_1x + \cdots$ begins with $b_0 \neq 0$; it is also true in the expansion about a new origin, as in a previous paragraph. In the latter case the difficulty cannot be avoided, but in the case of the expansion of a function of a function it is sometimes possible to make a preliminary change which materially simplifies the final result in that the coefficients become finite series. Thus here

$$e^{\cos x} = e^{1+z} = ee^z, \qquad z = \cos x - 1 = -\tfrac{1}{2}x^2 + \tfrac{1}{24}x^4 - \tfrac{1}{720}x^6 + \cdots,$$
$$z^2 = \tfrac{1}{4}x^4 - \tfrac{1}{24}x^6 + \cdots, \qquad z^3 = -\tfrac{1}{8}x^6 + \cdots, \qquad z^4, z^5, z^6 = 0 + \cdots,$$
$$e^z = 1 + (-\tfrac{1}{2}x^2 + \tfrac{1}{24}x^4 - \tfrac{1}{720}x^6 + \cdots) + \tfrac{1}{2}(\tfrac{1}{4}x^4 - \tfrac{1}{24}x^6 + \cdots) + \tfrac{1}{6}(-\tfrac{1}{8}x^6 + \cdots) + \cdots,$$
$$e^{\cos x} = ee^z = e(1 - \tfrac{1}{2}x^2 + \tfrac{1}{6}x^4 - \tfrac{31}{720}x^6 + \cdots).$$

The coefficients are now exact and the computation to x^6 turns out to be easier than to x^2 by the previous method; the advantage introduced by the change would be even greater if the expansion were to be carried several terms farther.

The quotient of two power series $f(x)$ by $g(x)$, if $g(0) \neq 0$, may be obtained by the ordinary algorism of division as

$$\frac{f(x)}{g(x)} = \frac{a_0 + a_1x + a_2x^2 + \cdots}{b_0 + b_1x + b_2x^2 + \cdots} = c_0 + c_1x + c_2x^2 + \cdots, \qquad b_0 \neq 0.$$

For in the first place as $g(0) \neq 0$, the quotient is analytic in the neighborhood of $x = 0$ and may be developed into a power series. It therefore merely remains to show that the coefficients $c_0, c_1, c_2, \cdots$ are those that would be obtained by division. Multiply

$$\begin{aligned}(a_0 + a_1x + a_2x^2 + \cdots) &= (c_0 + c_1x + c_2x^2 + \cdots)(b_0 + b_1x + b_2x^2 + \cdots)\\ &= b_0c_0 + (b_1c_0 + b_0c_1)x + (b_2c_0 + b_1c_1 + b_0c_2)x^2 + \cdots,\end{aligned}$$

and then equate coefficients of equal powers of x. Then

$$a_0 = b_0c_0, \qquad a_1 = b_1c_0 + b_0c_1, \qquad a_2 = b_2c_0 + b_1c_1 + b_0c_2, \cdots$$

is a set of equations to be solved for $c_0, c_1, c_2, \cdots$. The terms in $f(x)$ and $g(x)$ beyond x^n have no effect upon the values of $c_0, c_1, \cdots, c_n$, and hence these would be the same if $b_{n+1}, b_{n+2}, \cdots$ were replaced by $0, 0, \cdots$, and $a_{n+1}, a_{n+2}, \cdots, a_{2n}, a_{2n+1}, \cdots$ by such values $a'_{n+1}, a'_{n+2}, \cdots, a'_{2n}, 0, \cdots$ as would make the division come out even; the coefficients $c_0, c_1, \cdots, c_n$ are therefore precisely those obtained in dividing the series.

If y is developed into a power series in x as

$$y = f(x) = a_0 + a_1x + a_2x^2 + \cdots, \qquad a_1 \neq 0, \tag{17}$$

then x may be developed into a power series in $y - a_0$ as

$$x = f^{-1}(y - a_0) = b_1(y - a_0) + b_2(y - a_0)^2 + \cdots. \tag{18}$$

For since $a_1 \neq 0$, the function $f(x)$ has a nonvanishing derivative for $x = 0$ and hence the inverse function $f^{-1}(y - a_0)$ is analytic near $x = 0$ or $y = a_0$ and can be developed (p. 477). The method of undetermined coefficients may be used to find $b_1, b_2, \cdots$. This process of finding (18) from (17) is called the *reversion* of (17). For the actual work it is simpler to replace $(y - a_0)/a_1$ by t so that

$$t = x + a_2'x^2 + a_3'x^3 + a_4'x^4 + \cdots, \qquad a_i' = a_i/a_1,$$

and $$x = t + b_2't^2 + b_3't^3 + b_4't^4 + \cdots, \qquad b_i' = b_ia_1^i.$$

Let the assumed value of x be substituted in the series for t; rearrange the terms according to powers of t and equate the corresponding coefficients. Thus

$$\begin{aligned}t = t &+ (b_2' + a_2')t^2 + (b_3' + 2b_2'a_2' + a_3')t^3\\ &+ (b_4' + 2b_3'a_2' + b_2'^2a_2' + 3b_2'a_3' + a_4')t^4 + \cdots\end{aligned}$$

or $\quad b_2' = -a_2', \qquad b_3' = 2a_2'^2 - a_3', \qquad b_4' = -5a_2'^3 + 5a_2'a_3' - a_4', \cdots.$

170. For some few purposes, which are tolerably important, *a formal operational method* of treating series is so useful as to be almost indispensable. If the series be taken in the form

$$1 + a_1x + \frac{a_2}{2!}x^2 + \frac{a_3}{3!}x^3 + \cdots + \frac{a_n}{n!}x^n + \cdots,$$

with the factorials which occur in Maclaurin's development and with unity as the initial term, the series may be written as

$$e^{ax} = 1 + a^1x + \frac{a^2}{2!}x^2 + \frac{a^3}{3!}x^3 + \cdots + \frac{a^n}{n!}x^n + \cdots,$$

provided that a^i be interpreted as the formal equivalent of a_i. The product of two series would then formally suggest

$$e^{ax}e^{bx} = e^{(a+b)x} = 1 + (a+b)^1x + \frac{1}{2!}(a+b)^2x^2 + \cdots, \tag{19}$$

and if the coefficients be transformed by setting $a^ib^j = a_ib_j$, then

$$\left(1 + a_1x + \frac{a_2}{2!}x^2 + \cdots\right)\left(1 + b_1x + \frac{b_2}{2!}x^2 + \cdots\right)$$
$$= 1 + (a_1 + b_1)x + \frac{a_2 + 2a_1b_1 + b_2}{2!}x^2 + \cdots.$$

This as a matter of fact is the formula for the product of two series and hence justifies the suggestion contained in (19).

For example suppose that the development of

$$\frac{x}{e^x - 1} = 1 + B_1x + \frac{B_2}{2!}x^2 + \frac{B_3}{3!}x^3 + \cdots$$

were desired. As the development begins with 1, the formal method may be applied and the result is found to be

$$\frac{x}{e^x - 1} = e^{Bx}, \qquad x = e^{(B+1)x} - e^{Bx}, \tag{20}$$

$$x = x + [(B+1)^2 - B^2]\frac{x^2}{2!} + [(B+1)^3 - B^3]\frac{x^3}{3!} + \cdots, \tag{21}$$

$$(B+1)^2 - B^2 = 0, \quad (B+1)^3 - B^3 = 0, \cdots, \quad (B+1)^k - B^k = 0, \cdots,$$

$$\text{or } 2B_1 + 1 = 0, \quad 3B_2 + 3B_1 + 1 = 0, \quad 4B_3 + 6B_2 + 4B_1 + 1 = 0, \cdots,$$

$$\text{or} \qquad B_1 = -\tfrac{1}{2}, \quad B_2 = \tfrac{1}{6}, \quad B_3 = 0, \quad B_4 = -\tfrac{1}{30}, \cdots.$$

The formal method leads to a set of equations from which the successive B's may quickly be determined. Note that

$$\frac{x}{e^x - 1} + \frac{x}{2} = \frac{x}{2}\frac{e^x + 1}{e^x - 1} = \frac{x}{2}\coth\frac{x}{2} = -\frac{x}{2}\coth\left(-\frac{x}{2}\right) \tag{22}$$

is an even function of x, and that consequently all the B's with odd indices except B_1 are zero. This will facilitate the calculation. The first eight even B's are respectively

$$\tfrac{1}{6},\quad -\tfrac{1}{30},\quad \tfrac{1}{42},\quad -\tfrac{1}{30},\quad \tfrac{5}{66},\quad -\tfrac{691}{2730},\quad \tfrac{7}{6},\quad -\tfrac{3617}{510}. \qquad (23)$$

The numbers B, or their absolute values, are called *the Bernoullian numbers*. An independent justification for the method of formal calculation may readily be given. For observe that $e^x e^{Bx} = e^{(B+1)x}$ of (20) is true when B is regarded as an independent variable. Hence if this identity be arranged according to powers of B, the coefficient of each power must vanish. It will therefore not disturb the identity if any numbers whatsoever are substituted for $B^1, B^2, B^3, \cdots$; the particular set $B_1, B_2, B_3, \cdots$ may therefore be substituted; the series may be rearranged according to powers of x, and the coefficients of like powers of x may be equated to 0, — as in (21) to get the desired equations.

If an infinite series be written without the factorials as

$$1 + a_1x + a_2x^2 + a_3x^3 + \cdots + a_nx^n + \cdots,$$

a possible symbolic expression for the series is

$$\frac{1}{1-ax} = 1 + a^1x + a^2x^2 + a^3x^3 + \cdots, \qquad a^i = a_i.$$

If the substitution $y = x/(1+x)$ or $x = y/(1-y)$ be made,

$$\frac{1}{1-ax} = \frac{1}{1-a\dfrac{y}{1-y}} = \frac{1-y}{1-(1+a)y}. \qquad (24)$$

Now if the left-hand and right-hand expressions be expanded and a be regarded as an independent variable restricted to values which make $|ax| < 1$, the series obtained will both converge absolutely and may be arranged according to powers of a. Corresponding coefficients will then be equal and the identity will therefore not be disturbed if a_i replaces a^i. Hence

$$1 + a_1x + a_2x^2 + \cdots = (1-y)\left[1 + (1+a)y + (1+a)^2y^2 + \cdots\right],$$

provided that both series converge absolutely for $a_i = a^i$. Then

$$1 + a_1x + a_2x^2 + a_3x^3 + \cdots = 1 + ay + a(1+a)y^2 + a(1+a)^2y^3 + \cdots$$
$$= 1 + a_1y + (a_1 + a_2)y^2 + (a_1 + 2a_2 + a_3)y^3 + \cdots,$$

or

$$a_1x + a_2x^2 + a_3x^3 + \cdots = a_1y + (a_1 + a_2)y^2 + (a_1 + 2a_2 + a_3)y^3 + \cdots. \qquad (25)$$

This transformation is known as *Euler's transformation.* Its great advantage for computation lies in the fact that sometimes the second series converges much more rapidly than the first. This is especially true when the coefficients of the first series are such as to make the coefficients in the new series small. Thus from (25)

$$\log(1+x) = x - \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 - \tfrac{1}{4}x^4 + \tfrac{1}{5}x^5 - \tfrac{1}{6}x^6 + \cdots$$
$$= y + \tfrac{1}{2}y^2 + \tfrac{1}{3}y^3 + \tfrac{1}{4}y^4 + \tfrac{1}{5}y^5 + \tfrac{1}{6}y^6 + \cdots.$$

To compute log 2 to three decimals from the first series would require several hundred terms; eight terms are enough with the second series. An additional advantage of the new series is that it may continue to converge after the original series has ceased to converge. In this case the two series can hardly be said to be equal; but the second series of course remains equal to the (continuation of the) function defined by the first. Thus log 3 may be computed to three decimals with about a dozen terms of the second series, but cannot be computed from the first.

EXERCISES

1. By the multiplication of series prove the following relations:

(α) $(1 + x + x^2 + x^3 + \cdots)^2 = (1 + 2x + 3x^2 + 4x^3 + \cdots) = (1-x)^{-2}$,

(β) $\cos^2 x + \sin^2 x = 1$, (γ) $e^x e^y = e^{x+y}$, (δ) $2\sin^2 x = 1 - \cos 2x$.

2. Find the Maclaurin development to terms in x^6 for the functions:

(α) $e^x \cos x$, (β) $e^x \sin x$, (γ) $(1+x)\log(1+x)$, (δ) $\cos x \sin^{-1} x$.

3. Group the terms of the expansion of $\cos x$ in two different ways to show that $\cos 1 > 0$ and $\cos 2 < 0$. Why does it then follow that $\cos\xi = 0$ where $1 < \xi < 2$?

4. Establish the developments (Peirce's Nos. 785–789) of the functions:

(α) $e^{\sin x}$, (β) $e^{\tan x}$, (γ) $e^{\sin^{-1} x}$, (δ) $e^{\tan^{-1} x}$.

5. Show that if $g(x) = b_m x^m + b_{m+1}x^{m+1} + \cdots$ and $f(0) \neq 0$, then

$$\frac{f(x)}{g(x)} = \frac{a_0 + a_1 x + a_2 x^2 + \cdots}{b_m x^m + b_{m+1}x^{m+1} + \cdots} = \frac{c_{-m}}{x^m} + \frac{c_{-m+1}}{x^{m-1}} + \cdots + \frac{c_{-1}}{x} + c_0 + c_1 x + \cdots$$

and the development of the quotient has negative powers of x.

6. Develop to terms in x^6 the following functions:

(α) $\sin(k\sin x)$, (β) $\log\cos x$, (γ) $\sqrt{\cos x}$, (δ) $(1 - k^2\sin^2 x)^{-\frac{1}{2}}$.

7. Carry the reversion of these series to terms in the fifth power:

(α) $y = \sin x = x - \tfrac{1}{6}x^3 + \cdots$, ($\beta$) $y = \tan^{-1} x = x - \tfrac{1}{3}x^3 + \cdots$,

(γ) $y = e^x = 1 + x + \tfrac{1}{2}x^2 + \cdots$, ($\delta$) $y = 2x + 3x^2 + 4x^3 + 5x^4 + \cdots$.

8. Find the smallest root of these series by the method of reversion:

$$(\alpha)\ \frac{1}{2}=\int_0^x e^{-x^2}dx = x-\frac{1}{3}x^3+\frac{1}{3!\,5}x^5-\frac{1}{3!\,7}x^7+\cdots,$$

$$(\beta)\ \frac{1}{4}=\int_0^x \cos x^2 dx,\qquad (\gamma)\ \frac{1}{10}=\int_0^x\frac{dx}{\sqrt{(1-x^2)(1-\frac{1}{4}x^2)}}.$$

9. By the formal method obtain the general equations for the coefficients in the developments of these functions and compute the first five that do not vanish:

$$(\alpha)\ \frac{\sin x}{e^x-1},\qquad (\beta)\ \frac{2e^x}{e^x+1},\qquad (\gamma)\ \frac{x^3}{1-2xe^x+e^{2x}}.$$

10. Obtain the general expressions for the following developments:

$$(\alpha)\ \coth x=\frac{1}{x}+\frac{x}{3}-\frac{x^3}{45}+\frac{2x^5}{945}-\cdots+\frac{B_{2n}(2x)^{2n}}{(2n)!\,x}-\cdots,$$

$$(\beta)\ \cot x=\frac{1}{x}-\frac{x}{3}-\frac{x^3}{45}-\frac{2x^5}{945}-\cdots+(-1)^n\frac{B_{2n}(2x)^{2n}}{(2n)!\,x}-\cdots,$$

$$(\gamma)\ \log\sin x=\log x-\frac{x^2}{6}-\frac{x^4}{180}-\frac{x^6}{2835}-\cdots+(-1)^n\frac{B_{2n}(2x)^{2n}}{2n\cdot(2n)!}-\cdots,$$

$$(\delta)\ \log\sinh x=\log x+\frac{x^2}{6}-\frac{x^4}{180}+\frac{x^6}{2835}-\cdots+\frac{B_{2n}(2x)^{2n}}{2n\cdot(2n)!}-\cdots.$$

11. The Eulerian numbers E_{2n} are the coefficients in the expansion of sech x. Establish the defining equations and compute the first four as $-1, 5, -61, 1385$.

12. Write the expansions for $\sec x$ and $\log\tan(\frac{1}{4}\pi+\frac{1}{2}x)$.

13. From the identity $\frac{1}{e^x-1}-\frac{2}{e^{2x}-1}=\frac{1}{e^x+1}$ derive the expansions:

$$(\alpha)\ \frac{e^x}{e^x+1}=\frac{1}{2}+B_2(2^2-1)\frac{x}{2!}+B_4(2^4-1)\frac{x^3}{4!}+\cdots+B_{2n}(2^{2n}-1)\frac{x^{2n-1}}{2n!}+\cdots,$$

$$(\beta)\ \frac{1}{e^x+1}=\frac{1}{2}-B_2(2^2-1)\frac{x}{2!}-B_4(2^4-1)\frac{x^3}{4!}-\cdots-B_{2n}(2^{2n}-1)\frac{x^{2n-1}}{2n!}+\cdots,$$

$$(\gamma)\ \tanh x=(2^2-1)2^2B_2\frac{x}{2!}+(2^4-1)2^4B_4\frac{x^3}{4!}+\cdots+(2^{2n}-1)2^{2n}B_{2n}\frac{x^{2n-1}}{2n!}+\cdots,$$

$$(\delta)\ \tan x=x+\frac{x^3}{3}+\frac{2x^5}{15}+\frac{17x^7}{315}+\cdots+(-1)^{n-1}(2^{2n}-1)2^{2n}B_{2n}\frac{x^{2n-1}}{2n!}+\cdots,$$

$$(\epsilon)\ \log\cos x=-\frac{x^2}{2}-\frac{x^4}{12}-\frac{x^6}{45}-\cdots-(-1)^{n-1}(2^{2n}-1)2^{2n}B_{2n}\frac{x^{2n}}{2n\cdot 2n!}-\cdots,$$

$$(\zeta)\ \log\tan x=\log x+\frac{x^2}{3}+\frac{7x^4}{60}+\cdots+(-1)^{n-1}(2^{2n-1}-1)2^{2n}B_{2n}\frac{x^{2n}}{n\cdot 2n!}+\cdots,$$

$$(\eta)\ \csc x=\frac{1}{2}\left(\cot\frac{x}{2}+\tan\frac{x}{2}\right)=\frac{1}{x}+\frac{x}{3!}+\cdots+(-1)^{n-1}2(2^{2n-1}-1)B_{2n}\frac{x^{2n}}{2n!},$$

(θ) $\log\cosh x$, (ι) $\log\tanh x$, (κ) csch x, (λ) $\sec^2 x$.

Observe that the Bernoullian numbers afford a general development for all the trigonometric and hyperbolic functions and their logarithms with the exception of the sine and cosine (which have known developments) and the secant (which requires the Eulerian numbers). The importance of these numbers is therefore apparent.

14. The coefficients $P_1(y)$, $P_2(y)$, $\cdots$, $P_n(y)$ in the development

$$\frac{e^{yx}-1}{e^x-1} = y + P_1(y)x + P_2(y)x^2 + \cdots + P_n(y)x^n + \cdots$$

are called Bernoulli's polynomials. Show that $(n+1)!\,P_n(y) = (B+y)^{n+1} - B^{n+1}$ and thus compute the first six polynomials in y.

15. If $y = N$ is a positive integer, the quotient in Ex. 14 is simple. Hence

$$n!\,P_n(N) = 1 + 2^n + 3^n + \cdots + (N-1)^n$$

is easily shown. With the aid of the polynomials found above compute:

(α) $1 + 2^4 + 3^4 + \cdots + 10^4$, $\quad(\beta)$ $1 + 2^5 + 3^5 + \cdots + 9^5$,

(γ) $1 + 2^2 + 3^2 + \cdots + (N-1)^2$, $\quad(\delta)$ $1 + 2^3 + 3^3 + \cdots + (N-1)^3$.

16. Interpret $\dfrac{1}{1-ax}\,\dfrac{1}{1-bx} = \dfrac{1}{x(a-b)}\left[\dfrac{1}{1-ax} - \dfrac{1}{a-bx}\right] = \sum \dfrac{a^{n+1}-b^{n+1}}{a-b}x^n$.

17. From $\displaystyle\int_0^\infty e^{-(1-ax)t}dt = \frac{1}{1-ax}$ establish formally

$$1 + a_1x + a_2x^2 + a_3x^3 + \cdots = \int_0^\infty e^{-t}F(xt)\,dt = \frac{1}{x}\int_0^\infty e^{-\frac{u}{x}}F(u)\,du,$$

where
$$F(u) = 1 + a_1u + \frac{1}{2!}a_2u^2 + \frac{1}{3!}a_3u^3 + \cdots.$$

Show that the integral will converge when $0 < x < 1$ provided $|a_i| \leqq 1$.

18. If in a series the coefficients $a_i = \displaystyle\int_0^1 t^i f(t)\,dt$, show

$$1 + a_1x + a_2x^2 + a_3x^3 + \cdots = \int_0^1 \frac{f(t)}{1-xt}\,dt.$$

19. Note that Exs. 17 and 18 convert a series into an integral. Show

$$(\alpha)\quad 1 + \frac{x}{2^p} + \frac{x^2}{3^p} + \frac{x^3}{4^p} + \cdots = \frac{1}{\Gamma(p)}\int_0^1 \frac{(-\log t)^{p-1}}{1-xt}dt \quad\text{by}\quad \frac{\Gamma(p)}{n^p} = \int_0^\infty e^{-n\xi}\xi^{p-1}d\xi,$$

$$(\beta)\quad \frac{1}{1+1^2} + \frac{x}{1+2^2} + \frac{x^2}{1+3^2} + \cdots = -\int_0^1 \frac{\sin\log t}{1-xt}dt \quad\text{by}\quad \frac{1}{1+n^2} = \int_0^\infty e^{-n\xi}\sin\xi\,d\xi,$$

$$(\gamma)\quad 1 + \frac{a}{b}x + \frac{a(a+1)}{b(b+1)}x^2 + \frac{a(a+1)(a+2)}{b(b+1)(b+2)}x^3 + \cdots$$
$$= \frac{\Gamma(b)}{\Gamma(a)\,\Gamma(b-a)}\int_0^1 \frac{t^{a-1}(1-t)^{b-a-1}}{1-xt}dt.$$

20. In case the coefficients in a series are alternately positive and negative show that Euler's transformed series may be written

$$a_1x - a_2x^2 + a_3x^3 - a_4x^4 + \cdots = a_1y + \Delta a_1y^2 + \Delta^2 a_1y^3 + \Delta^3 a_1 y^4 + \cdots$$

where $\Delta a_1 = a_1 - a_2$, $\Delta^2 a_1 = \Delta a_1 - \Delta a_2 = a_1 - 2a_2 + a_3, \cdots$ are the successive first, second, $\cdots$ differences of the numerical coefficients.

21. Compute the values of these series by the method of Ex. 20 with $x = 1$, $y = \frac{1}{2}$. Add the first few terms and apply the method of differences to the next few as indicated:

(α) $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = 0.69315$, add 8 terms and take 7 more,

(β) $1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}} + \cdots = 0.6049$, add 5 terms and take 7 more,

(γ) $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = 0.78539813$, add 10 and take 11 more,

(δ) Prove $\left(1 + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \cdots\right) = \frac{2^{p-1}}{2^{p-1} - 1}\left(1 - \frac{1}{2^p} + \frac{1}{3^p} - \frac{1}{4^p} + \cdots\right)$

and compute for $p = 1.01$ with the aid of five-place tables.

22. If an infinite series converges absolutely, show that any infinite series the terms of which are selected from the terms of the given series must also converge. What if the given series converged, but not absolutely?

23. Note that the proof concerning term-by-term integration (p. 432) would not hold if the interval were infinite. Discuss this case with especial references to justifying if possible the formal evaluations of Exs. 12 (α), (δ), p. 439.

24. Check the formula of Ex. 17 by termwise integration. Evaluate

$$\frac{1}{x}\int_0^\infty e^{-\frac{u}{x}} J_0(bu)\,du = 1 - \tfrac{1}{2}b^2x^2 + \tfrac{1}{2}\cdot\tfrac{3}{2}\frac{b^4x^4}{2\,!} - \cdots = (1 + b^2x^2)^{-\frac{1}{2}}$$

by the inverse transformation. See Exs. 8 and 15, p. 399.