CHAPTER VIII

THE COMMONER ORDINARY DIFFERENTIAL EQUATIONS

89. Integration by separating the variables. If a differential equation of the first order may be solved for y' so that

$$y' = \phi(x, y) \quad \text{or} \quad M(x, y)\,dx + N(x, y)\,dy = 0 \tag{1}$$

(where the functions ϕ, M, N are single valued or where only one specific branch of each function is selected in case the solution leads to multiple valued functions), the differential equation involves only the first power of the derivative and is said to be of the first degree. If, furthermore, it so happens that the functions ϕ, M, N are products of functions of x and functions of y so that the equation (1) takes the form

$$y' = \phi_1(x)\,\phi_2(y) \quad \text{or} \quad M_1(x)\,M_2(y)\,dx + N_1(x)\,N_2(y)\,dy = 0, \tag{2}$$

it is clear that the variables may be separated in the manner

$$\frac{dy}{\phi_2(y)} = \phi_1(x)\,dx \quad \text{or} \quad \frac{M_1(x)}{N_1(x)}\,dx + \frac{N_2(y)}{M_2(y)}\,dy = 0, \tag{2'}$$

and the integration is then immediately performed by integrating each side of the equation. It was in this way that the numerous problems considered in Chap. VII were solved.

As an example consider the equation $yy' + xy^2 = x$. Here

$$ydy + x(y^2 - 1)\,dx = 0 \quad \text{or} \quad \frac{ydy}{y^2 - 1} + xdx = 0,$$

and $$\tfrac{1}{2}\log(y^2 - 1) + \tfrac{1}{2}x^2 = C \quad \text{or} \quad (y^2 - 1)\,e^{x^2} = C.$$

The second form of the solution is found by taking the exponential of both sides of the first form after multiplying by 2.

In some differential equations (1) in which the variables are not immediately separable as above, the introduction of some change of variable, whether of the dependent or independent variable or both, may lead to a differential equation in which the new variables are separated and the integration may be accomplished. The selection of the proper change of variable is in general a matter for the exercise of ingenuity; succeeding paragraphs, however, will point out some special types of equations for which a definite type of substitution is known to accomplish the separation.

As an example consider the equation $xdy - ydx = x\sqrt{x^2+y^2}\,dx$, where the variables are clearly not separable without substitution. The presence of $\sqrt{x^2+y^2}$ suggests a change to polar coördinates. The work of finding the solution is:

$$x = r\cos\theta,\quad y = r\sin\theta,\quad dx = \cos\theta dr - r\sin\theta d\theta,\quad dy = \sin\theta dr + r\cos\theta d\theta;$$

then $$xdy - ydx = r^2 d\theta,\qquad x\sqrt{x^2+y^2}\,dx = r^2\cos\theta d(r\cos\theta).$$

Hence the differential equation may be written in the form

$$r^2 d\theta = r^2\cos\theta d(r\cos\theta)\quad\text{or}\quad \sec\theta d\theta = d(r\cos\theta),$$

and $$\log\tan\left(\tfrac{1}{2}\theta + \tfrac{1}{4}\pi\right) = r\cos\theta + C\quad\text{or}\quad \log\frac{1+\sin\theta}{\cos\theta} = x + C.$$

Hence $$\frac{\sqrt{x^2+y^2}+y}{x} = Ce^x\qquad\text{(on substitution for }\theta\text{)}.$$

Another change of variable which works, is to let $y = vx$. Then the work is:

$$x(vdx + xdv) - vxdx = x^2\sqrt{1+v^2}\,dx\ \text{or}\ dv = \sqrt{1+v^2}\,dx.$$

Then $$\frac{dv}{\sqrt{1+v^2}} = dx,\qquad \sinh^{-1}v = x + C,\qquad y = x\sinh(x+C).$$

This solution turns out to be shorter and the answer appears in neater form than before obtained. The great difference of form that may arise in the answer when different methods of integration are employed, is a noteworthy fact, and renders a set of answers practically worthless; two solvers may frequently waste more time in trying to get their answers reduced to a common form than each would spend in solving the problem in two ways.

90. If in the equation $y' = \phi(x, y)$ the function ϕ turns out to be $\phi(y/x)$, a function of y/x alone, that is, if the functions M and N are homogeneous functions of x, y and of the same order (§ 53), the differential equation is said to be *homogeneous* and the change of variable $y = vx$ or $x = vy$ will always result in separating the variables. The statement may be tabulated as:

$$\text{if}\qquad \frac{dy}{dx} = \phi\left(\frac{y}{x}\right),\qquad \text{substitute}\ \begin{cases} y = vx \\ \text{or}\ x = vy. \end{cases}\qquad (3)$$

A sort of corollary case is given in Ex. 6 below.

As an example take $y\left(1 + e^{\frac{x}{y}}\right)dx + e^{\frac{x}{y}}(y - x)\,dy = 0$, of which the homogeneity is perhaps somewhat disguised. Here it is better to choose $x = vy$. Then

$$(1 + e^v)\,dx + e^v(1 - v)\,dy = 0\quad\text{and}\quad dx = vdy + ydv.$$

Hence $$(v + e^v)\,dy + y(1 + e^v)\,dv = 0\quad\text{or}\quad \frac{dy}{y} + \frac{1+e^v}{v+e^v}dv = 0.$$

Hence $$\log y + \log(v + e^v) = C\quad\text{or}\quad x + ye^{\frac{x}{y}} = C.$$

If the differential equation may be arranged so that

$$\frac{dy}{dx} + X_1(x)\,y = X_2(x)\,y^n \quad \text{or} \quad \frac{dx}{dy} + Y_1(y)\,x = Y_2(y)\,x^n, \tag{4}$$

where the second form differs from the first only through the interchange of x and y and where X_1 and X_2 are functions of x alone and Y_1 and Y_2 functions of y, the equation is called a *Bernoulli equation;* and in particular if $n = 0$, so that the dependent variable does not occur on the right-hand side, the equation is called *linear.* The substitution which separates the variables in the respective cases is

$$y = ve^{-\int X_1(x)\,dx} \quad \text{or} \quad x = ve^{-\int Y_1(y)\,dy}. \tag{5}$$

To show that the separation is really accomplished and to find a general formula for the solution of any Bernoulli or linear equation, the substitution may be carried out formally. For

$$\frac{dy}{dx} = \frac{dv}{dx}\,e^{-\int X_1 dx} - vX_1 e^{-\int X dx}.$$

The substitution of this value in the equation gives

$$\frac{dv}{dx}\,e^{-\int X_1 dx} = X_2 v^n e^{-n\int X_1 dx} \quad \text{or} \quad \frac{dv}{v^n} = X_2 e^{(1-n)\int X_1 dx}\,dx.$$

Hence $$v^{1-n} = (1-n)\int X_2 e^{(1-n)\int X_1 dx}\,dx, \quad \text{when} \quad n \neq 1,^*$$

or $$y^{1-n} = (1-n)\,e^{(n-1)\int X_1 dx}\left[\int X_2 e^{(1-n)\int X_1 dx}\,dx\right]. \tag{6}$$

There is an analogous form for the second form of the equation.

The equation $(x^2y^3 + xy)\,dy = dx$ may be treated by this method by writing it as

$$\frac{dx}{dy} - yx = y^3x^2 \quad \text{so that} \quad Y_1 = -y,\ Y_2 = y^3,\ n = 2.$$

Then let $$x = ve^{-\int -y dy} = ve^{\frac{1}{2}y^2}.$$

Then $$\frac{dx}{dy} - yx = \frac{dv}{dy}e^{\frac{1}{2}y^2} + vye^{\frac{1}{2}y^2} - yve^{\frac{1}{2}y^2} = \frac{dv}{dy}e^{\frac{1}{2}y^2}$$

and $$\frac{dv}{dy}e^{\frac{1}{2}y^2} = y^3v^2e^{y^2} \quad \text{or} \quad \frac{dv}{v^2} = y^3e^{\frac{1}{2}y^2}dy,$$

and $$-\frac{1}{v} = (y^2 - 2)\,e^{\frac{1}{2}y^2} + C \quad \text{or} \quad \frac{1}{x} = 2 - y^2 + Ce^{-\frac{1}{2}y^2}.$$

This result could have been obtained by direct substitution in the formula

$$x^{1-n} = (1-n)\,e^{(n-1)\int Y_1 dy}\left[\int Y_2 e^{(1-n)\int Y_1 dy}\,dy\right],$$

but actually to carry the method through is far more instructive.

* If $n = 1$, the variables are separated in the original equation.

EXERCISES

1. Solve the equations (variables immediately separable):

(α) $(1+x)y+(1-y)xy'=0$, *Ans.* $xy = Ce^{x-y}$.

(β) $a(xdy+2ydx)=xydy$, (γ) $\sqrt{1-x^2}\,dy+\sqrt{1-y^2}\,dx=0$,

(δ) $(1+y^2)dx-\left(y+\sqrt{1+y}\right)(1+x)^{\frac{3}{2}}dy=0$.

2. By various ingenious changes of variable, solve:

(α) $(x+y)^2y'=a^2$, *Ans.* $x+y=a\tan(y/a+C)$.

(β) $(x-y^2)dx+2xydy=0$, (γ) $xdy-ydx=(x^2+y^2)dx$,

(δ) $y'=x-y$, (ϵ) $yy'+y^2+x+1=0$.

3. Solve these homogeneous equations:

(α) $\left(2\sqrt{xy}-x\right)y'+y=0$, *Ans.* $\sqrt{x/y}+\log y=C$.

(β) $xe^{\frac{y}{x}}+y-xy'=0$, *Ans.* $y+x\log\log C/x=0$.

(γ) $(x^2+y^2)dy=xydx$, (δ) $xy'-y=\sqrt{x^2+y^2}$.

4. Solve these Bernoulli or linear equations:

(α) $y'+y/x=y^2$, *Ans.* $xy\log Cx+1=0$.

(β) $y'-y\csc x=\cos x-1$, *Ans.* $y=\sin x+C\tan\frac{1}{2}x$.

(γ) $xy'+y=y^2\log x$, *Ans.* $y^{-1}=\log x+1+Cx$.

(δ) $(1+y^2)dx+(\tan^{-1}y-x)dy$, (ϵ) $ydx+(ax^2y^n-2x)dy=0$,

(ζ) $xy'-ay=x+1$, (η) $yy'+\frac{1}{2}y^2=\cos x$.

5. Show that the substitution $y=vx$ always separates the variables in the homogeneous equation $y'=\phi(y/x)$ and derive the general formula for the integral.

6. Let a differential equation be reducible to the form

$$\frac{dy}{dx}=\phi\left(\frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}\right),\qquad \begin{array}{l} a_1b_2-a_2b_1\neq 0,\\ \text{or}\ \ a_1b_2-a_2b_1=0.\end{array}$$

In case $a_1b_2-a_2b_1\neq 0$, the two lines $a_1x+b_1y+c_1=0$ and $a_2x+b_2y+c_2=0$ will meet in a point. Show that a transformation to this point as origin makes the new equation homogeneous and hence soluble. In case $a_1b_2-a_2b_1=0$, the two lines are parallel and the substitution $z=a_2x+b_2y$ or $z=a_1x+b_1y$ will separate the variables.

7. By the method of Ex. 6 solve the equations:

(α) $(3y-7x+7)dx+(7y-3x+3)dy=0$, *Ans.* $(y-x+1)^2(y+x-1)^7=C$.

(β) $(2x+3y-5)y'+(3x+2y-5)=0$, (γ) $(4x+3y+1)dx+(x+y+1)dy=0$,

(δ) $(2x+y)=y'(4x+2y-1)$, (ϵ) $\dfrac{dy}{dx}=\left(\dfrac{x-y-1}{2x-2y+1}\right)^2$.

8. Show that if the equation may be written as $yf(xy)dx+xg(xy)dy=0$, where f and g are functions of the product xy, the substitution $v=xy$ will separate the variables.

9. By virtue of Ex. 8 integrate the equations:

(α) $(y+2xy^2-x^2y^3)dx+2x^2ydy=0$, *Ans.* $x+x^2y=C(1-xy)$.

(β) $(y+xy^2)dx+(x-x^2y)dy=0$, (γ) $(1+xy)xy^2dx+(xy-1)xdy=0$.

10. By any method that is applicable solve the following. If more than one method is applicable, state what methods, and any apparent reasons for choosing one:

(α) $y' + y\cos x = y^n \sin 2x$, ($\beta$) $(2x^2y + 3y^3)\,dx = (x^3 + 2xy^2)\,dy$,
(γ) $(4x + 2y - 1)y' + 2x + y + 1 = 0$, ($\delta$) $yy' + xy^2 = x$,
(ϵ) $y'\sin y + \sin x \cos y = \sin x$, ($\zeta$) $\sqrt{a^2 + x^2}\,(1 - y') = x + y$,
(η) $(x^3y^3 + x^2y^2 + xy + 1)\,y + (x^3y^3 - x^2y^2 - xy + 1)\,xy'$, ($\theta$) $y' = \sin(x - y)$,
(ι) $xydy - y^2dx = (x + y)^2 e^{-\frac{y}{x}} dx$, ($\kappa$) $(1 - y^2)\,dx = axy\,(x + 1)\,dy$.

91. Integrating factors. If the equation $Mdx + Ndy = 0$ by a suitable rearrangement of the terms can be put in the form of a sum of total differentials of certain functions $u, v, \cdots$, say

$$du + dv + \cdots = 0, \quad \text{then} \quad u + v + \cdots = C \tag{7}$$

is surely the solution of the equation. In this case the equation is called an *exact differential equation.* It frequently happens that although the equation cannot itself be so arranged, yet the equation obtained from it by multiplying through with a certain factor $\mu(x, y)$ may be so arranged. The factor $\mu(x, y)$ is then called an *integrating factor* of the given equation. Thus in the case of variables separable, an integrating factor is $1/M_2N_1$; for

$$\frac{1}{M_2N_1}\left[M_1M_2\,dx + N_1N_2\,dy\right] = \frac{M_1(x)}{N_1(x)}\,dx + \frac{N_2(y)}{M_2(y)}\,dy = 0\,; \tag{8}$$

and the integration is immediate. Again, the linear equation may be treated by an integrating factor. Let

$$dy + X_1ydx = X_2dx \quad \text{and} \quad \mu = e^{\int X_1dx}; \tag{9}$$

then

$$e^{\int X_1dx}\,dy + X_1e^{\int X_1dx}\,ydx = e^{\int X_1dx}\,X_2dx \tag{10}$$

or

$$d\left[ye^{\int X_1dx}\right] = e^{\int X_1dx}\,X_2dx, \quad \text{and} \quad ye^{\int X_1dx} = \int e^{\int X_1dx}\,X_2dx. \tag{11}$$

In the case of variables separable the use of an integrating factor is therefore implied in the process of separating the variables. In the case of the linear equation the use of the integrating factor is somewhat shorter than the use of the substitution for separating the variables. In general it is not possible to hit upon an integrating factor by inspection and not practicable to obtain an integrating factor by analysis, but the integration of an equation is so simple when the factor is known, and the equations which arise in practice so frequently do have simple integrating factors, that it is worth while to examine the equation to see if the factor cannot be determined by inspection and trial. To aid in the work, the differentials of the simpler functions such as

$$dxy = xdy + ydx, \qquad \tfrac{1}{2}\, d(x^2 + y^2) = xdx + ydy,$$

$$d\frac{y}{x} = \frac{xdy - ydx}{x^2}, \qquad d\tan^{-1}\frac{x}{y} = \frac{ydx - xdy}{x^2 + y^2}, \tag{12}$$

should be borne in mind.

Consider the equation $(x^4e^x - 2\,mxy^2)\,dx + 2\,mx^2ydy = 0$. Here the first term x^4e^xdx will be a differential of a function of x no matter what function of x may be assumed as a trial μ. With $\mu = 1/x^4$ the equation takes the form

$$e^xdx + 2\,m\left(\frac{ydy}{x^2} - \frac{y^2dx}{x^3}\right) = de^x + md\frac{y^2}{x^2} = 0.$$

The integral is therefore seen to be $e^x + my^2/x^2 = C$ without more ado. It may be noticed that this equation is of the Bernoulli type and that an integration by that method would be considerably longer and more tedious than this use of an integrating factor.

Again, consider $(x + y)\,dx - (x - y)\,dy = 0$ and let it be written as

$$xdx + ydy + ydx - xdy = 0; \quad \text{try} \quad \mu = 1/(x^2 + y^2);$$

then $$\frac{xdx + ydy}{x^2 + y^2} + \frac{ydx - xdy}{x^2 + y^2} = 0 \quad \text{or} \quad \frac{1}{2}d\log(x^2 + y^2) + d\tan^{-1}\frac{x}{y} = 0,$$

and the integral is $\log\sqrt{x^2 + y^2} + \tan^{-1}(x/y) = C$. Here the terms $xdx + ydy$ strongly suggested $x^2 + y^2$ and the known form of the differential of $\tan^{-1}(x/y)$ corroborated the idea. This equation comes under the homogeneous type, but the use of the integrating factor considerably shortens the work of integration.

92. The attempt has been to write $Mdx + Ndy$ or $\mu(Mdx + Ndy)$ as the sum of total differentials $du + dv + \cdots$, that is, as the differential dF of the function $u + v + \cdots$, so that the solution of the equation $Mdx + Ndy = 0$ could be obtained as $F = C$. When the expressions are complicated, the attempt may fail in practice even where it theoretically should succeed. It is therefore of importance to establish conditions under which a differential expression like $Pdx + Qdy$ shall be the total differential dF of some function, and to find a means of obtaining F when the conditions are satisfied. This will now be done.

Suppose $$Pdx + Qdy = dF = \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy; \tag{13}$$

then $$P = \frac{\partial F}{\partial x}, \qquad Q = \frac{\partial F}{\partial y}, \qquad \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} = \frac{\partial^2 F}{\partial x \partial y}.$$

Hence if $Pdx + Qdy$ is a total differential dF, it follows (as in § 52) that the relation $P'_y = Q'_x$ must hold. Now conversely if this relation does hold, it may be shown that $Pdx + Qdy$ is the total differential of a function, and that this function is

$$F = \int_{x_0}^{x} P(x, y)\,dx + \int Q(x_0, y)\,dy$$

or
$$F = \int_{y_0}^{y} Q(x, y)\,dy + \int P(x, y_0)\,dx, \tag{14}$$

where the fixed value x_0 or y_0 will naturally be so chosen as to simplify the integrations as much as possible.

To show that these expressions may be taken as F it is merely necessary to compute their derivatives for identification with P and Q. Now

$$\frac{\partial F}{\partial x} = \frac{\partial}{\partial x}\int_{x_0}^{x} P(x, y)\,dx + \frac{\partial}{\partial x}\int Q(x_0, y)\,dy = P(x, y),$$

$$\frac{\partial F}{\partial y} = \frac{\partial}{\partial y}\int_{x_0}^{x} P(x, y)\,dx + \frac{\partial}{\partial y}\int Q(x_0, y)\,dy = \frac{\partial}{\partial y}\int P dx + Q(x_0, y).$$

These differentiations, applied to the first form of F, require only the fact that the derivative of an integral is the integrand. The first turns out satisfactorily. The second must be simplified by interchanging the order of differentiation by y and integration by x (Leibniz's Rule, § 119) and by use of the fundamental hypothesis that $P'_y = Q'_x$.

$$\frac{\partial}{\partial y}\int_{x_0}^{x} P dx + Q(x_0, y) = \int_{x_0}^{x} \frac{\partial P}{\partial y}\,dx + Q(x_0, y)$$

$$= \int_{x_0}^{x} \frac{\partial Q}{\partial x}\,dx + Q(x_0, y) = Q(x, y)\Big|_{x_0}^{x} + Q(x_0, y) = Q(x, y).$$

The identity of P and Q with the derivatives of F is therefore established. The second form of F would be treated similarly.

Show that $(x^2 + \log y)\,dx + x/y\,dy = 0$ is an exact differential equation and obtain the solution. Here it is first necessary to apply the test $P'_y = Q'_x$. Now

$$\frac{\partial}{\partial y}(x^2 + \log y) = \frac{1}{y} \quad \text{and} \quad \frac{\partial}{\partial x}\frac{x}{y} = \frac{1}{y}.$$

Hence the test is satisfied and the integral is obtained by applying the formula:

$$\int_0^x (x^2 + \log y)\,dx + \int \frac{0}{y}\,dy = \frac{1}{3}x^3 + x\log y = C$$

or
$$\int_1^y \frac{x}{y}\,dy + \int (x^2 + \log 1)\,dx = x\log y + \frac{1}{3}x^3 = C.$$

It should be noticed that the choice of $x_0 = 0$ simplifies the integration in the first case because the substitution of the lower limit 0 is easy and because the second integral vanishes. The choice of $y_0 = 1$ introduces corresponding simplifications in the second case.

Derive the *partial differential equation which any integrating factor of the differential equation $Mdx + Ndy = 0$ must satisfy.* If μ is an integrating factor, then

$$\mu Mdx + \mu Ndy = dF \quad \text{and} \quad \frac{\partial \mu M}{\partial y} = \frac{\partial \mu N}{\partial x}.$$

Hence
$$M\frac{\partial \mu}{\partial y} - N\frac{\partial \mu}{\partial x} = \mu\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) \qquad (15)$$

is the desired equation. To determine the integrating factor by solving this equation would in general be as difficult as solving the original equation; in some special cases, however, this equation is useful in determining μ.

93. It is now convenient to tabulate a list of different types of differential equations for which an integrating factor of a standard form can be given. With the knowledge of the factor, the equations may then be integrated by (14) or by inspection.

EQUATION $Mdx + Ndy = 0$:	FACTOR μ:
I. Homogeneous $Mdx + Ndy = 0$,	$\dfrac{1}{Mx + Ny}$.
II. Bernoulli $dy + X_1ydx = X_2y^ndx$,	$y^{-n}e^{(1-n)\int X_1dx}$.
III. $M = yf(xy)$, $N = xg(xy)$,	$\dfrac{1}{Mx - Ny}$.
IV. If $\dfrac{\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}}{N} = f(x)$,	$e^{\int f(x)dx}$.
V. If $\dfrac{\dfrac{\partial N}{\partial x} - \dfrac{\partial M}{\partial y}}{M} = f(y)$,	$e^{\int f(y)dy}$.
VI. Type $x^\alpha y^\beta(mydx + nxdy) = 0$,	$\begin{cases} x^{km-1-\alpha}y^{kn-1-\beta}, \\ k \text{ arbitrary.} \end{cases}$
VII. $x^\alpha y^\beta(mydx + nxdy) + x^\gamma y^\delta(pydx + qxdy) = 0$,	$\begin{cases} x^{km-1-\alpha}y^{kn-1-\beta}, \\ k \text{ determined.} \end{cases}$

The use of the integrating factor often is simpler than the substitution $y = vx$ in the homogeneous equation. It is practically identical with the substitution in the Bernoulli type. In the third type it is often shorter than the substitution. The remaining types have had no substitution indicated for them. The proofs that the assigned forms of the factor are right are given in the examples below or are left as exercises.

To show that $\mu = (Mx + Ny)^{-1}$ is an integrating factor for the homogeneous case, it is possible simply to substitute in the equation (15), which μ must satisfy, and show that the equation actually holds by virtue of the fact that M and N are homogeneous of the same degree, — this fact being used to simplify the result by Euler's Formula (30) of § 53. But it is easier to proceed directly to show

$$\frac{\partial}{\partial y}\frac{M}{Mx+Ny}=\frac{\partial}{\partial x}\left(\frac{N}{Mx+Ny}\right) \quad\text{or}\quad \frac{\partial}{\partial y}\left(\frac{1}{x}\frac{1}{1+\phi}\right)=\frac{\partial}{\partial x}\left(\frac{1}{y}\frac{\phi}{1+\phi}\right),\quad\text{where}\quad \phi=\frac{Ny}{Mx}.$$

Owing to the homogeneity, ϕ is a function of y/x alone. Differentiate.

$$\frac{\partial}{\partial y}\left(\frac{1}{x}\frac{1}{1+\phi}\right)=-\frac{1}{x}\frac{\phi'}{(1+\phi)^2}\frac{1}{x}=\frac{1}{y}\frac{\phi'}{(1+\phi)^2}\cdot\frac{-y}{x^2}=\frac{\partial}{\partial x}\left(\frac{1}{y}\frac{\phi}{1+\phi}\right).$$

As this is an evident identity, the theorem is proved.

To find the condition that the integrating factor may be a function of x only and to find the factor when the condition is satisfied, the equation (15) which μ satisfies may be put in the more compact form by dividing by μ.

$$M\frac{1}{\mu}\frac{\partial\mu}{\partial y}-N\frac{1}{\mu}\frac{\partial\mu}{\partial x}=\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\quad\text{or}\quad M\frac{\partial\log\mu}{\partial y}-N\frac{\partial\log\mu}{\partial x}=\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}. \qquad (15')$$

Now if μ (and hence $\log\mu$) is a function of x alone, the first term vanishes and

$$\frac{d\log\mu}{dx}=\frac{M'_y-N'_x}{N}=f(x)\quad\text{or}\quad \log\mu=\int f(x)\,dx.$$

This establishes the rule of type IV above and further shows that in no other case can μ be a function of x alone. The treatment of type V is clearly analogous.

Integrate the equation $x^4y(3\,ydx+2\,xdy)+x^2(4\,ydx+3\,xdy)=0$. This is of type VII; an integrating factor of the form $\mu=x^\rho y^\sigma$ will be assumed and the exponents ρ, σ will be determined so as to satisfy the condition that the equation be an exact differential. Here

$$P=\mu M=3\,x^{\rho+4}y^{\sigma+2}+4\,x^{\rho+2}y^{\sigma+1},\qquad Q=\mu N=2\,x^{\rho+5}y^{\sigma+1}+3\,x^{\rho+3}y^{\sigma}.$$

Then
$$P'_y=3(\sigma+2)\,x^{\rho+4}y^{\sigma+1}+4(\sigma+1)\,x^{\rho+2}y^{\sigma}$$
$$=2(\rho+5)\,x^{\rho+4}y^{\sigma+1}+3(\rho+3)\,x^{\rho+2}y^{\sigma}=Q'_x.$$

Hence if
$$3(\sigma+2)=2(\rho+5)\quad\text{and}\quad 4(\sigma+1)=3(\rho+3),$$

the relation $P'_y=Q'_x$ will hold. This gives $\sigma=2$, $\rho=1$. Hence $\mu=xy^2$,

and
$$\int_0^x(3\,x^5y^4+4\,x^3y^3)\,dx+\int 0dy=\tfrac{1}{2}\,x^6y^4+x^4y^3=C$$

is the solution. The work might be shortened a trifle by dividing through in the first place by x^2. Moreover the integration can be performed at sight without the use of (14).

94. Several of the most important facts relative to integrating factors and solutions of $Mdx+Ndy=0$ will now be stated as theorems and the proofs will be indicated below.

1. If an integrating factor is known, the corresponding solution may be found; and conversely if the solution is known, the corresponding integrating factor may be found. Hence the existence of either implies the existence of the other.

2. If $F=C$ and $G=C$ are two solutions of the equation, either must be a function of the other, as $G=\Phi(F)$; and any function of either is

a solution. If μ and ν are two integrating factors of the equation, the ratio μ/ν is either constant or a solution of the equation; and the product of μ by any function of a solution, as $\mu\Phi(F)$, is an integrating factor of the equation.

3. The normal derivative dF/dn of a solution obtained from the factor μ is the product $\mu\sqrt{M^2+N^2}$ (see § 48).

It has already been seen that if an integrating factor μ is known, the corresponding solution $F = C$ may be found by (14). Now if the solution is known, the equation

$$dF = F'_x dx + F'_y dy = \mu(Mdx + Ndy) \quad \text{gives} \quad F'_x = \mu M,\ F'_y = \mu N;$$

and hence μ may be found from either of these equations as the quotient of a derivative of F by a coefficient of the differential equation. The statement 1 is therefore proved. It may be remarked that the discussion of approximate solutions to differential equations (§§ 86–88), combined with the theory of limits (beyond the scope of this text), affords a demonstration that any equation $Mdx + Ndy = 0$, where M and N satisfy certain restrictive conditions, has a solution; and hence it may be inferred that such an equation has an integrating factor.

If μ be eliminated from the relations $F'_x = \mu M$, $F'_y = \mu N$ found above, it is seen that

$$MF'_y - NF'_x = 0, \quad \text{and similarly,} \quad MG'_y - NG'_x = 0, \tag{16}$$

are the conditions that F and G should be solutions of the differential equation. Now these are two simultaneous homogeneous equations of the first degree in M and N. If M and N are eliminated from them, there results the equation

$$F'_y G'_x - F'_x G'_y = 0 \quad \text{or} \quad \begin{vmatrix} F'_x & F'_y \\ G'_x & G'_y \end{vmatrix} = J(F, G) = 0, \tag{16'}$$

which shows (§ 62) that F and G are functionally related as required. To show that any function $\Phi(F)$ is a solution, consider the equation

$$M\Phi'_y - N\Phi'_x = (MF'_y - NF'_x)\Phi'.$$

As F is a solution, the expression $MF'_y - NF'_x$ vanishes by (16), and hence $M\Phi'_y - N\Phi'_x$ also vanishes, and Φ is a solution of the equation as is desired. The first half of 2 is proved.

Next, if μ and ν are two integrating factors, equation (15′) gives

$$M\frac{\partial \log\mu}{\partial y} - N\frac{\partial \log\mu}{\partial x} = M\frac{\partial \log\nu}{\partial y} - N\frac{\partial \log\nu}{\partial x} \quad \text{or} \quad M\frac{\partial \log\mu/\nu}{\partial y} - N\frac{\partial \log\mu/\nu}{\partial x} = 0.$$

On comparing with (16) it then appears that $\log(\mu/\nu)$ must be a solution of the equation and hence μ/ν itself must be a solution. The inference, however, would not hold if μ/ν reduced to a constant. Finally if μ is an integrating factor leading to the solution $F = C$, then

$$dF = \mu(Mdx + Ndy), \quad \text{and hence} \quad \mu\Phi(F)(Mdx + Ndy) = d\int \Phi(F)\,dF.$$

It therefore appears that the factor $\mu\Phi(F)$ makes the equation an exact differential and must be an integrating factor. Statement 2 is therefore wholly proved.

The third proposition is proved simply by differentiation and substitution. For

$$\frac{dF}{dn}=\frac{\partial F}{\partial x}\frac{dx}{dn}+\frac{\partial F}{\partial y}\frac{dy}{dn}=\mu M\frac{dx}{dn}+\mu N\frac{dy}{dn}.$$

And if τ denotes the inclination of the curve $F=C$, it follows that

$$\tan\tau=\frac{dy}{dx}=-\frac{M}{N},\qquad \sin\tau=\frac{dy}{dn}=\frac{N}{\sqrt{M^2+N^2}},\qquad -\cos\tau=\frac{dx}{dn}=\frac{M}{\sqrt{M^2+N^2}}.$$

Hence $dF/dn=\mu\sqrt{M^2+N^2}$ and the proposition is proved.

EXERCISES

1. Find the integrating factor by inspection and integrate:

(α) $xdy-ydx=(x^2+y^2)dx$, (β) $(y^2-xy)dx+x^2dy=0$,
(γ) $ydx-xdy+\log x dx=0$, (δ) $y(2xy+e^x)dx-e^xdy=0$,
(ϵ) $(1+xy)ydx+(1-xy)xdy=0$, (ζ) $(x-y^2)dx+2xydy=0$,
(η) $(xy^2+y)dx-xdy=0$, (θ) $a(xdy+2ydx)=xydy$,
(ι) $(x^2+y^2)(xdx+ydy)+\sqrt{1+(x^2+y^2)}(ydx-xdy)=0$,
(κ) $x^2ydx-(x^3+y^3)dy=0$, (λ) $xdy-ydx=x\sqrt{x^2-y^2}dy$.

2. Integrate these linear equations with an integrating factor:

(α) $y'+ay=\sin bx$, (β) $y'+y\cot x=\sec x$,
(γ) $(x+1)y'-2y=(x+1)^4$, (δ) $(1+x^2)y'+y=e^{\tan^{-1}x}$,

and (β), (δ), (ζ) of Ex. 4, p. 206.

3. Show that the expression given under II, p. 210, is an integrating factor for the Bernoulli equation, and integrate the following equations by that method:

(α) $y'-y\tan x=y^4\sec x$, (β) $3y^2y'+y^3=x-1$,
(γ) $y'+y\cos x=y^n\sin 2x$, (δ) $dx+2xydy=2ax^3y^3dy$,

and (α), (γ), (ϵ), (η) of Ex. 4, p. 206.

4. Show the following are exact differential equations and integrate:

(α) $(3x^2+6xy^2)dx+(6x^2y+4y^2)dy=0$, ($\beta$) $\sin x\cos ydx+\cos x\sin ydy=0$,
(γ) $(6x-2y+1)+(2y-2x-3)dy=0$, (δ) $(x^3+3xy^2)dx+(y^3+3x^2y)dy=0$,
(ϵ) $\frac{2xy+1}{y}dx+\frac{y-x}{y^2}dy=0$, ($\zeta$) $\left(1+e^{\frac{x}{y}}\right)dx+e^{\frac{x}{y}}\left(1-\frac{x}{y}\right)dy=0$,
(η) $e^x(x^2+y^2+2x)dx+2ye^xdy=0$, ($\theta$) $(y\sin x-1)dx+(y-\cos x)dy=0$.

5. Show that $(Mx-Ny)^{-1}$ is an integrating factor for type III. Determine the integrating factors of the following equations, thus render them exact, and integrate:

(α) $(y+x)dx+xdy=0$, (β) $(y^2-xy)dx+x^2dy=0$,
(γ) $(x^2+y^2)dx-2xydy=0$, (δ) $(x^2y^2+xy)ydx+(x^2y^2-1)xdy=0$,
(ϵ) $(\sqrt{xy}-1)xdy-(\sqrt{xy}+1)ydx=0$, ($\zeta$) $x^3dx+(3x^2y+2y^3)dy=0$,

and Exs. 3 and 9, p. 206.

6. Show that the factor given for type VI is right, and that the form given for type VII is right if k satisfies $k(qm-pn)=q(\alpha-\gamma)-p(\beta-\delta)$.

7. Integrate the following equations of types IV–VII:

(α) $(y^4 + 2y)dx + (xy^3 + 2y^4 - 4x)dy = 0$, ($\beta$) $(x^2 + y^2 + 1)dx - 2xydy = 0$,
(γ) $(3x^2 + 6xy 3y^2)dx + (2x^2 + 3xy)dy = 0$, ($\delta$) $(2x^2y^2 + y) - (x^3y - 3x)y' = 0$,
(ϵ) $(2x^2y - 3y^4)dx + (3x^3 + 2xy^3)dy = 0$,
(ζ) $(2 - y')\sin(3x - 2y) + y' \sin(x - 2y) = 0$.

8. By virtue of proposition 2 above, it follows that if an equation is exact and homogeneous, or exact and has the variables separable, or homogeneous and under types IV–VII, so that two different integrating factors may be obtained, the solution of the equation may be obtained without integration. Apply this to finding the solutions of Ex. 4 (β), (δ), (γ); Ex. 5 (α), (γ).

9. Discuss the apparent exceptions to the rules for types I, III, VII, that is, when $Mx + Ny = 0$ or $Mx - Ny \doteq 0$ or $qm - pn = 0$.

10. Consider this rule for integrating $Mdx + Ndy = 0$ when the equation is known to be exact: Integrate Mdx regarding y as constant, differentiate the result regarding y as variable, and subtract from N; then integrate the difference with respect to y. In symbols,

$$C = \int (Mdx + Ndy) = \int Mdx + \int \left(N - \frac{\partial}{\partial y}\int Mdx\right)dy.$$

Apply this instead of (14) to Ex. 4. Observe that in no case should either this formula or (14) be applied when the integral is obtainable by inspection.

95. Linear equations with constant coefficients. The type

$$a_0 \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_{n-1}\frac{dy}{dx} + a_n y = X(x) \tag{17}$$

of differential equation of the nth order which is of the first degree in y and its derivatives is called a *linear* equation. For the present only the case where the coefficients $a_0, a_1, \cdots, a_{n-1}, a_n$ are constant will be treated, and for convenience it will be assumed that the equation has been divided through by a_0 so that the coefficient of the highest derivative is 1. Then if differentiation be denoted by D, the equation may be written *symbolically* as

$$(D^n + a_1 D^{n-1} + \cdots + a_{n-1}D + a_n)y = X, \tag{17'}$$

where the symbol D combined with constants follows many of the laws of ordinary algebraic quantities (see § 70).

The simplest equation would be of the first order. Here

$$\frac{dy}{dx} - a_1 y = X \quad \text{and} \quad y = e^{a_1 x}\int e^{-a_1 x}Xdx, \tag{18}$$

as may be seen by reference to (11) or (6). Now if $D - a_1$ be treated as an algebraic symbol, the solution may be indicated as

$$(D - a_1)y = X \quad \text{and} \quad y = \frac{1}{D - a_1}X, \tag{18'}$$

where the operator $(D - a_1)^{-1}$ is the *inverse* of $D - a_1$. The solution which has just been obtained shows that the interpretation which must be assigned to the inverse operator is

$$\frac{1}{D - a_1}(*) = e^{a_1 x}\int e^{-a_1 x}(*)\,dx, \tag{19}$$

where $(*)$ denotes the function of x upon which it operates. That the integrating operator is the inverse of $D - a_1$ may be proved by direct differentiation (see Ex. 7, p. 152).

This operational method may at once be extended to obtain the solution of equations of higher order. For consider

$$\frac{d^2y}{dx^2} + a_1\frac{dy}{dx} + a_2 y = X \quad \text{or} \quad (D^2 + a_1 D + a_2)\,y = X. \tag{20}$$

Let α_1 and α_2 be the roots of the equation $D^2 + a_1 D + a_2 = 0$ so that the differential equation may be written in the form

$$[D^2 - (\alpha_1 + \alpha_2)\,D + \alpha_1\alpha_2]\,y = X \quad \text{or} \quad (D - \alpha_1)(D - \alpha_2)\,y = X. \tag{20'}$$

The solution may now be evaluated by a succession of steps as

$$(D - \alpha_2)\,y = \frac{1}{D - \alpha_1}X = e^{\alpha_1 x}\int e^{-\alpha_1 x}X\,dx,$$

$$y = \frac{1}{D - \alpha_2}\left[\frac{1}{D - \alpha_1}X\right] = e^{\alpha_2 x}\int e^{-\alpha_2 x}\left[e^{\alpha_1 x}\int e^{-\alpha_1 x}X\,dx\right]$$

or

$$y = e^{\alpha_2 x}\int e^{(\alpha_1 - \alpha_2)x}\left[\int e^{-\alpha_1 x}X\,dx\right]dx. \tag{20''}$$

The solution of the equation is thus reduced to quadratures.

The extension of the method to an equation of any order is immediate. The first step in the solution is to solve the equation

$$D^n + a_1 D^{n-1} + \cdots + a_{n-1}D + a_n = 0$$

so that the differential equation may be written in the form

$$(D - \alpha_1)(D - \alpha_2)\cdots(D - \alpha_{n-1})(D - \alpha_n)\,y = X; \tag{17''}$$

whereupon the solution is comprised in the formula

$$y = e^{\alpha_n x}\int e^{(\alpha_{n-1} - \alpha_n)x}\int \cdots \int e^{(\alpha_1 - \alpha_2)x}\int e^{-\alpha_1 x}X\,(dx)^n, \tag{17'''}$$

where the successive integrations are to be performed by beginning upon the extreme right and working toward the left. Moreover, it appears that if the operators $D - \alpha_n$, $D - \alpha_{n-1}, \cdots, D - \alpha_2$, $D - \alpha_1$ were successively applied to this value of y, they would undo the work here done and lead back to the original equation. As n integrations are required, there will occur n arbitrary constants of integration in the answer for y.

As an example consider the equation $(D^3 - 4D)y = x^2$. Here the roots of the algebraic equation $D^3 - 4D = 0$ are $0, 2, -2$, and the solution for y is

$$y = \frac{1}{D}\frac{1}{D-2}\frac{1}{D+2}x^2 = \int e^{2x}\int e^{-2x}e^{-2x}\int e^{2x}x^2(dx)^3.$$

The successive integrations are very simple by means of a table. Then

$$\int e^{2x}x^2dx = \tfrac{1}{2}x^2e^{2x} - \tfrac{1}{2}xe^{2x} + \tfrac{1}{4}e^{2x} + C_1,$$

$$\int e^{-4x}\int e^{2x}x^2(dx)^2 = \int(\tfrac{1}{2}x^2e^{-2x} - \tfrac{1}{2}xe^{-2x} + \tfrac{1}{4}e^{-2x} + C_1e^{-4x})\,dx$$
$$= -\tfrac{1}{4}x^2e^{-2x} - \tfrac{1}{8}e^{-2x} + C_1e^{-4x} + C_2,$$

$$y = \int e^{2x}\int e^{-4x}\int e^{2x}x^2(dx)^3 = \int(-\tfrac{1}{4}x^2 - \tfrac{1}{8} + C_1e^{-2x} + C_2e^{2x})\,dx$$
$$= -\tfrac{1}{12}x^3 - \tfrac{1}{8}x + C_1e^{-2x} + C_2e^{2x} + C_3.$$

This is the solution. It may be noted that in integrating a term like C_1e^{-4x} the result may be written as C_1e^{-4x}, for the reason that C_1 is arbitrary anyhow; and, moreover, if the integration had introduced any terms such as $2e^{-2x}$, $\frac{1}{2}e^{2x}$, 5, these could be combined with the terms C_1e^{-2x}, C_2e^{2x}, C_3 to simplify the form of the results.

In case the roots are imaginary the procedure is the same. Consider

$$\frac{d^2y}{dx^2} + y = \sin x \quad \text{or} \quad (D^2+1)y = \sin x \quad \text{or} \quad (D+i)(D-i)y = \sin x.$$

Then $$y = \frac{1}{D-i}\frac{1}{D+i}\sin x = e^{ix}\int e^{-2ix}\int e^{ix}\sin x\,(dx)^2, \qquad i = \sqrt{-1}.$$

The formula for $\int e^{ax}\sin bx dx$, as given in the tables, is not applicable when $a^2 + b^2 = 0$, as is the case here, because the denominator vanishes. It therefore becomes expedient to write $\sin x$ in terms of exponentials. Then

$$y = e^{ix}\int e^{-2ix}\int e^{ix}\frac{e^{ix} - e^{-ix}}{2i}(dx)^2; \quad \text{for} \quad \sin x = \frac{e^{ix} - e^{-ix}}{2i}.$$

Now $$\frac{1}{2i}e^{ix}\int e^{-2ix}\int(e^{2ix} - 1)(dx)^2 = \frac{1}{2i}e^{ix}\int e^{-2ix}\left[\frac{1}{2i}e^{2ix} - x + C_1\right]dx$$
$$= \frac{1}{2i}e^{ix}\left[\frac{1}{2i}x + \frac{1}{2i}e^{-2ix}x - \frac{1}{4}e^{-2ix} + C_1e^{-2ix} + C_2\right]$$
$$= -\frac{x}{2}\frac{e^{ix} + e^{-ix}}{2} + C_1e^{-ix} + C_2e^{ix}.$$

Now $$C_1e^{-ix} + C_2e^{ix} = (C_2 + C_1)\frac{e^{ix} + e^{-ix}}{2} + (C_2 - C_1)\,i\,\frac{e^{ix} - e^{-ix}}{2i}.$$

Hence this expression may be written as $C_1\cos x + C_2\sin x$, and then

$$y = -\tfrac{1}{2}x\cos x + C_1\cos x + C_2\sin x.$$

The solution of such equations as these gives excellent opportunity to cultivate the art of manipulating trigonometric functions through exponentials (§ 74).

96. The general method of solution given above may be considerably simplified in case the function $X(x)$ has certain special forms. In the first place suppose $X = 0$, and let the equation be $P(D)y = 0$, where $P(D)$ denotes the symbolic polynomial of the nth degree in D. Suppose the roots of $P(D) = 0$ are $\alpha_1, \alpha_2, \cdots, \alpha_k$ and their respective multiplicities are $m_1, m_2, \cdots, m_k$, so that

$$(D-\alpha_k)^{m_k}\cdots(D-\alpha_2)^{m_2}(D-\alpha_1)^{m_1}y = 0$$

is the form of the differential equation. Now, as above, if

$$(D-\alpha_1)^{m_1}y = 0, \quad \text{then} \quad y = \frac{1}{(D-\alpha_1)^{m_1}}0 = e^{\alpha_1 x}\int\cdots\int 0\,(dx)^{m_1}.$$

Hence $\qquad y = e^{\alpha_1 x}(C_1 + C_2 x + C_3 x^2 + \cdots + C_{m_1}x^{m_1-1})$

is annihilated by the application of the operator $(D-\alpha_1)^{m_1}$, and therefore by the application of the whole operator $P(D)$, and must be a solution of the equation. As the factors in $P(D)$ may be written so that any one of them, as $(D-\alpha_i)^{m_i}$, comes last, it follows that to each factor $(D-\alpha_i)^{m_i}$ will correspond a solution

$$y_i = e^{\alpha_i x}(C_{i1} + C_{i2}x + \cdots + C_{im_i}x^{m_i-1}), \qquad P(D)y_i = 0,$$

of the equation. Moreover the sum of all these solutions,

$$y = \sum_{i=1}^{i=k} e^{\alpha_i x}(C_{i1} + C_{i2}x + \cdots + C_{im_i}x^{m_i-1}), \tag{21}$$

will be a solution of the equation; for in applying $P(D)$ to y,

$$P(D)y = P(D)y_1 + P(D)y_2 + \cdots + P(D)y_k = 0.$$

Hence the general rule may be stated that: *The solution of the differential equation $P(D)y = 0$ of the nth order may be found by multiplying each $e^{\alpha x}$ by a polynomial of $(m-1)$st degree in x (where α is a root of the equation $P(D) = 0$ of multiplicity m and where the coefficients of the polynomial are arbitrary) and adding the results.* Two observations may be made. First, the solution thus found contains n arbitrary constants and may therefore be considered as the general solution; and second, if there are imaginary roots for $P(D) = 0$, *the exponentials arising from the pure imaginary parts of the roots may be converted into trigonometric functions.*

As an example take $(D^4 - 2D^3 + D^2)y = 0$. The roots are 1, 1, 0, 0. Hence the solution is

$$y = e^x(C_1 + C_2 x) + (C_3 + C_4 x).$$

Again if $(D^4 + 4)y = 0$, the roots of $D^4 + 4 = 0$ are $\pm 1 \pm i$ and the solution is

$$y = C_1 e^{(1+i)x} + C_2 e^{(1-i)x} + C_3 e^{(-1+i)x} + C_4 e^{(-1-i)x}$$

or
$$y = e^x(C_1 e^{ix} + C_2 e^{-ix}) + e^{-x}(C_3 e^{ix} + C_4 e^{-ix})$$
$$= e^x(C_1 \cos x + C_2 \sin x) + e^{-x}(C_3 \cos x + C_4 \sin x),$$
where the new C's are not identical with the old C's. Another form is
$$y = e^x A \cos(x + \gamma) + e^{-x} B \cos(x + \delta),$$
where γ and δ, A and B, are arbitrary constants. For
$$C_1 \cos x + C_2 \sin x = \sqrt{C_1^2 + C_2^2}\left[\frac{C_1}{\sqrt{C_1^2 + C_2^2}} \cos x + \frac{C_2}{\sqrt{C_1^2 + C_2^2}} \sin x\right],$$
and if $\gamma = \tan^{-1}\left(-\frac{C_2}{C_1}\right)$, then $C_1 \cos x + C_2 \sin x = \sqrt{C_1^2 + C_2^2} \cos(x + \gamma)$.

Next if x is not zero but *if any one solution I can be found so that $P(D)\,I = X$, then a solution containing n arbitrary constants may be found by adding to I the solution of $P(D)y = 0$.* For if
$$P(D)\,I = X \quad \text{and} \quad P(D)\,y = 0, \quad \text{then} \quad P(D)\,(I + y) = X.$$
It therefore remains to devise means for finding one solution I. This solution I may be found by the long method of (17‴), where the integration may be shortened by omitting the constants of integration since only one, and not the general, value of the solution is needed. In the most important cases which arise in practice there are, however, some very short cuts to the solution I. The solution I of $P(D)y = X$ is called the *particular integral* of the equation and the general solution of $P(D)\,y = 0$ is called the *complementary function* for the equation $P(D)\,y = X$.

Suppose that X *is a polynomial in* x. Solve symbolically, arrange $P(D)$ in ascending powers of D, and divide out to powers of D equal to the order of the polynomial X. Then
$$P(D)\,I = X, \qquad I = \frac{1}{P(D)}\,X = \left[Q(D) + \frac{R(D)}{P(D)}\right] X, \tag{22}$$
where the remainder $R(D)$ is of *higher* order in D than X in x. Then
$$P(D)\,I = P(D)\,Q(D)\,X + R(D)\,X, \qquad R(D)\,X = 0.$$
Hence $Q(D)\,x$ may be taken as I, since $P(D)\,Q(D)\,X = P(D)\,I = X$. By this method the solution I may be found, when X is a polynomial, *as rapidly as $P(D)$ can be divided into* 1; the solution of $P(D)\,y = 0$ may be written down by (21); and the sum of I and this will be the required solution of $P(D)\,y = X$ containing n constants.

As an example consider $(D^3 + 4\,D^2 + 3\,D)\,y = x^2$. The work is as follows:
$$I = \frac{1}{3\,D + 4\,D^2 + D^3}\,x^2 = \frac{1}{D}\,\frac{1}{3 + 4\,D + D^2}\,x^2 = \frac{1}{D}\left[\frac{1}{3} - \frac{4}{9}D + \frac{13}{27}D^2 + \frac{R(D)}{P(D)}\right] x^2.$$

Hence $\quad I = Q(D)x^2 = \frac{1}{D}\left(\frac{1}{3} - \frac{4}{9}D + \frac{13}{27}D^2\right)x^2 = \frac{1}{9}x^3 - \frac{4}{9}x^2 + \frac{26}{27}x.$

For $D^3 + 4D^2 + 3D = 0$ the roots are $0, -1, -3$ and the complementary function or solution of $P(D)y = 0$ would be $C_1 + C_2e^{-x} + C_3e^{-3x}$. Hence the solution of the equation $P(D)y = x^2$ is

$$y = C_1 + C_2e^{-x} + C_3e^{-3x} + \tfrac{1}{9}x^3 - \tfrac{4}{9}x^2 + \tfrac{26}{27}x.$$

It should be noted that in this example D is a factor of $P(D)$ and has been taken out before dividing; this shortens the work. Furthermore note that, in interpreting $1/D$ as integration, the constant may be omitted because any one value of I will do.

97. Next suppose that $X = Ce^{ax}$. Now $De^{ax} = ae^{ax}$, $D^ke^{ax} = a^ke^{ax}$,

and $\quad P(D)e^{ax} = P(a)e^{ax}$; hence $\quad P(D)\left[\frac{C}{P(a)}e^{ax}\right] = Ce^{ax}.$

But $\quad P(D)I = Ce^{ax}$, and hence $\quad I = \frac{C}{P(a)}e^{ax} \qquad (23)$

is clearly a solution of the equation, provided a is not a root of $P(D) = 0$. If $P(a) = 0$, the division by $P(a)$ is impossible and the quest for I has to be directed more carefully. Let a be a root of multiplicity m so that $P(D) = (D - a)^m P_1(D)$. Then

$$P_1(D)(D - a)^m I = Ce^{ax}, \qquad (D - a)^m I = \frac{C}{P_1(a)}e^{ax},$$

and
$$I = \frac{C}{P_1(a)}e^{ax}\int\cdots\int(dx)^m = \frac{Ce^{ax}x^m}{P_1(a)\,m!}. \qquad (23')$$

For in the integration the constants may be omitted. It follows that when $X = Ce^{ax}$, the solution I may be found *by direct substitution.*

Now if X broke up into the sum of terms $X = X_1 + X_2 + \cdots$ and if solutions $I_1, I_2, \cdots$ were determined for each of the equations $P(D)I_1 = X_1$, $P(D)I_2 = X_2, \cdots$, the solution I corresponding to X would be the sum $I_1 + I_2 + \cdots$. Thus it is seen that the above short methods apply to equations in which X is a sum of terms of the form Cx^m or Ce^{ax}.

As an example consider $(D^4 - 2D^2 + 1)y = e^x$. The roots are $1, 1, -1, -1$, and $a = 1$. Hence the solution for I is written as

$$(D + 1)^2(D - 1)^2 I = e^x, \qquad (D - 1)^2 I = \tfrac{1}{4}e^x, \qquad I = \tfrac{1}{8}e^x x^2.$$

Then $\quad y = e^x(C_1 + C_2x) + e^{-x}(C_3 + C_4x) + \tfrac{1}{8}e^x x^2.$

Again consider $(D^2 - 5D + 6)y = x + e^{mx}$. To find the I_1 corresponding to x, divide.

$$I_1 = \frac{1}{6 - 5D + D^2}x = \left(\frac{1}{6} + \frac{5}{36}D + \cdots\right)x = \frac{1}{6}x + \frac{5}{36}.$$

To find the I_2 corresponding to e^{mx}, substitute. There are three cases,

$$I_2 = \frac{1}{m^2 - 5m + 6}e^{mx}, \qquad I_2 = xe^{3x}, \qquad I_2 = -xe^{2x},$$

according as m is neither 2 nor 3, or is 3, or is 2. Hence for the complete solution,

$$y = C_1e^{3x} + C_2e^{2x} + \frac{1}{6}x + \frac{5}{36} + \frac{1}{m^2 - 5m + 6}e^{mx},$$

when m is neither 2 nor 3; but in these special cases the results are

$$y = C_1e^{3x} + C_2e^{2x} + \tfrac{1}{6}x + \tfrac{5}{36} - xe^{2x}, \qquad y = C_1e^{3x} + C_2e^{2x} + \tfrac{1}{6}x + \tfrac{5}{36} + xe^{3x}.$$

The next case to consider is where X is of the form $\cos\beta x$ or $\sin\beta x$. If these trigonometric functions be expressed in terms of exponentials, the solution may be conducted by the method above; and this is perhaps the best method when $\pm\beta i$ are roots of the equation $P(D) = 0$. It may be noted that this method would apply also to the case where X might be of the form $e^{\alpha x}\cos\beta x$ or $e^{\alpha x}\sin\beta x$. Instead of splitting the trigonometric functions into two exponentials, it is possible to combine two trigonometric functions into an exponential. Thus, consider the equations

$$P(D)y = e^{\alpha x}\cos\beta x, \qquad P(D)y = e^{\alpha x}\sin\beta x,$$

and

$$P(D)y = e^{\alpha x}(\cos\beta x + i\sin\beta x) = e^{(\alpha+\beta i)x}. \tag{24}$$

The solution I of this last equation may be found and split into its real and imaginary parts, of which the real part is the solution of the equation involving the cosine, and the imaginary part the sine.

When X has the form $\cos\beta x$ or $\sin\beta x$ and $\pm\beta i$ are not roots of the equation $P(D) = 0$, there is a very short method of finding I. For

$$D^2\cos\beta x = -\beta^2\cos\beta x \quad\text{and}\quad D^2\sin\beta x = -\beta^2\sin\beta x.$$

Hence if $P(D)$ be written as $P_1(D^2) + DP_2(D^2)$ by collecting the even terms and the odd terms so that P_1 and P_2 are both even in D, the solution may be carried out symbolically as

$$I = \frac{1}{P(D)}\cos x = \frac{1}{P_1(D^2) + DP_2(D^2)}\cos x = \frac{1}{P_1(-\beta^2) + DP_2(-\beta^2)}\cos x,$$

or

$$I = \frac{P_1(-\beta^2) - DP_2(-\beta^2)}{[P_1(-\beta^2)]^2 + \beta^2[P_2(-\beta^2)]^2}\cos x. \tag{25}$$

By this device of substitution and of rationalization as if D were a surd, the differentiation is transferred to the numerator and can be performed. This method of procedure may be justified directly, or it may be made to depend upon that of the paragraph above.

Consider the example $(D^2 + 1)y = \cos x$. Here $\beta i = i$ is a root of $D^2 + 1 = 0$. As an operator D^2 is equivalent to -1, and the rationalization method will not work. If the first solution be followed, the method of solution is

$$I = \frac{1}{D^2+1}\frac{e^{ix}}{2} + \frac{1}{D^2+1}\frac{e^{-ix}}{2} = \frac{1}{D-i}\frac{e^{ix}}{4i} - \frac{1}{D+i}\frac{e^{-ix}}{4i} = \frac{1}{4i}[xe^{ix} - xe^{-ix}] = \frac{1}{2}x\sin x.$$

If the second suggestion be followed, the solution may be found as follows:

$$(D^2+1)I = \cos x + i\sin x = e^{ix}, \qquad I = \frac{1}{D^2+1}e^{ix} = \frac{xe^{ix}}{2i}.$$

Now $$I = \frac{x}{2i}(\cos x + i\sin x) = \frac{1}{2}x\sin x - \frac{1}{2}ix\cos x.$$

Hence $I = \frac{1}{2}x\sin x$ for $(D^2+1)I = \cos x$,

and $I = -\frac{1}{2}x\cos x$ for $(D^2+1)I = \sin x$.

The complete solution is $y = C_1\cos x + C_2\sin x + \frac{1}{2}x\sin x$,

and for $(D^2+1)y = \sin x$, $y = C_1\cos x + C_2\sin x - \frac{1}{2}x\cos x$.

As another example take $(D^2 - 3D + 2)y = \cos x$. The roots are 1, 2, neither is equal to $\pm\beta i = \pm i$, and the method of rationalization is practicable. Then

$$I = \frac{1}{D^2-3D+2}\cos x = \frac{1}{1-3D}\cos x = \frac{1+3D}{10}\cos x = \frac{1}{10}(\cos x - 3\sin x).$$

The complete solution is $y = C_1e^{-x} + C_2e^{-2x} + \frac{1}{10}(\cos x - 3\sin x)$. The extreme simplicity of this substitution-rationalization method is noteworthy.

EXERCISES

1. By the general method solve the equations:

(α) $\dfrac{d^2y}{dx^2} + 4\dfrac{dy}{dx} + 3y = 2e^{2x}$, ($\beta$) $\dfrac{d^3y}{dx^3} - 3\dfrac{d^2y}{dx^2} + 3\dfrac{dy}{dx} - y = e^x$,

(γ) $(D^2 - 4D + 2)y = x$, (δ) $(D^3 + D^2 - 4D + 4)y = x$,

(ϵ) $(D^3 + 5D^2 + 6D)y = x$, (ζ) $(D^2 + D + 1)y = xe^x$,

(η) $(D^2 + D + 1)y = \sin 2x$, (θ) $(D^2 - 4)y = x + e^{2x}$,

(ι) $(D^2 + 3D + 2)y = x + \cos x$, ($\kappa$) $(D^4 - 4D^2)y = 1 - \sin x$,

(λ) $(D^2 + 1)y = \cos x$, (μ) $(D^2 + 1)y = \sec x$, (ν) $(D^2 + 1)y = \tan x$.

2. By the rule write the solutions of these equations:

(α) $(D^2 + 3D + 2)y = 0$, (β) $(D^3 + 3D^2 + D - 5)y = 0$,

(γ) $(D - 1)^3y = 0$, (δ) $(D^4 + 2D^2 + 1)y = 0$,

(ϵ) $(D^3 - 3D^2 + 4)y = 0$, (ζ) $(D^4 - D^3 - 9D^2 - 11D - 4)y = 0$,

(η) $(D^3 - 6D^2 + 9D)y = 0$, (θ) $(D^4 - 4D^3 + 8D^2 - 8D + 4)y = 0$,

(ι) $(D^5 - 2D^4 + D^3)y = 0$, (κ) $(D^3 - D^2 + D)y = 0$,

(λ) $(D^4 - 1)^2y = 0$, (μ) $(D^5 - 13D^3 + 26D^2 + 82D + 104)y = 0$.

3. By the short method solve (γ), (δ), (ϵ) of Ex. 1, and also:

(α) $(D^4 - 1)y = x^4$, (β) $(D^3 - 6D^2 + 11D - 6)y = x$,

(γ) $(D^3 + 3D^2 + 2D)y = x^2$, (δ) $(D^3 - 3D^2 - 6D + 8)y = x$,

(ϵ) $(D^3 + 8)y = x^4 + 2x + 1$, ($\zeta$) $(D^3 - 3D^2 - D + 3)y = x^2$,

(η) $(D^4 - 2D^3 + D^2)y = x$, (θ) $(D^4 + 2D^3 + 3D^2 + 2D + 1)y = 1 + x + x^2$,

(ι) $(D^3 - 1)y = x^2$, (κ) $(D^4 - 2D^3 + D^2)y = x^3$.

4. By the short method solve (α), (β), (θ) of Ex. 1, and also:

(α) $(D^2 - 3D + 2)y = e^x$, (β) $(D^4 - D^3 - 3D^2 + 5D - 2)y = e^{3x}$,

(γ) $(D^2 - 2D + 1)y = e^x$, (δ) $(D^3 - 3D^2 + 4)y = e^{3x}$,

(ϵ) $(D^2 + 1)y = 2e^x + x^3 - x$, ($\zeta$) $(D^3 + 1)y = 3 + e^{-x} + 5e^{2x}$,

(η) $(D^4 + 2D^2 + 1)y = e^x + 4$, ($\theta$) $(D^3 + 3D^2 + 3D + 1)y = 2e^{-x}$,

(ι) $(D^2 - 2D)y = e^{2x} + 1$, (κ) $(D^3 + 2D^2 + D)y = e^{2x} + x^2 + x$,

(λ) $(D^2 - a^2)y = e^{ax} + e^{bx}$, ($\mu$) $(D^2 - 2aD + a^2)y = e^x + 1$.

5. Solve by the short method (η), (ι), (κ) of Ex. 1, and also:

(α) $(D^2 - D - 2)y = \sin x$,
(β) $(D^2 + 2D + 1)y = 3e^{2x} - \cos x$,
(γ) $(D^2 + 4)y = x^2 + \cos x$,
(δ) $(D^3 + D^2 - D - 1)y = \cos 2x$,
(ϵ) $(D^2 + 1)^2 y = \cos x$,
(ζ) $(D^3 - D^2 + D - 1)y = \cos x$,
(η) $(D^2 - 5D + 6)y = \cos x - e^{2x}$,
(θ) $(D^3 - 2D^2 - 3D)y = 3x^2 + \sin x$,
(ι) $(D^2 - 1)^2 y = \sin x$,
(κ) $(D^2 + 3D + 2)y = e^{2x} \sin x$,
(λ) $(D^4 - 1)y = e^x \cos x$,
(μ) $(D^3 - 3D^2 + 4D - 2)y = e^x + \cos x$,
(ν) $(D^2 - 2D + 4)y = e^x \sin x$,
(o) $(D^2 + 4)y = \sin 3x + e^x + x^2$,
(π) $(D^6 + 1)y = \sin \frac{3}{2}x \sin \frac{1}{2}x$,
(ρ) $(D^3 + 1)y = e^{2x} \sin x + e^{\frac{x}{2}} \sin \frac{x\sqrt{3}}{2}$,
(σ) $(D^2 + 4)y = \sin^2 x$,
(τ) $(D^4 + 32D + 48)y = xe^{-2x} + e^{2x} \cos 2^{\frac{3}{2}} x$.

6. If X has the form $e^{\alpha x}X_1$, show that $I = \frac{1}{P(D)} e^{\alpha x} X_1 = e^{\alpha x} \frac{1}{P(D + \alpha)} X_1$. This enables the solution of equations where X_1 is a polynomial to be obtained by a short method; it also gives a way of treating equations where X is $e^{\alpha x} \cos \beta x$ or $e^{\alpha x} \sin \beta x$, but is not an improvement on (24); finally, combined with the second suggestion of (24), it covers the case where X is the product of a sine or cosine by a polynomial. Solve by this method, or partly by this method, (ζ) of Ex. 1; (κ), (λ), (ν), (ρ), (τ) of Ex. 5; and also

(α) $(D^2 - 2D + 1)y = x^2 e^{3x}$,
(β) $(D^3 + 3D^2 + 3D + 1)y = (2 - x^2)e^{-x}$,
(γ) $(D^2 + n^2)y = x^4 e^x$,
(δ) $(D^4 - 2D^3 - 3D^2 + 4D + 4)y = x^2 e^x$,
(ϵ) $(D^3 - 7D - 6)y = e^{2x}(1 + x)$,
(ζ) $(D - 1)^2 y = e^x + \cos x + x^2 e^x$,
(η) $(D - 1)^3 y = x - x^3 e^x$,
(θ) $(D^2 + 2)y = x^2 e^{3x} + e^x \cos 2x$,
(ι) $(D^3 - 1)y = xe^x + \cos^2 x$,
(κ) $(D^2 - 1)y = x \sin x + (1 + x^2)e^x$,
(λ) $(D^2 + 4)y = x \sin x$,
(μ) $(D^4 + 2D^2 + 1)y = x^2 \cos ax$,
(ν) $(D^2 + 4)y = (x \sin x)^2$,
(o) $(D^2 - 2D + 4)^2 y = xe^x \cos \sqrt{3}x$.

7. Show that the substitution $x = e^t$, Ex. 9, p. 152, changes equations of the type

$$x^n D^n y + a_1 x^{n-1} D^{n-1} y + \cdots + a_{n-1} x D y + a_n y = X(x) \tag{26}$$

into equations with constant coefficients; also that $ax + b = e^t$ would make a similar simplification for equations whose coefficients were powers of $ax + b$. Hence integrate:

(α) $(x^2D^2 - xD + 2)y = x \log x$,
(β) $(x^3D^3 - x^2D^2 + 2xD - 2)y = x^3 + 3x$,
(γ) $[(2x-1)^3 D^3 + (2x-1)D - 2]y = 0$,
(δ) $(x^2D^2 + 3xD + 1)y = (1 - x)^{-2}$,
(ϵ) $(x^3D^3 + xD - 1)y = x \log x$,
(ζ) $[(x + 1)^2 D^2 - 4(x + 1)D + 6]y = x$,
(η) $(x^2D + 4xD + 2)y = e^x$,
(θ) $(x^3D^2 - 3x^2D + x)y = \log x \sin \log x + 1$,
(ι) $(x^4D^4 + 6x^3D^3 + 4x^2D^2 - 2xD - 4)y = x^2 + 2\cos \log x$.

8. If L be self-induction, R resistance, C capacity, i current, q charge upon the plates of a condenser, and $f(t)$ the electromotive force, then the differential equations for the circuit are

$$(\alpha)\quad \frac{d^2q}{dt^2} + \frac{R}{L}\frac{dq}{dt} + \frac{q}{LC} = \frac{1}{L}f(t), \qquad (\beta)\quad \frac{d^2i}{dt^2} + \frac{R}{L}\frac{di}{dt} + \frac{i}{LC} = \frac{1}{L}f'(t).$$

Solve (α) when $f(t) = e^{-at} \sin bt$ and (β) when $f(t) = \sin bt$. Reduce the trigonometric part of the particular solution to the form $K \sin (bt + \gamma)$. Show that if R is small and b is nearly equal to $1/\sqrt{LC}$, the amplitude K is large.

98. Simultaneous linear equations with constant coefficients. If there be given two (or in general n) linear equations with constant coefficients in two (or in general n) dependent variables and one independent variable t, the symbolic method of solution may still be used to advantage. Let the equations be

$$\begin{aligned}(a_0D^n + a_1D^{n-1} + \cdots + a_n)\,x + (b_0D^m + b_1D^{m-1} + \cdots + b_m)\,y = R\,(t),\\ (c_0D^p + c_1D^{p-1} + \cdots + c_p)\,x + (d_0D^q + d_1D^{q-1} + \cdots + d_q)\,y = S\,(t),\end{aligned} \qquad (27)$$

when there are two variables and where D denotes differentiation by t. The equations may also be written more briefly as

$$P_1(D)\,x + Q_1(D)\,y = R \quad \text{and} \quad P_2(D)\,x + Q_2(D)\,y = S.$$

The ordinary algebraic process of solution for x and y may be employed because it depends only on such laws as are satisfied equally by the symbols D, $P_1(D)$, $Q_1(D)$, and so on.

Hence the solution for x and y is found by multiplying by the appropriate coefficients and adding the equations.

$$\begin{array}{r|r|l} Q_2(D) & -P_2(D) & P_1(D)\,x + Q_1(D)\,y = R,\\ -Q_1(D) & P_1(D) & P_2(D)\,x + Q_2(D)\,y = S.\end{array}$$

Then
$$\begin{aligned}[P_1(D)\,Q_2(D) - P_2(D)\,Q_1(D)]\,x = Q_2(D)\,R - Q_1(D)\,S,\\ [P_1(D)\,Q(D) - P_2(D)\,Q_1(D)]\,y = P_1(D)\,S - P_2(D)\,R.\end{aligned} \qquad (27')$$

It will be noticed that the coefficients by which the equations are multiplied (written on the left) are so chosen as to make the coefficients of x and y in the solved form the same in sign as in other respects. It may also be noted that the order of P and Q in the symbolic products is immaterial. By expanding the operator $P_1(D)\,Q_2(D) - P_2(D)\,Q_1(D)$ a certain polynomial in D is obtained and by applying the operators to R and S as indicated certain functions of t are obtained. Each equation, whether in x or in y, is quite of the form that has been treated in §§ **95–97.**

As an example consider the solution for x and y in the case of

$$2\frac{d^2x}{dt^2} - \frac{dy}{dt} - 4x = 2t, \qquad 2\frac{dx}{dt} + 4\frac{dy}{dt} - 3y = 0;$$

or
$$(2D^2 - 4)\,x - Dy = 2t, \qquad 2Dx + (4D - 3)\,y = 0.$$

Solve
$$\begin{array}{r|r|l} 4D - 3 & -2D & (2D^2 - 4)\,x - Dy = 2t\\ D & 2D^2 - 4 & 2Dx + (4D - 3)\,y = 0.\end{array}$$

Then
$$\begin{aligned}&[(4D - 3)(2D^2 - 4) + 2D^2]\,x = (4D - 3)\,2t,\\ &[2D^2 + (2D^2 - 4)(4D - 3)]\,y = -(2D)\,2t,\end{aligned}$$

or
$$4(2D^3 - D^2 - 4D + 3)\,x = 8 - 6t, \qquad 4(2D^3 - D^2 - 4D + 3)\,y = -4.$$

The roots of the polynomial in D are 1, 1, $-1\frac{1}{2}$; and the particular solution I_x for x is $-\frac{1}{2}t$, and I_y for y is $-\frac{1}{3}$. Hence the solutions have the form

$$x = (C_1 + C_2t)\,e^t + C_3e^{-\frac{3}{2}t} - \tfrac{1}{2}t, \qquad y = (K_1 + K_2t)\,e^t + K_3e^{-\frac{3}{2}t} - \tfrac{1}{3}.$$

The arbitrary constants which are introduced into the solutions for x and y are not independent nor are they identical. *The solutions must be substituted into one of the equations to establish the necessary relations between the constants.* It will be noticed that in general the order of the equation in D for x and for y is the sum of the orders of the highest derivatives which occur in the two equations, — in this case, $3 = 2 + 1$. The order may be diminished by cancellations which occur in the formal algebraic solutions for x and y. In fact it is conceivable that the coefficient $P_1Q_2 - P_2Q_1$ of x and y in the solved equations should vanish and the solution become illusory. This case is of so little consequence in practice that it may be dismissed with the statement that the solution is then either impossible or indeterminate; that is, either there are no functions x and y of t which satisfy the two given differential equations, or there are an infinite number in each of which other things than the constants of integration are arbitrary.

To finish the example above and determine one set of arbitrary constants in terms of the other, substitute in the second differential equation. Then

$$2\left(C_1e^t + C_2e^t + C_2te^t - \tfrac{3}{2}C_3e^{-\frac{3}{2}t} - \tfrac{1}{2}\right) + 4\left(K_1e^t + K_2e^t + K_2te^t - \tfrac{3}{2}K_3e^{-\frac{3}{2}t}\right)$$
$$- 3\left(K_1e^t + K_2te^t + K_3e^{-\frac{3}{2}t} - \tfrac{1}{3}\right) = 0,$$

or $$e^t(2C_1 + 2C_2 + K_1 + K_2) + te^t(2C_2 + K_2) - 3e^{-\frac{3}{2}t}(C_3 + 3K_3) = 0.$$

As the terms e^t, te^t, $e^{-\frac{3}{2}t}$ are independent, the linear relation between them can hold only if each of the coefficients vanishes. Hence

$$C_3 + 3K_3 = 0, \qquad 2C_2 + K_2 = 0, \qquad 2C_1 + 2C_2 + K_1 + K_2 = 0,$$

and $$C_3 = -3K_3, \qquad 2C_2 = -K_2, \qquad 2C_1 = -K_1.$$

Hence $x = (C_1 + C_2t)e^t - 3K_3e^{-\frac{3}{2}t} - \frac{1}{2}t, \qquad y = -2(C_1 + C_2t)e^t + K_3e^{-\frac{3}{2}t} - \frac{1}{3}$

are the finished solutions, where C_1, C_2, K_3 are three arbitrary constants of integration and might equally well be denoted by C_1, C_2, C_3, or K_1, K_2, K_3.

99. One of the most important applications of the theory of simultaneous equations with constant coefficients is to *the theory of small vibrations about a state of equilibrium in a conservative* dynamical system.* If $q_1, q_2, \cdots, q_n$ are n coördinates (see Exs. 19–20, p. 112) which specify the position of the system measured relatively

* The potential energy V is defined as $-dV = dW = Q_1dq_1 + Q_2dq_2 + \cdots + Q_ndq_n$, where

$$Q_i = X_1\frac{\partial x_1}{\partial q_i} + Y_1\frac{\partial y_1}{\partial q_i} + Z_1\frac{\partial z_1}{\partial q_i} + \cdots + X_n\frac{\partial x_n}{\partial q_i} + Y_n\frac{\partial y_n}{\partial q_i} + Z_n\frac{\partial z_n}{\partial q_i}.$$

This is the immediate extension of Q_1 as given in Ex. 19, p. 112. Here dW denotes the differential of work and $dW = \Sigma\mathbf{F}_i\cdot d\mathbf{r}_i = \Sigma(X_idx_i + Y_idy_i + Z_idz_i)$. To find Q_i it is generally quickest to compute dW from this relation with dx_i, dy_i, dz_i expressed in terms of the differentials $dq_1, \cdots, dq_n$. The generalized forces Q_i are then the coefficients of dq_i. If there is to be a potential V, the differential dW must be exact. It is frequently easy to find V directly in terms of $q_1, \cdots, q_n$ rather than through the mediation of $Q_1, \cdots, Q_n$; when this is not so, it is usually better to leave the equations in the form $\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_i} - \frac{\partial T}{\partial q_i} = Q_i$ rather than to introduce V and L.

to a position of stable equilibrium in which all the q's vanish, the development of the potential energy by Maclaurin's Formula gives

$$V(q_1, q_2, \cdots, q_n) = V_0 + V_1(q_1, q_2, \cdots, q_n) + V_2(q_1, q_2, \cdots, q_n) + \cdots,$$

where the first term is constant, the second is linear, and the third is quadratic, and where the supposition that the q's take on only small values, owing to the restriction to small vibrations, shows that each term is infinitesimal with respect to the preceding. Now the constant term may be neglected in any expression of potential energy. As the position when all the q's are 0 is assumed to be one of equilibrium, the forces

$$Q_1 = -\frac{\partial V}{\partial q_1}, \quad Q_2 = -\frac{\partial V}{\partial q_2}, \quad \cdots, \quad Q_n = -\frac{\partial V}{\partial q_n}$$

must all vanish when the q's are 0. This shows that the coefficients, $(\partial V/\partial q_i)_0 = 0$, of the linear expression are all zero. Hence the first term in the expansion is the quadratic term, and relative to it the higher terms may be disregarded. As the position of equilibrium is stable, the system will tend to return to the position where all the q's are 0 when it is slightly displaced from that position. It follows that the quadratic expression must be definitely positive.

The kinetic energy is always a quadratic function of the velocities $\dot q_1, \dot q_2, \cdots, \dot q_n$ with coefficients which may be functions of the q's. If each coefficient be expanded by the Maclaurin Formula and only the first or constant term be retained, the kinetic energy becomes a quadratic function with constant coefficients. Hence the Lagrangian function (cf. § 160)

$$L = T - V = T(\dot q_1, \dot q_2, \cdots, \dot q_n) - V(q_1, q_2, \cdots, q_n),$$

when substituted in the formulas for the motion of the system, gives

$$\frac{d}{dt}\frac{\partial L}{\partial \dot q_1} - \frac{\partial L}{\partial q_1} = 0, \quad \frac{d}{dt}\frac{\partial L}{\partial \dot q_2} - \frac{\partial L}{\partial q_2} = 0, \quad \cdots, \quad \frac{d}{dt}\frac{\partial L}{\partial \dot q_n} - \frac{\partial L}{\partial q_n} = 0,$$

a set of equations of the second order with constant coefficients. The equations moreover involve the operator D only through its square, and the roots of the equation in D must be either real or pure imaginary. The pure imaginary roots introduce trigonometric functions in the solution and represent vibrations. If there were real roots, which would have to occur in pairs, the positive root would represent a term of exponential form which would increase indefinitely with the time, — a result which is at variance both with the assumption of stable equilibrium and with the fact that the energy of the system is constant.

When there is friction in the system, the forces of friction are supposed to vary with the velocities for small vibrations. In this case there exists a dissipative function $F(\dot q_1, \dot q_2, \cdots, \dot q_n)$ which is quadratic in the velocities and may be assumed to have constant coefficients. The equations of motion of the system then become

$$\frac{d}{dt}\frac{\partial L}{\partial \dot q_1} - \frac{\partial L}{\partial q_1} + \frac{\partial F}{\partial \dot q_1} = 0, \quad \cdots, \quad \frac{d}{dt}\frac{\partial L}{\partial \dot q_n} - \frac{\partial L}{\partial q_n} + \frac{\partial F}{\partial \dot q_n} = 0,$$

which are still linear with constant coefficients but involve first powers of the operator D. It is physically obvious that the roots of the equation in D must be negative if real, and must have their real parts negative if the roots are complex; for otherwise the energy of the motion would increase indefinitely with the time, whereas it is known to be steadily dissipating its initial energy. It may be added that if, in addition to the internal forces arising from the potential V and the frictional forces arising from the dissipative function F, there are other forces impressed on the system, these forces would remain to be inserted upon the right-hand side of the equations of motion just given.

The fact that the equations for small vibrations lead to equations with constant coefficients by neglecting the higher powers of the variables gives the important physical theorem of the superposition of small vibrations. The theorem is: If with a certain set of initial conditions, a system executes a certain motion; and if with a different set of initial conditions taken at the same initial time, the system executes a second motion; then the system may execute the motion which consists of merely adding or superposing these motions at each instant of time; and in particular this combined motion will be that which the system would execute under initial conditions which are found by simply adding the corresponding values in the two sets of initial conditions. This theorem is of course a mere corollary of the linearity of the equations.

EXERCISES

1. Integrate the following systems of equations:

(α) $Dx - Dy + x = \cos t,\qquad D^2x - Dy + 3x - y = e^{2t},$

(β) $3Dx + 3x + 2y = e^t,\qquad 4x - 3Dy + 3y = 3t,$

(γ) $D^2x - 3x - 4y = 0,\qquad D^2y + x + y = 0,$

(δ) $\dfrac{dx}{y - 7x} = \dfrac{-dy}{2x + 5y} = dt,\qquad$ (ϵ) $-dt = \dfrac{dx}{3x + 4y} = \dfrac{dy}{2x + 5y},$

(ζ) $tDx + 2(x - y) = 1,\qquad tDy + x + 5y = t,$

(η) $Dx = ny - mz,\qquad Dy = lz - nx,\qquad Dz = mx - ly,$

(θ) $D^2x - 3x - 4y + 3 = 0,\qquad D^2y + x - 8y + 5 = 0,$

(ι) $D^4x - 4D^3y + 4D^2x - x = 0,\qquad D^4y - 4D^3x + 4D^2y - y = 0.$

2. A particle vibrates without friction upon the inner surface of an ellipsoid. Discuss the motion. Take the ellipsoid as

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{(z-c)^2}{c^2} = 1; \quad \text{then} \quad x = C\sin\left(\frac{\sqrt{cg}}{a}t + C_1\right), \quad y = K\sin\left(\frac{\sqrt{cg}}{b}t + K_1\right).$$

3. Same as Ex. 2 when friction varies with the velocity.

4. Two heavy particles of equal mass are attached to a light string, one at the middle, one at one end, and are suspended by attaching the other end of the string to a fixed point. If the particles are slightly displaced and the oscillations take place without friction in a vertical plane containing the fixed point, discuss the motion.

5. If there be given two electric circuits without capacity, the equations are

$$L_1\frac{di_1}{dt} + M\frac{di_2}{dt} + R_1i_1 = E_1, \qquad L_2\frac{di_2}{dt} + M\frac{di_1}{dt} + R_2i_2 = E_2,$$

where i_1, i_2 are the currents in the circuits, L_1, L_2 are the coefficients of self-induction, R_1, R_2 are the resistances, and M is the coefficient of mutual induction. (α) Integrate the equations when the impressed electromotive forces E_1, E_2 are zero in both circuits. (β) Also when $E_2 = 0$ but $E_1 = \sin pt$ is a periodic force. (γ) Discuss the cases of loose coupling, that is, where M^2/L_1L_2 is small; and the case of close coupling, that is, where M^2/L_1L_2 is nearly unity. What values for p are especially noteworthy when the damping is small?

6. If the two circuits of Ex. 5 have capacities C_1, C_2 and if q_1, q_2 are the charges on the condensers so that $i_1 = dq_1/dt$, $i_2 = dq_2/dt$ are the currents, the equations are

$$L_1\frac{d^2q_1}{dt^2} + M\frac{d^2q_2}{dt^2} + R_1\frac{dq_1}{dt} + \frac{q_1}{C_1} = E_1, \qquad L_2\frac{d^2q_2}{dt^2} + M\frac{d^2q_1}{dt^2} + R_2\frac{dq_2}{dt} + \frac{q_2}{C_2} = E_2.$$

Integrate when the resistances are negligible and $E_1 = E_2 = 0$. If $T_1 = 2\pi\sqrt{C_1L_1}$ and $T_2 = 2\pi\sqrt{C_2L_2}$ are the periods of the individual separate circuits and $\Theta = 2\pi M\sqrt{C_1C_2}$, and if $T_1 = T_2$, show that $\sqrt{T^2 + \Theta^2}$ and $\sqrt{T^2 - \Theta^2}$ are the independent periods in the coupled circuits.

7. A uniform beam of weight 6 lb. and length 2 ft. is placed orthogonally across a rough horizontal cylinder 1 ft. in diameter. To each end of the beam is suspended a weight of 1 lb. upon a string 1 ft. long. Solve the motion produced by giving one of the weights a slight horizontal velocity. Note that in finding the kinetic energy of the beam, the beam may be considered as rotating about its middle point (§ 39).