PART I. DIFFERENTIAL CALCULUS

CHAPTER III

TAYLOR'S FORMULA AND ALLIED TOPICS

31. Taylor's Formula. The object of Taylor's Formula is to express the value of a function $f(x)$ in terms of the values of the function and its derivatives at some one point $x = a$. Thus

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \cdots + \frac{(x-a)^{n-1}}{(n-1)!}f^{(n-1)}(a) + R. \qquad (1)$$

Such an expansion is necessarily true because the remainder R may be considered as defined by the equation; the real significance of the formula must therefore lie in the possibility of finding a simple expression for R, and there are several.

Theorem. On the hypothesis that $f(x)$ and its first n derivatives exist and are continuous over the interval $a \leqq x \leqq b$, the function may be expanded in that interval into a polynomial in $x - a$,

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \cdots + \frac{(x-a)^{n-1}}{(n-1)!}f^{(n-1)}(a) + R, \qquad (1)$$

with the remainder R expressible in any one of the forms

$$R = \frac{(x-a)^n}{n!}f^{(n)}(\xi) = \frac{h^n(1-\theta)^{n-1}}{(n-1)!}f^{(n)}(\xi) = \frac{1}{(n-1)!}\int_0^h t^{n-1}f^{(n)}(a+h-t)\,dt, \qquad (2)$$

where $h = x - a$ and $a < \xi < x$ or $\xi = a + \theta h$ where $0 < \theta < 1$.

A first proof may be made to depend on Rolle's Theorem as indicated in Ex. 8, p. 49. Let x be regarded for the moment as constant, say equal to b. Construct the function $\psi(x)$ there indicated. Note that $\psi(a) = \psi(b) = 0$ and that the derivative $\psi'(x)$ is merely

$$\psi'(x) = -\frac{(b-x)^{n-1}}{(n-1)!}f^{(n)}(x) + n\frac{(b-x)^{n-1}}{(b-a)^n}\Big[f(b) - f(a) - (b-a)f'(a) - \cdots - \frac{(b-a)^{n-1}}{(n-1)!}f^{(n-1)}(a)\Big].$$

By Rolle's Theorem $\psi'(\xi) = 0$. Hence if ξ be substituted above, the result is

$$f(b) = f(a) + (b-a)f'(a) + \cdots + \frac{(b-a)^{n-1}}{(n-1)!}f^{(n-1)}(a) + \frac{(b-a)^n}{n!}f^{(n)}(\xi),$$

after striking out the factor $-(b-\xi)^{n-1}$, multiplying by $(b-a)^n/n$, and transposing $f(b)$. The theorem is therefore proved with the first form of the remainder. *This proof does not require the continuity of the nth derivative nor its existence at a and at b.*

The second form of the remainder may be found by applying Rolle's Theorem to

$$\psi(x) = f(b) - f(x) - (b-x)f'(x) - \cdots - \frac{(b-x)^{n-1}}{(n-1)!}f^{(n-1)}(x) - (b-x)P,$$

where P is determined so that $R = (b-a)P$. Note that $\psi(b) = 0$ and that by Taylor's Formula $\psi(a) = 0$. Now

$$\psi'(x) = -\frac{(b-x)^{n-1}}{(n-1)!}f^{(n)}(x) + P \quad \text{or} \quad P = f^{(n)}(\xi)\frac{(b-\xi)^{n-1}}{(n-1)!} \quad \text{since} \quad \psi'(\xi) = 0.$$

Hence if ξ be written $\xi = a + \theta h$ where $h = b - a$, then $b - \xi = b - a - \theta h = (b-a)(1-\theta)$.

And $$R = (b-a)P = (b-a)\frac{(b-a)^{n-1}(1-\theta)^{n-1}}{(n-1)!}f^{(n)}(\xi) = \frac{(b-a)^n(1-\theta)^{n-1}}{(n-1)!}f^{(n)}(\xi).$$

The second form of R is thus found. In this work as before, the result is proved for $x = b$, the end point of the interval $a \leqq x \leqq b$. But as the interval could be considered as terminating at any of its points, the proof clearly applies to any x in the interval.

A second proof of Taylor's Formula, and the easiest to remember, consists in integrating the nth derivative n times from a to x. The successive results are

$$\int_a^x f^{(n)}(x)\,dx = f^{n-1}(x)\Big]_a^x = f^{(n-1)}(x) - f^{(n-1)}(a).$$

$$\int_a^x\int_a^x f^{(n)}(x)\,dx^2 = \int_a^x f^{(n-1)}(x)\,dx - \int_a^x f^{(n-1)}(a)\,dx$$
$$= f^{(n-2)}(x) - f^{(n-2)}(a) - (x-a)f^{(n-1)}(a).$$

$$\int_a^x\int_a^x\int_a^x f^{(n)}(x)\,dx^3 = f^{(n-3)}(x) - f^{(n-3)}(a) - (x-a)f^{(n-2)}(a) - \frac{(x-a)^2}{2!}f^{(n-1)}(a).$$

$$\int_a^x\cdots\int_a^x f^{(n)}(x)\,dx^n = f(x) - f(a) - (x-a)f'(a) - \frac{(x-a)^2}{2!}f''(a) - \cdots - \frac{(x-a)^{n-1}}{(n-1)!}f^{(n-1)}(a).$$

The formula is therefore proved with R in the form $\int_a^x\cdots\int_a^x f^{(n)}(x)\,dx^n$. To transform this to the ordinary form, the Law of the Mean may be applied ((65), § 16). For

$$m(x-a) < \int_a^x f^{(n)}(x)\,dx < M(x-a), \qquad m\frac{(x-a)^n}{n!} < \int_a^x\cdots\int_a^x f^{(n)}(x)\,dx^n < M\frac{(x-a)^n}{n!},$$

where m is the least and M the greatest value of $f^{(n)}(x)$ from a to x. There is then some intermediate value $f^{(n)}(\xi) = \mu$ such that

$$\int_a^x \cdots \int_a^x f^{(n)}(x)\,dx^n = \frac{(x-a)^n}{n!} f^{(n)}(\xi).$$

This proof requires that the nth derivative be continuous and is less general.

The third proof is obtained by applying successive integrations by parts to the obvious identity $f(a+h) - f(a) = \int_0^h f'(a+h-t)\,dt$ to make the integrand contain higher derivatives.

$$\begin{aligned} f(a+h) - f(a) &= \int_0^h f'(a+h-t)\,dt = tf'(a+h-t)\Big]_0^h + \int_0^h tf''(a+h-t)\,dt \\ &= hf'(a) + \tfrac{1}{2}t^2 f''(a+h-t)\Big]_0^h + \int_0^h \tfrac{1}{2}t^2 f'''(a+h-t)\,dt \\ &= hf'(a) + \frac{h^2}{2!}f''(a) + \cdots + \frac{h^{n-1}}{(n-1)!}f^{(n-1)}(a) + \int_0^h \frac{t^{n-1}}{(n-1)!} f^{(n)}(a+h-t)\,dt. \end{aligned}$$

This, however, is precisely Taylor's Formula with the third form of remainder.

If the point a about which the function is expanded is $x = 0$, the expansion will take the form known as Maclaurin's Formula:

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \cdots + \frac{x^{n-1}}{(n-1)!}f^{(n-1)}(0) + R, \quad (3)$$

$$R = \frac{x^n}{n!}f^{(n)}(\theta x) = \frac{x^n}{(n-1)!}(1-\theta)^{n-1}f^{(n)}(\theta x) = \frac{1}{(n-1)!}\int_0^x t^{n-1} f^{(n)}(x-t)\,dt.$$

32. Both Taylor's Formula and its special case, Maclaurin's, express a function as a polynomial in $h = x - a$, of which all the coefficients except the last are constants while the last is not constant but depends on h both explicitly and through the unknown fraction θ which itself is a function of h. If, however, the nth derivative is continuous, the coefficient $f^{(n)}(a + \theta h)/n!$ must remain finite, and if the form of the derivative is known, it may be possible actually to assign limits between which $f^{(n)}(a + \theta h)/n!$ lies. This is of great importance in making approximate calculations as in Exs. 8 ff. below; for it sets a limit to the value of R for any value of n.

Theorem. There is only one possible expansion of a function into a polynomial in $h = x - a$ of which all the coefficients except the last are constant and the last finite; and hence if such an expansion is found in any manner, it must be Taylor's (or Maclaurin's).

To prove this theorem consider two polynomials of the nth order

$$c_0 + c_1h + c_2h^2 + \cdots + c_{n-1}h^{n-1} + c_nh^n = C_0 + C_1h + C_2h^2 + \cdots + C_{n-1}h^{n-1} + C_nh^n,$$

which represent the same function and hence are equal for all values of h from 0 to $b - a$. It follows that the coefficients must be equal. For let h approach 0.

The terms containing h will approach 0 and hence c_0 and C_0 may be made as nearly equal as desired; and as they are constants, they must be equal. Strike them out from the equation and divide by h. The new equation must hold for all values of h from 0 to $b - a$ with the possible exception of 0. Again let $h \doteq 0$ and now it follows that $c_1 = C_1$. And so on, with all the coefficients. The two developments are seen to be identical, and hence identical with Taylor's.

To illustrate the application of the theorem, let it be required to find the expansion of $\tan x$ about 0 when the expansions of $\sin x$ and $\cos x$ about 0 are given.

$$\sin x = x - \tfrac{1}{6}x^3 + \tfrac{1}{120}x^5 + Px^7, \qquad \cos x = 1 - \tfrac{1}{2}x^2 + \tfrac{1}{24}x^4 + Qx^6,$$

where P and Q remain finite in the neighborhood of $x = 0$. In the first place note that $\tan x$ clearly has an expansion; for the function and its derivatives (which are combinations of $\tan x$ and $\sec x$) are finite and continuous until x approaches $\frac{1}{2}\pi$. By division,

$$\begin{array}{r|l}
 & x + \tfrac{1}{3}x^3 + \tfrac{2}{15}x^5 \\
1 - \tfrac{1}{2}x^2 + \tfrac{1}{24}x^4 + Qx^6 & x - \tfrac{1}{6}x^3 + \tfrac{1}{120}x^5 \;\vdots + Px^7 \\
 & x - \tfrac{1}{2}x^3 + \tfrac{1}{24}x^5 \;\vdots + Qx^7 \\
 & \tfrac{1}{3}x^3 - \tfrac{1}{30}x^5 \;\vdots + (P - Q)x^7 \\
 & \tfrac{1}{3}x^3 - \tfrac{1}{6}x^5 \;\vdots + \tfrac{1}{72}x^7 + \tfrac{1}{3}Qx^9 \\
 & \tfrac{2}{15}x^5 \;\vdots \cdots\cdots\cdots
\end{array}$$

Hence $\tan x = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \dfrac{S'}{\cos x}x^7$, where S is the remainder in the division and is an expression containing P, Q, and powers of x; it must remain finite if P and Q remain finite. The quotient $S/\cos x$ which is the coefficient of x^7 therefore remains finite near $x = 0$, and the expression for $\tan x$ is the Maclaurin expansion up to terms of the sixth order, plus a remainder.

In the case of functions compounded from simple functions of which the expansion is known, this method of obtaining the expansion by algebraic processes upon the known expansions treated as polynomials is generally shorter than to obtain the result by differentiation. The computation may be abridged by omitting the last terms and work such as follows the dotted line in the example above; but if this is done, care must be exercised against carrying the algebraic operations too far or not far enough. In Ex. 5 below, the last terms should be put in and carried far enough to insure that the desired expansion has neither more nor fewer terms than the circumstances warrant.

EXERCISES

1. Assume $R = (b - a)^k P$; show $R = \dfrac{h^n(1-\theta)^{n-k}}{(n-1)!\,k} f^{(n)}(\xi)$.

2. Apply Ex. 5, p. 29, to compare the third form of remainder with the first.

3. Obtain, by differentiation and substitution in (1), three nonvanishing terms:

(α) $\sin^{-1}x$, $a = 0$, (β) $\tanh x$, $a = 0$, (γ) $\tan x$, $a = \frac{1}{4}\pi$,

(δ) $\csc x$, $a = \frac{1}{3}\pi$, (ϵ) $e^{\sin x}$, $a = 0$, (ζ) $\log \sin x$, $a = \frac{1}{2}\pi$.

4. Find the nth derivatives in the following cases and write the expansion:

(α) $\sin x$, $a = 0$, (β) $\sin x$, $a = \frac{1}{2}\pi$, (γ) c^x, $a = 0$,

(δ) c^x, $a = 1$, (ϵ) $\log x$, $a = 1$, (ζ) $(1 + x)^k$, $a = 0$.

5. By algebraic processes find the Maclaurin expansion to the term in x^5:

(α) $\sec x$, (β) $\tanh x$, (γ) $-\sqrt{1-x^2}$,
(δ) $e^x \sin x$, (ϵ) $[\log(1-x)]^2$, (ζ) $+\sqrt{\cosh x}$,
(η) $e^{\sin x}$, (θ) $\log\cos x$, (ι) $\log\sqrt{1+x^2}$.

The expansions needed in this work may be found by differentiation or taken from B. O. Peirce's "Tables." In (γ) and (ζ) apply the binomial theorem of Ex. 4 (ζ). In (η) let $y = \sin x$, expand e^y, and substitute for y the expansion of $\sin x$. In (θ) let $\cos x = 1 - y$. In all cases show that the coefficient of the term in x^6 really remains finite when $x \doteq 0$.

6. If $f(a+h) = c_0 + c_1 h + c_2 h^2 + \cdots + c_{n-1}h^{n-1} + c_n h^n$, show that in

$$\int_0^h f(a+h)\,dh = c_0 h + \frac{c_1}{2}h^2 + \frac{c_2}{3}h^3 + \cdots + \frac{c_{n-1}}{n}h^n + \int_0^h c_n h^n dh$$

the last term may really be put in the form Ph^{n+1} with P finite. Apply Ex. 5, p. 29.

7. Apply Ex. 6 to $\sin^{-1}x = \int_0^x \frac{dx}{\sqrt{1-x^2}}$, etc., to find developments of

(α) $\sin^{-1}x$, (β) $\tan^{-1}x$, (γ) $\sinh^{-1}x$,
(δ) $\log\frac{1+x}{1-x}$, (ϵ) $\int_0^x e^{-x^2}dx$, (ζ) $\int_0^x \frac{\sin x}{x}dx$.

In all these cases the results may be found if desired to n terms.

8. Show that the remainder in the Maclaurin development of e^x is less than $x^n e^x/n\,!$; and hence that the error introduced by disregarding the remainder in computing e^x is less than $x^n e^x/n\,!$. How many terms will suffice to compute e to four decimals? How many for e^5 and for $e^{0.1}$?

9. Show that the error introduced by disregarding the remainder in computing $\log(1+x)$ is not greater than x^n/n if $x > 0$. How many terms are required for the computation of $\log 1\frac{1}{2}$ to four places? of $\log 1.2$? Compute the latter.

10. The hypotenuse of a triangle is 20 and one angle is 31°. Find the sides by expanding $\sin x$ and $\cos x$ about $a = \frac{1}{6}\pi$ as linear functions of $x - \frac{1}{6}\pi$. Examine the term in $(x - \frac{1}{6}\pi)^2$ to find a maximum value to the error introduced by neglecting it.

11. Compute to 6 places: (α) $e^{\frac{1}{5}}$, (β) $\log 1.1$, (γ) $\sin 30'$, (δ) $\cos 30'$. During the computation one place more than the desired number should be carried along in the arithmetic work for safety.

12. Show that the remainder for $\log(1+x)$ is less than $x^n/n(1+x)^n$ if $x < 0$. Compute (α) $\log 0.9$ to 5 places, (β) $\log 0.8$ to 4 places.

13. Show that the remainder for $\tan^{-1}x$ is less than x^n/n where n may always be taken as odd. Compute to 4 places $\tan^{-1}\frac{1}{5}$.

14. The relation $\frac{1}{4}\pi = \tan^{-1}1 = 4\tan^{-1}\frac{1}{5} - \tan^{-1}\frac{1}{239}$ enables $\frac{1}{4}\pi$ to be found easily from the series for $\tan^{-1}x$. Find $\frac{1}{4}\pi$ to 7 places (intermediate work carried to 8 places).

15. *Computation of logarithms.* (α) If $a = \log\frac{10}{9}$, $b = \log\frac{25}{24}$, $c = \log\frac{81}{80}$, then

$$\log 2 = 7a - 2b + 3c, \qquad \log 3 = 11a - 3b + 5c, \qquad \log 5 = 16a - 4b + 7c.$$

Now $a = -\log(1-\frac{1}{10})$, $b = -\log(1-\frac{4}{100})$, $c = \log(1+\frac{1}{80})$ are readily computed and hence log 2, log 3, log 5 may be found. Carry the calculations of a, b, c to 10 places and deduce the logarithms of 2, 3, 5, 10, retaining only 8 places. Compare Peirce's "Tables," p. 109.

(β) Show that the error in the series for $\log\dfrac{1+x}{1-x}$ is less than $\dfrac{2x^n}{n(1-x)^n}$. Compute log 2 corresponding to $x = \frac{1}{3}$ to 4 places, $\log 1\frac{2}{3}$ to 5 places, $\log 1\frac{2}{5}$ to 6 places.

(γ) Show $\log\dfrac{p}{q} = 2\left[\dfrac{p-q}{p+q} + \dfrac{1}{3}\left(\dfrac{p-q}{p+q}\right)^3 + \cdots + \dfrac{1}{2n-1}\left(\dfrac{p-q}{p+q}\right)^{2n-1} + R_{2n+1}\right]$, give an estimate of R_{2n+1}, and compute to 10 figures log 3 and log 7 from log 2 and log 5 of Peirce's "Tables" and from

$$4\log 3 - 4\log 2 - \log 5 = \log\frac{81}{80}, \qquad 4\log 7 - 5\log 2 - \log 3 - 2\log 5 = \log\frac{7^4}{7^4-1}.$$

16. Compute Ex. 7 (ϵ) to 4 places for $x = 1$ and to 6 places for $x = \frac{1}{2}$.

17. Compute $\sin^{-1} 0.1$ to seconds and $\sin^{-1}\frac{1}{3}$ to minutes.

18. Show that in the expansion of $(1+x)^k$ the remainder, as x is $>$ or < 0, is

$$R_n < \left|\frac{k\cdot(k-1)\cdots(k-n)}{1\cdot 2\cdots n}x^n\right| \quad \text{or} \quad R_n < \left|\frac{k\cdot(k-1)\cdots(k-n)}{1\cdot 2\cdots n}\frac{x^n}{(1+x)^{n-k}}\right|, \quad n > k.$$

Hence compute to 5 figures $\sqrt{103}$, $\sqrt{98}$, $\sqrt[3]{28}$, $\sqrt[5]{250}$, $\sqrt[10]{1000}$.

19. Sometimes the remainder cannot be readily found but the terms of the expansion appear to be diminishing so rapidly that all after a certain point appear negligible. Thus use Peirce's "Tables," Nos. 774–789, to compute to four places (estimated) the values of tan 6°, log cos 10°, csc 3°, sec 2°.

20. Find to within 1% the area under $\cos(x^2)$ and $\sin(x^2)$ from 0 to $\frac{1}{2}\pi$.

21. A unit magnetic pole is placed at a distance L from the center of a magnet of pole strength M and length $2l$, where l/L is small. Find the force on the pole if (α) the pole is in the line of the magnet and if (β) it is in the perpendicular bisector.

Ans. (α) $\dfrac{4Ml}{L^3}(1+\epsilon)$ with ϵ about $2\left(\dfrac{l}{L}\right)^2$, ($\beta$) $\dfrac{2Ml}{L^3}(1-\epsilon)$ with ϵ about $\dfrac{3}{2}\left(\dfrac{l}{L}\right)^2$.

22. The formula for the distance of the horizon is $D = \sqrt{\frac{3}{2}h}$ where D is the distance in miles and h is the altitude of the observer in feet. Prove the formula and show that the error is about $\frac{1}{2}$% for heights up to a few miles. Take the radius of the earth as 3960 miles.

23. Find an approximate formula for the dip of the horizon in minutes below the horizontal if h in feet is the height of the observer.

24. If S is a circular arc and C its chord and c the chord of half the arc, prove $S = \frac{1}{3}(8c - C)(1+\epsilon)$ where ϵ is about $S^4/7680R^4$ if R is the radius.

25. If two quantities differ from each other by a small fraction ϵ of their value, show that their geometric mean will differ from their arithmetic mean by about $\frac{1}{8}\epsilon^2$ of its value.

26. The algebraic method may be applied to finding expansions of some functions which become infinite. (Thus if the series for $\cos x$ and $\sin x$ be divided to find $\cot x$, the initial term is $1/x$ and becomes infinite at $x = 0$ just as $\cot x$ does.

Such expansions are not Maclaurin developments but are analogous to them. The function $x \cot x$ would, however, have a Maclaurin development and the expansion found for $\cot x$ is this development divided by x.) Find the developments about $x = 0$ to terms in x^4 for

(α) $\cot x$, (β) $\cot^2 x$, (γ) $\csc x$, (δ) $\csc^3 x$,
(ϵ) $\cot x \csc x$, (ζ) $1/(\tan^{-1} x)^2$, (η) $(\sin x - \tan x)^{-1}$

27. Obtain the expansions:

(α) $\log \sin x = \log x - \frac{1}{6}x^2 - \frac{1}{180}x^4 + R$, ($\beta$) $\log \tan x = \log x + \frac{1}{3}x^2 + \frac{7}{90}x^4 + \cdots$,
(γ) likewise for log vers x.

33. Indeterminate forms, infinitesimals, infinites. If two functions $f(x)$ and $\phi(x)$ are defined for $x = a$ and if $\phi(a) \neq 0$, the quotient f/ϕ is defined for $x = a$. But if $\phi(a) = 0$, the quotient f/ϕ is not defined for a. If in this case f and ϕ are defined and continuous in the neighborhood of a and $f(a) \neq 0$, the quotient will become infinite as $x \doteq a$; whereas if $f(a) = 0$, the behavior of the quotient f/ϕ is not immediately apparent but gives rise to the indeterminate form $0/0$. In like manner if f and ϕ become infinite at a, the quotient f/ϕ is not defined, as neither its numerator nor its denominator is defined; thus arises the indeterminate form ∞/∞. The question of determining or evaluating an indeterminate form is merely the question of finding out whether the quotient f/ϕ approaches a limit (and if so, what limit) or becomes positively or negatively infinite when x approaches a.

THEOREM. *L'Hospital's Rule.* If the functions $f(x)$ and $\phi(x)$, which give rise to the indeterminate form $0/0$ or ∞/∞ when $x \doteq a$, are continuous and differentiable in the interval $a < x \leqq b$ and if b can be taken so near to a that $\phi'(x)$ does not vanish in the interval and if the quotient f'/ϕ' of the derivatives approaches a limit or becomes positively or negatively infinite as $x \doteq a$, then the quotient f/ϕ will approach that limit or become positively or negatively infinite as the case may be. Hence *an indeterminate form* $0/0$ or ∞/∞ *may be replaced by the quotient of the derivatives of numerator and denominator.*

CASE I. $f(a) = \phi(a) = 0$. The proof follows from Cauchy's Formula, Ex. 6, p. 49.

For
$$\frac{f(x)}{\phi(x)} = \frac{f(x) - f(a)}{\phi(x) - \phi(a)} = \frac{f'(\xi)}{\phi'(\xi)}, \qquad a < \xi < x.$$

Now if $x \doteq a$, so must ξ, which lies between x and a. Hence if the quotient on the right approaches a limit or becomes positively or negatively infinite, the same is true of that on the left. The necessity of inserting the restrictions that f and ϕ shall be continuous and differentiable and that ϕ' shall not have a root indefinitely near to a is apparent from the fact that Cauchy's Formula is proved only for functions that satisfy these conditions. If the derived form f'/ϕ' should also be indeterminate, the rule could again be applied and the quotient f''/ϕ'' would replace f'/ϕ' with the understanding that proper restrictions were satisfied by f', ϕ', and ϕ''.

CASE II. $f(a) = \phi(a) = \infty$. Apply Cauchy's Formula as follows:

$$\frac{f(x)-f(b)}{\phi(x)-\phi(b)} = \frac{f(x)}{\phi(x)}\frac{1-f(b)/f(x)}{1-\phi(b)/\phi(x)} = \frac{f'(\xi)}{\phi'(\xi)}, \qquad \begin{matrix} a<x<b, \\ x<\xi<b, \end{matrix}$$

where the middle expression is merely a different way of writing the first. Now suppose that $f'(x)/\phi'(x)$ approaches a limit when $x \doteq a$. It must then be possible to take b so near to a that $f'(\xi)/\phi'(\xi)$ differs from that limit by as little as desired, no matter what value ξ may have between a and b. Now as f and ϕ become infinite when $x \doteq a$, it is possible to take x so near to a that $f(b)/f(x)$ and $\phi(b)/\phi(x)$ are as near zero as desired. The second equation above then shows that $f(x)/\phi(x)$, multiplied by a quantity which differs from 1 by as little as desired, is equal to a quantity $f'(\xi)/\phi'(\xi)$ which differs from the limit of $f'(x)/\phi'(x)$ as $x \doteq a$ by as little as desired. Hence f/ϕ must approach the same limit as f'/ϕ'. Similar reasoning would apply to the supposition that f'/ϕ' became positively or negatively infinite, and the theorem is proved. It may be noted that, by Theorem 16 of § 27, the form f'/ϕ' is sure to be indeterminate. The advantage of being able to differentiate therefore lies wholly in the possibility that the new form be more amenable to algebraic transformation than the old.

The other indeterminate forms $0\cdot\infty$, 0^0, 1^∞, ∞^0, $\infty-\infty$ may be reduced to the foregoing by various devices which may be indicated as follows:

$$0\cdot\infty = \frac{0}{\frac{1}{\infty}} = \frac{\infty}{\frac{1}{0}}, \quad 0^0 = e^{\log 0^0} = e^{0\log 0} = e^{0\cdot\infty}, \quad \cdots, \quad \infty-\infty = \log e^{\infty-\infty} = \log\frac{e^\infty}{e^\infty}.$$

The case where the variable becomes infinite instead of approaching a finite value a is covered in Ex. 1 below. The theory is therefore completed.

Two methods which frequently may be used to shorten the work of evaluating an indeterminate form are *the method of E-functions* and *the application of Taylor's Formula*. By definition *an E-function for the point $x = a$ is any continuous function which approaches a finite limit other than 0 when $x \doteq a$.* Suppose then that $f(x)$ or $\phi(x)$ or both may be written as the products $E_1 f_1$ and $E_2\phi_1$. Then the method of treating indeterminate forms need be applied only to f_1/ϕ_1 and the result multiplied by $\lim E_1/E_2$. For example,

$$\lim_{x\doteq a}\frac{x^3-a^3}{\sin(x-a)} = \lim_{x\doteq a}(x^2+ax+a^2)\lim_{x\doteq a}\frac{x-a}{\sin(x-a)} = 3a^2\lim_{x\doteq a}\frac{x-a}{\sin(x-a)} = 3a^2.$$

Again, suppose that in the form 0/0 both numerator and denominator may be developed about $x = a$ by Taylor's Formula. The evaluation is immediate. Thus

$$\frac{\tan x - \sin x}{x^2\log(1+x)} = \frac{(x+\frac{1}{3}x^3+Px^5)-(x-\frac{1}{6}x^3+Qx^5)}{x^2(x-\frac{1}{2}x^2+Rx^3)} = \frac{\frac{1}{2}+(P-Q)x^2}{1-\frac{1}{2}x+Rx^2};$$

and now if $x \doteq 0$, the limit is at once shown to be simply $\frac{1}{2}$.

When the functions become infinite at $x = a$, the conditions requisite for Taylor's Formula are not present and there is no Taylor expansion. Nevertheless an expansion may sometimes be obtained by the algebraic method (§ 32) and may frequently be used to advantage. To illustrate, let it be required to evaluate $\cot x - 1/x$ which is of the form $\infty-\infty$ when $x \doteq 0$. Here

$$\cot x = \frac{\cos x}{\sin x} = \frac{1+\frac{1}{2}x^2+Px^4}{x-\frac{1}{6}x^3+Qx^5} = \frac{1}{x}\frac{1-\frac{1}{2}x^2+Px^4}{1-\frac{1}{6}x^2+Qx^4} = \frac{1}{x}\left(1-\frac{1}{3}x^2+Sx^4\right),$$

where S remains finite when $x \doteq 0$. If this value be substituted for $\cot x$, then

$$\lim_{x \doteq 0}\left(\cot x - \frac{1}{x}\right) = \lim_{x \doteq 0}\left(\frac{1}{x} - \frac{1}{3}x + Sx^3 - \frac{1}{x}\right) = \lim_{x \doteq 0}\left(-\frac{1}{3}x + Sx^3\right) = 0.$$

34. *An infinitesimal is a variable which is ultimately to approach the limit zero ; an infinite is a variable which is to become either positively or negatively infinite.* Thus the increments Δy and Δx are finite quantities, but when they are to serve in the definition of a derivative they must ultimately approach zero and hence may be called infinitesimals. The form $0/0$ represents the quotient of two infinitesimals; * the form ∞/∞, the quotient of two infinites; and $0 \cdot \infty$, the product of an infinitesimal by an infinite. If any infinitesimal α is chosen as the *primary infinitesimal,* a second infinitesimal β is said to be *of the same order* as α if the limit of the quotient β/α exists and is not zero when $\alpha \doteq 0$; whereas if the quotient β/α becomes zero, β is said to be an infinitesimal *of higher order* than α, but *of lower order* if the quotient becomes infinite. If in particular the limit β/α^n exists and is not zero when $\alpha \doteq 0$, then β is said to be *of the nth order relative to* α. The determination of the order of one infinitesimal relative to another is therefore essentially a problem in indeterminate forms. Similar definitions may be given in regard to infinites.

THEOREM. If the quotient β/α of two infinitesimals approaches a limit or becomes infinite when $\alpha \doteq 0$, the quotient β'/α' of two infinitesimals which differ respectively from β and α by infinitesimals of higher order will approach the same limit or become infinite.

THEOREM. *Duhamel's Theorem.* If the sum $\Sigma\alpha_i = \alpha_1 + \alpha_2 + \cdots + \alpha_n$ of n positive infinitesimals approaches a limit when their number n becomes infinite, the sum $\Sigma\beta_i = \beta_1 + \beta_2 + \cdots + \beta_n$, where each β_i differs uniformly from the corresponding α_i by an infinitesimal of higher order, will approach the same limit.

As $\alpha' - \alpha$ is of higher order than α and $\beta' - \beta$ of higher order than β,

$$\lim \frac{\alpha' - \alpha}{\alpha} = 0, \quad \lim \frac{\beta' - \beta}{\beta} = 0 \quad \text{or} \quad \frac{\alpha'}{\alpha} = 1 + \eta, \quad \frac{\beta'}{\beta} = 1 + \zeta,$$

where η and ζ are infinitesimals. Now $\alpha' = \alpha(1 + \eta)$ and $\beta' = \beta(1 + \zeta)$. Hence

$$\frac{\beta'}{\alpha'} = \frac{\beta}{\alpha}\frac{1 + \zeta}{1 + \eta} \quad \text{and} \quad \lim \frac{\beta'}{\alpha'} = \frac{\beta}{\alpha},$$

provided β/α approaches a limit; whereas if β/α becomes infinite, so will β'/α'. In a more complex fraction such as $(\beta - \gamma)/\alpha$ it is *not* permissible to replace β

* It cannot be emphasized too strongly that in the symbol 0/0 the 0's are merely symbolic for a mode of variation just as ∞ is; they are not actual 0's and some other notation would be far preferable, likewise for $0 \cdot \infty$, 0^0, etc.

and γ individually by infinitesimals of higher order; for $\beta - \gamma$ may itself be of higher order than β or γ. Thus $\tan x - \sin x$ is an infinitesimal of the third order relative to x although $\tan x$ and $\sin x$ are only of the first order. To replace $\tan x$ and $\sin x$ by infinitesimals which differ from them by those of the second order or even of the third order would generally alter the limit of the ratio of $\tan x - \sin x$ to x^3 when $x \doteq 0$.

To prove Duhamel's Theorem the β's may be written in the form

$$\beta_i = \alpha_i(1 + \eta_i), \qquad i = 1, 2, \cdots, n, \qquad |\eta_i| < \epsilon,$$

where the η's are infinitesimals and where all the η's simultaneously may be made less than the assigned ϵ owing to the uniformity required in the theorem. Then

$$|(\beta_1 + \beta_2 + \cdots + \beta_n) - (\alpha_1 + \alpha_2 + \cdots + \alpha_n)| = |\eta_1\alpha_1 + \eta_2\alpha_2 + \cdots + \eta_n\alpha_n| < \epsilon\Sigma\alpha.$$

Hence the sum of the β's may be made to differ from the sum of the α's by less than $\epsilon\Sigma\alpha$, a quantity as small as desired, and as $\Sigma\alpha$ approaches a limit by hypothesis, so $\Sigma\beta$ must approach the same limit. The theorem may clearly be extended to the case where the α's are not all positive provided the sum $\Sigma|\alpha_i|$ of the absolute values of the α's approaches a limit.

35. If $y = f(x)$, the *differential* of y is defined as

$$dy = f'(x)\Delta x, \qquad \text{and hence} \qquad dx = 1 \cdot \Delta x. \tag{4}$$

From this definition of dy and dx it appears that $dy/dx = f'(x)$, where the quotient dy/dx is the quotient of two finite quantities of which dx may be assigned at pleasure. This is true if x is the independent variable. If x and y are both expressed in terms of t,

$$x = x(t), \qquad y = y(t), \qquad dx = D_t x\, dt, \qquad dy = D_t y\, dt;$$

and
$$\frac{dy}{dx} = \frac{D_t y}{D_t x} = D_x y, \qquad \text{by virtue of (4), § 2.}$$

From this appears the important theorem: *The quotient dy/dx is the derivative of y with respect to x no matter what the independent variable may be.* It is this theorem which really justifies writing the derivative as a fraction and treating the component differentials according to the rules of ordinary fractions. For higher derivatives this is not so, as may be seen by reference to Ex. 10.

As Δy and Δx are regarded as infinitesimals in defining the derivative, it is natural to regard dy and dx as infinitesimals. The difference $\Delta y - dy$ may be put in the form

$$\Delta y - dy = \left[\frac{f(x + \Delta x) - f(x)}{\Delta x} - f'(x)\right]\Delta x, \tag{5}$$

wherein it appears that, when $\Delta x \doteq 0$, the bracket approaches zero. Hence arises the theorem: *If x is the independent variable and if Δy and dy are regarded as infinitesimals, the difference $\Delta y - dy$ is an infinitesimal of higher order than Δx.* This has an application to the subject of change of variable in a definite integral. For if $x = \phi(t)$, then $dx = \phi'(t)\,dt$, and apparently

$$\int_a^b f(x)\,dx = \int_{t_1}^{t_2} f[\phi(t)]\,\phi'(t)\,dt,$$

where $\phi(t_1) = a$ and $\phi(t_2) = b$, so that t ranges from t_1 to t_2 when x ranges from a to b.

But this substitution is too hasty; for the dx written in the integrand is really Δx, which differs from dx by an infinitesimal of higher order when x is not the independent variable. The true condition may be seen by comparing the two sums

$$\sum f(x_i)\,\Delta x_i, \qquad \sum f[\phi(t_i)]\,\phi'(t_i)\,\Delta t_i, \qquad \Delta t = dt,$$

the limits of which are the two integrals above. Now as Δx differs from $dx = \phi'(t)\,dt$ by an infinitesimal of higher order, so $f(x)\,\Delta x$ will differ from $f[\phi(t)]\,\phi'(t)\,dt$ by an infinitesimal of higher order, and with the proper assumptions as to continuity the difference will be uniform. Hence if the infinitesimals $f(x)\,\Delta x$ be all positive, Duhamel's Theorem may be applied to justify the formula for change of variable. To avoid the restriction to positive infinitesimals it is well to replace Duhamel's Theorem by the new

THEOREM. *Osgood's Theorem.* Let $\alpha_1, \alpha_2, \cdots, \alpha_n$ be n infinitesimals and let α_i differ uniformly by infinitesimals of higher order than Δx from the elements $f(x_i)\,\Delta x_i$ of the integrand of a definite integral $\int_a^b f(x)\,dx$, where f is continuous; then the sum $\Sigma\alpha = \alpha_1 + \alpha_2 + \cdots + \alpha_n$ approaches the value of the definite integral as a limit when the number n becomes infinite.

Let $\alpha_i = f(x_i)\,\Delta x_i + \zeta_i \Delta x_i$, where $|\zeta_i| < \epsilon$ owing to the uniformity demanded.

Then $$\left|\sum \alpha_i - \sum f(x_i)\,\Delta x_i\right| = \left|\sum \zeta_i \Delta x_i\right| < \epsilon \sum \Delta x_i = \epsilon(b-a).$$

But as f is continuous, the definite integral exists and one can make

$$\left|\sum f(x_i)\,\Delta x_i - \int_a^b f(x)\,dx\right| < \epsilon, \quad \text{and hence} \quad \left|\sum \alpha_i - \int_a^b f(x)\,dx\right| < \epsilon(b - a + 1).$$

It therefore appears that $\Sigma\alpha_i$ may be made to differ from the integral by as little as desired, and $\Sigma\alpha_i$ must then approach the integral as a limit. Now if this theorem be applied to the case of the change of variable and if it be assumed that $f[\phi(t)]$ and $\phi'(t)$ are continuous, the infinitesimals Δx_i and $dx_i = \phi'(t_i)\,dt_i$ will differ uniformly (compare Theorem 18 of § 27 and the above theorem on $\Delta y - dy$) by an infinitesimal of higher order, and so will the infinitesimals $f(x_i)\,\Delta x_i$ and $f[\phi(t_i)]\,\phi'(t_i)\,dt_i$. Hence the change of variable suggested by the hasty substitution is justified.

EXERCISES

1. Show that l'Hospital's Rule applies to evaluating the indeterminate form $f(x)/\phi(x)$ when x becomes infinite and both f and ϕ either become zero or infinite.

2. Evaluate the following forms by differentiation. Examine the quotients for left-hand and for right-hand approach; sketch the graphs in the neighborhood of the points.

$(\alpha)\ \lim\limits_{x \doteq 0} \frac{a^x - b^x}{x}$, $(\beta)\ \lim\limits_{x \doteq \frac{1}{4}\pi} \frac{\tan x - 1}{x - \frac{1}{4}\pi}$, $(\gamma)\ \lim\limits_{x \doteq 0} x \log x$,

$(\delta)\ \lim\limits_{x = \infty} xe^{-x}$, $(\epsilon)\ \lim\limits_{x \doteq 0} (\cot x)^{\sin x}$, $(\zeta)\ \lim\limits_{x \doteq 1} x^{\frac{1}{1-x}}$.

3. Evaluate the following forms by the method of expansions:

$(\alpha)\ \lim\limits_{x \doteq 0} \left(\frac{1}{x^2} - \cot^2 x\right)$, $(\beta)\ \lim\limits_{x \doteq 0} \frac{e^x - e^{\tan x}}{x - \tan x}$, $(\gamma)\ \lim\limits_{x \doteq 1} \frac{\log x}{1 - x}$,

$(\delta)\ \lim\limits_{x \doteq 0} (\operatorname{csch} x - \csc x)$, $(\epsilon)\ \lim\limits_{x \doteq 0} \frac{x \sin(\sin x) - \sin^2 x}{x^6}$, $(\zeta)\ \lim\limits_{x \doteq 0} \frac{e^x - e^{-x} - 2x}{x - \sin x}$.

4. Evaluate by any method:

$(\alpha)\ \lim\limits_{x \doteq 0} \frac{e^x - e^{-x} + 2 \sin x - 4x}{x^5}$, $(\beta)\ \lim\limits_{x \doteq 0} \left(\frac{\tan x}{x}\right)^{\frac{1}{x^2}}$,

$(\gamma)\ \lim\limits_{x \doteq 0} \frac{x \cos^3 x - \log(1 + x) - \sin^{-1} \frac{1}{2} x^2}{x^3}$, $(\delta)\ \lim\limits_{x \doteq \frac{1}{2}\pi} \frac{\log(x - \frac{1}{2}\pi)}{\tan x}$,

$(\epsilon)\ \lim\limits_{x = \infty} \left[x\left(1 + \frac{1}{x}\right)^x - ex^2 \log\left(1 + \frac{1}{x}\right)\right]$.

5. Give definitions for order as applied to infinites, noting that higher order would mean becoming infinite to a greater degree just as it means becoming zero to a greater degree for infinitesimals. State and prove the theorem relative to quotients of infinites analogous to that given in the text for infinitesimals. State and prove an analogous theorem for the product of an infinitesimal and infinite.

6. Note that if the quotient of two infinites has the limit 1, the difference of the infinites is an infinite of lower order. Apply this to the proof of the resolution in partial fractions of the quotient $f(x)/F(x)$ of two polynomials in case the roots of the denominator are all real. For if $F(x) = (x - a)^k F_1(x)$, the quotient is an infinite of order k in the neighborhood of $x = a$; but the difference of the quotient and $f(a)/(x - a)^k F_1(a)$ will be of lower integral order — and so on.

7. Show that when $x = +\infty$, the function e^x is an infinite of higher order than x^n no matter how large n. Hence show that if $P(x)$ is any polynomial, $\lim\limits_{x = \infty} P(x) e^{-x} = 0$ when $x = +\infty$.

8. Show that $(\log x)^m$ when x is infinite is a weaker infinite than x^n no matter how large m or how small n, supposed positive, may be. What is the graphical interpretation?

9. If P is a polynomial, show that $\lim\limits_{x \doteq 0} P\left(\frac{1}{x}\right) e^{-\frac{1}{x^2}} = 0$. Hence show that the Maclaurin development of $e^{-\frac{1}{x^2}}$ is $f(x) = e^{-\frac{1}{x^2}} = \frac{x^n}{n!} f^{(n)}(\theta x)$ if $f(0)$ is defined as 0.

10. The higher differentials are defined as $d^n y = f^{(n)}(x)(dx)^n$ where x is taken as the independent variable. Show that $d^k x = 0$ for $k > 1$ if x is the independent variable. Show that the higher derivatives $D_x^2 y$, $D_x^3 y, \cdots$ are not the quotients d^2y/dx^2, $d^3y/dx^3, \cdots$ if x and y are expressed in terms of a third variable, but that the relations are

$$D_x^2 y = \frac{d^2y dx - d^2x dy}{dx^3}, \qquad D_x^3 y = \frac{dx\,(dx d^3y - dy d^3x) - 3\,d^2x\,(dx d^2y - dy d^2x)}{dx^5}, \quad \cdots.$$

The fact that the quotient $d^n y/dx^n$, $n > 1$, is not the derivative when x and y are expressed parametrically militates against the usefulness of the higher differentials and emphasizes the advantage of working with derivatives. The notation $d^n y/dx^n$ is, however, used for the derivative. Nevertheless, as indicated in Exs. 16–19, higher differentials may be used if proper care is exercised.

11. Compare the conception of higher differentials with the work of Ex. 5, p. 48.

12. Show that in a circle the difference between an infinitesimal arc and its chord is of the third order relative to either arc or chord.

13. Show that if β is of the nth order with respect to α, and γ is of the first order with respect to α, then β is of the nth order with respect to γ.

14. Show that the order of a product of infinitesimals is equal to the sum of the orders of the infinitesimals when all are referred to the same primary infinitesimal α. Infer that in a product each infinitesimal may be replaced by one which differs from it by an infinitesimal of higher order than it without affecting the order of the product.

15. Let A and B be two points of a unit circle and let the angle AOB subtended at the center be the primary infinitesimal. Let the tangents at A and B meet at T, and OT cut the chord AB in M and the arc AB in C. Find the trigonometric expression for the infinitesimal difference $TC - CM$ and determine its order.

16. Compute $d^2(x \sin x) = (2\cos x - x \sin x)\,dx^2 + (\sin x + x\cos x)\,d^2x$ by taking the differential of the differential. Thus find the second derivative of $x \sin x$ if x is the independent variable and the second derivative with respect to t if $x = 1 + t^2$.

17. Compute the first, second, and third differentials, $d^2x \neq 0$.

(α) $x^2 \cos x$, (β) $\sqrt{1-x}\log(1-x)$, (γ) $xe^{2x}\sin x$.

18. In Ex. 10 take y as the independent variable and hence express $D_x^2 y$, $D_x^3 y$ in terms of $D_y x$, $D_y^2 x$. Cf. Ex. 10, p. 14.

19. Make the changes of variable in Exs. 8, 9, 12, p. 14, by the method of differentials, that is, by replacing the derivatives by the corresponding differential expressions where x is not assumed as independent variable and by replacing these differentials by their values in terms of the new variables where the higher differentials of the new independent variable are set equal to 0.

20. Reconsider some of the exercises at the end of Chap. I, say, 17–19, 22, 23, 27, from the point of view of Osgood's Theorem instead of the Theorem of the Mean.

21. Find the areas of the bounding surfaces of the solids of Ex. 11, p. 18.

22. Assume the law $F = kmm'/r^2$ of attraction between particles. Find the attraction of :

(α) a circular wire of radius a and of mass M on a particle m at a distance r from the center of the wire along a perpendicular to its plane ; *Ans.* $kMmr(a^2 + r^2)^{-\frac{3}{2}}$.

(β) a circular disk, etc., as in (α) ; *Ans.* $2kMma^{-2}(1 - r/\sqrt{r^2 + a^2})$.

(γ) a semicircular wire on a particle at its center ; *Ans.* $2kMm/\pi a^2$.

(δ) a finite rod upon a particle not in the line of the rod. The answer should be expressed in terms of the angle the rod subtends at the particle.

(ϵ) two parallel equal rods, forming the opposite sides of a rectangle, on each other.

23. Compare the method of derivatives (§ 7), the method of the Theorem of the Mean (§ 17), and the method of infinitesimals above as applied to obtaining the formulas for (α) area in polar coördinates, (β) mass of a rod of variable density, (γ) pressure on a vertical submerged bulkhead, (δ) attraction of a rod on a particle. Obtain the results by each method and state which method seems preferable for each case.

24. Is the substitution $dx = \phi'(t)\,dt$ in the indefinite integral $\int f(x)\,dx$ to obtain the indefinite integral $\int f[\phi(t)]\,\phi'(t)\,dt$ justifiable immediately ?

36. Infinitesimal analysis. To work rapidly in the applications of calculus to problems in geometry and physics and to follow readily the books written on those subjects, it is necessary to have some familiarity with working directly with infinitesimals. It is possible by making use of the Theorem of the Mean and allied theorems to retain in every expression its complete exact value; but if that expression is an infinitesimal which is ultimately to enter into a quotient or a limit of a sum, any infinitesimal which is of higher order than that which is ultimately kept will not influence the result and may be discarded at any stage of the work if the work may thereby be simplified. A few theorems worked through by the infinitesimal method will serve partly to show how the method is used and partly to establish results which may be of use in further work. The theorems which will be chosen are:

1. The increment Δx and the differential dx of a variable differ by an infinitesimal of higher order than either.

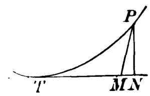
2. If a tangent is drawn to a curve, the perpendicular from the curve to the tangent is of higher order than the distance from the foot of the perpendicular to the point of tangency.

3. An infinitesimal arc differs from its chord by an infinitesimal of higher order relative to the arc.

4. If one angle of a triangle, none of whose angles are infinitesimal, differs infinitesimally from a right angle and if h is the side opposite and if ϕ is another angle of the triangle, then the side opposite ϕ is $h \sin \phi$ except for an infinitesimal of the second order and the adjacent side is $h \cos \phi$ except for an infinitesimal of the first order.

The first of these theorems has been proved in § 35. The second follows from it and from the idea of tangency. For take the x-axis coincident with the tangent or parallel to it. Then the perpendicular is Δy and the distance from its foot to the point of tangency is Δx. The quotient $\Delta y/\Delta x$ approaches 0 as its limit because the tangent is horizontal; and the theorem is proved. *The theorem would remain true if the perpendicular were replaced by a line making a constant angle with the tangent and the distance from the point of tangency to the foot of the perpendicular were replaced by the distance to the foot of the oblique line.* For if $\angle PMN = \theta$,

$$\frac{PM}{TM} = \frac{PN \csc\theta}{TN - PN\cot\theta} = \frac{PN}{TN}\,\frac{\csc\theta}{1 - \dfrac{PN}{TN}\cot\theta},$$

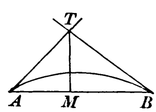


and therefore when P approaches T with θ constant, PM/TM approaches zero and PM is of higher order than TM.

The third theorem follows without difficulty from the assumption or theorem that the arc has a length intermediate between that of the chord and that of the sum of the two tangents at the ends of the chord. Let θ_1 and θ_2 be the angles between the chord and the tangents. Then

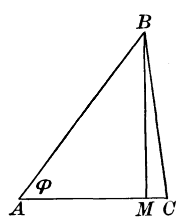
$$\frac{s - AB}{AM + MB} < \frac{AT + TB - AB}{AM + MB} = \frac{AM(\sec\theta_1 - 1) + MB(\sec\theta_2 - 1)}{AM + MB}. \tag{6}$$

Now as AB approaches 0, both $\sec\theta_1 - 1$ and $\sec\theta_2 - 1$ approach 0 and their coefficients remain necessarily finite. Hence the difference between the arc and the chord is an infinitesimal of higher order than the chord. As the arc and chord are therefore of the same order, the difference is of higher order than the arc. This result enables one to replace the arc by its chord and vice versa in discussing infinitesimals of the first order, and for such purposes to consider an infinitesimal arc as straight. In discussing infinitesimals of the second order, this substitution would not be permissible except in view of the further theorem given below in § 37, and even then the substitution will hold only as far as the lengths of arcs are concerned and not in regard to directions.



For the fourth theorem let θ be the angle by which C departs from $90°$ and with the perpendicular BM as radius strike an arc cutting BC. Then by trigonometry

$$AC = AM + MC = h\cos\phi + BM\tan\theta,$$
$$BC = h\sin\phi + BM(\sec\theta - 1).$$



Now $\tan\theta$ is an infinitesimal of the first order with respect to θ; for its Maclaurin development begins with θ. And $\sec\theta - 1$ is an infinitesimal of the second order; for its development begins with a term in θ^2. The theorem is therefore proved. This theorem is frequently applied to infinitesimal triangles, that is, triangles in which h is to approach 0.

37. As a further discussion of the third theorem it may be recalled that by definition the length of the arc of a curve is the limit of the length of an inscribed polygon, namely,

$$s = \lim_{n=\infty}\left(\sqrt{\Delta x_1^2 + \Delta y_1^2} + \sqrt{\Delta x_2^2 + \Delta y_2^2} + \cdots + \sqrt{\Delta x_n^2 + \Delta y_n^2}\right).$$

Now $$\sqrt{\Delta x^2+\Delta y^2}-\sqrt{dx^2+dy^2}=\frac{\Delta x^2+\Delta y^2-dx^2-dy^2}{\sqrt{\Delta x^2+\Delta y^2}+\sqrt{dx^2+dy^2}}$$
$$=\frac{(\Delta x-dx)(\Delta x+dx)+(\Delta y-dy)(\Delta y+dy)}{\sqrt{\Delta x^2+\Delta y^2}+\sqrt{dx^2+dy^2}},$$

and $$\frac{\sqrt{\Delta x^2+\Delta y^2}-\sqrt{dx^2+dy^2}}{\sqrt{\Delta x^2+\Delta y^2}}=\frac{(\Delta x-dx)}{\sqrt{\Delta x^2+\Delta y^2}}\frac{\Delta x+dx}{\sqrt{\Delta x^2+\Delta y^2}+\sqrt{dx^2+dy^2}}$$
$$+\frac{(\Delta y-dy)}{\sqrt{\Delta x^2+\Delta y^2}}\frac{\Delta y+dy}{\sqrt{\Delta x^2+\Delta y^2}+\sqrt{dx^2+dy^2}}.$$

But $\Delta x-dx$ and $\Delta y-dy$ are infinitesimals of higher order than Δx and Δy. Hence the right-hand side must approach zero as its limit and hence $\sqrt{\Delta x^2+\Delta y^2}$ differs from $\sqrt{dx^2+dy^2}$ by an infinitesimal of higher order and may replace it in the sum

$$s=\lim_{n=\infty}\sum\sqrt{\Delta x_i^2+\Delta y_i^2}=\lim_{n=\infty}\sum\sqrt{dx^2+dy^2}=\int_{x_0}^{x_1}\sqrt{1+y'^2}\,dx.$$

The length of the arc measured from a fixed point to a variable point is a function of the upper limit and the differential of arc is

$$ds=d\int_{x_0}^{x}\sqrt{1+y'^2}\,dx=\sqrt{1+y'^2}\,dx=\sqrt{dx^2+dy^2}.$$

To find the order of the difference between the arc and its chord let the origin be taken at the initial point and the x-axis tangent to the curve at that point. The expansion of the arc by Maclaurin's Formula gives

$$s(x)=s(0)+xs'(0)+\tfrac{1}{2}x^2s''(0)+\tfrac{1}{6}x^3s'''(\theta x),$$

where $$s(0)=0,\qquad s'(0)=\sqrt{1+y'^2}\,\Big|_0=1,\qquad s''(0)=\frac{y'y''}{\sqrt{1+y'^2}}\Big|_0=0.$$

Owing to the choice of axes, the expansion of the curve reduces to

$$y=f(x)=y(0)+xy'(0)+\tfrac{1}{2}x^2y''(\theta x)=\tfrac{1}{2}x^2y''(\theta x),$$

and hence the chord of the curve is

$$c(x)=\sqrt{x^2+y^2}=x\sqrt{1+\tfrac{1}{4}x^2[y''(\theta x)]^2}=x(1+x^2P),$$

where P is a complicated expression arising in the expansion of the radical by Maclaurin's Formula. The difference

$$s(x)-c(x)=[x+\tfrac{1}{6}x^3s'''(\theta x)]-[x(1+x^2P)]=x^3(\tfrac{1}{6}s'''(\theta x)-P).$$

This is an infinitesimal of at least the third order relative to x. Now as both $s(x)$ and $c(x)$ are of the first order relative to x, it follows that the difference $s(x)-c(x)$ must also be of the third order relative to either $s(x)$ or $c(x)$. Note that the proof assumes that y'' is finite at the point considered. This result, which has been found analytically, follows more simply though perhaps less rigorously from the fact that $\sec\theta_1-1$ and $\sec\theta_2-1$ in (6) are infinitesimals of the second order with θ_1 and θ_2.

38. The theory of *contact of plane curves* may be treated by means of Taylor's Formula and stated in terms of infinitesimals. Let two curves $y=f(x)$ and $y=g(x)$ be tangent at a given point and let the origin be chosen at that point with the x-axis tangent to the curves. The Maclaurin developments are

$$y = f(x) = \frac{1}{2}f''(0)x^2 + \cdots + \frac{1}{(n-1)!}x^{n-1}f^{(n-1)}(0) + \frac{1}{n!}x^{(n)}f^{(n)}(0) + \cdots$$

$$y = g(x) = \frac{1}{2}g''(0)x^2 + \cdots + \frac{1}{(n-1)!}x^{n-1}g^{(n-1)}(0) + \frac{1}{n!}x^{n}g^{(n)}(0) + \cdots.$$

If these developments agree up to but not including the term in x^n, the difference between the ordinates of the curves is

$$f(x) - g(x) = \frac{1}{n!}x^n[f^{(n)}(0) - g^{(n)}(0)] + \cdots, \qquad f^{(n)}(0) \neq g^{(n)}(0),$$

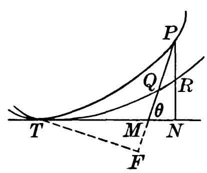
and is an infinitesimal of the nth order with respect to x. The curves are then said to have *contact of order* $n-1$ at their point of tangency. In general when two curves are tangent, the derivatives $f''(0)$ and $g''(0)$ are unequal and the curves have simple contact or *contact of the first order*.

The problem may be stated differently. Let PM be a line which makes a constant angle θ with the x-axis. Then, when P approaches T, if RQ be regarded as straight, the proportion

$$\lim (PR : PQ) = \lim (\sin \angle PQR : \sin \angle PRQ) = \sin\theta : 1$$

shows that PR and PQ are of the same order. Clearly also the lines TM and TN are of the same order. Hence if

$$\lim \frac{PR}{(TN)^n} \neq 0, \infty, \quad \text{then} \quad \lim \frac{PQ}{(TM)^n} \neq 0, \infty.$$



Hence if two curves have contact of the $(n-1)$st order, the segment of a line intercepted between the two curves is of the nth order with respect to the distance from the point of tangency to its foot. It would also be of the nth order with respect to the perpendicular TF from the point of tangency to the line.

In view of these results it is not necessary to assume that the two curves have a special relation to the axis. Let two curves $y = f(x)$ and $y = g(x)$ intersect when $x = a$, and assume that the tangents at that point are not parallel to the y-axis. Then

$$y = y_0 + (x-a)f'(a) + \cdots + \frac{(x-a)^{n-1}}{(n-1)!}f^{(n-1)}(a) + \frac{(x-a)^n}{n!}f^{(n)}(a) + \cdots$$

$$y = y_0 + (x-a)g'(a) + \cdots + \frac{(x-a)^{n-1}}{(n-1)!}f^{(n-1)}(a) + \frac{(x-a)^n}{n!}g^{(n)}(a) + \cdots$$

will be the Taylor developments of the two curves. If the difference of the ordinates for equal values of x is to be an infinitesimal of the nth order with respect to $x-a$ which is the perpendicular from the point of tangency to the ordinate, then the Taylor developments must agree up to but not including the terms in x^n. This is the condition for contact of order $n-1$.

As the difference between the ordinates is

$$f(x)-g(x)=\frac{1}{n!}(x-a)^n[f^{(n)}(a)-g^{(n)}(a)]+\cdots,$$

the difference will change sign or keep its sign when x passes through a according as n is odd or even, because for values sufficiently near to x the higher terms may be neglected. Hence *the curves will cross each other if the order of contact is even*, but *will not cross each other if the order of contact is odd.* If the values of the ordinates are equated to find the points of intersection of the two curves, the result is

$$0=\frac{1}{n!}(x-a)^n\{[f^{(n)}(a)-g^{(n)}(a)]+\cdots\}$$

and shows that $x=a$ is a root of multiplicity n. Hence it is said that two curves have in common as many coincident points as the order of their contact plus one. This fact is usually stated more graphically by saying that *the curves have n consecutive points in common.* It may be remarked that what Taylor's development carried to n terms does, is to give a polynomial which has contact of order $n-1$ with the function that is developed by it.

As a problem on contact consider the determination of the circle which shall have contact of the second order with a curve at a given point (a, y_0). Let

$$y=y_0+(x-a)f'(a)+\tfrac{1}{2}(x-a)^2f''(a)+\cdots$$

be the development of the curve and let $y'=f'(a)=\tan\tau$ be the slope. If the circle is to have contact with the curve, its center must be at some point of the normal. Then if R denotes the assumed radius, the equation of the circle may be written as

$$(x-a)^2+2R\sin\tau\,(x-a)+(y-y_0)^2-2R\cos\tau\,(y-y_0)=0,$$

where it remains to determine R so that the development of the circle will coincide with that of the curve as far as written. Differentiate the equation of the circle.

$$\frac{dy}{dx}=\frac{R\sin\tau+(x-a)}{R\cos\tau-(y-y_0)},\qquad \left(\frac{dy}{dx}\right)_{a,\,y_0}=\tan\tau=f'(a),$$

$$\frac{d^2y}{dx^2}=\frac{[R\cos\tau-(y-y_0)]^2+[R\sin\tau+(x-a)]^2}{[R\cos\tau-(y-y_0)]^3},\qquad \left(\frac{d^2y}{dx^2}\right)_{a,\,y_0}=\frac{1}{R\cos^3\tau},$$

and

$$y=y_0+(x-a)f'(a)+\tfrac{1}{2}(x-a)^2\frac{1}{R\cos^3\tau}+\cdots$$

is the development of the circle. The equation of the coefficients of $(x-a)^2$,

$$\frac{1}{R\cos^3\tau}=f''(a), \quad \text{gives} \quad R=\frac{\sec^3\tau}{f''(a)}=\frac{\{1+[f'(a)]^2\}^{\frac{3}{2}}}{f''(a)}.$$

This is the well known formula for the radius of curvature and shows that the circle of curvature has contact of at least the second order with the curve. The circle is sometimes called the osculating circle instead of the circle of curvature.

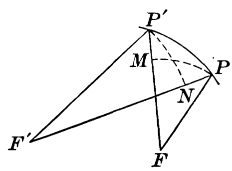
39. Three theorems, one in geometry and two in kinematics, will now be proved to illustrate the direct application of the infinitesimal methods to such problems. The choice will be:

1. The tangent to the ellipse is equally inclined to the focal radii drawn to the point of contact.

2. The displacement of any rigid body in a plane may be regarded at any instant as a rotation through an infinitesimal angle about some point unless the body is moving parallel to itself.

3. The motion of a rigid body in a plane may be regarded as the rolling of one curve upon another.

For the first problem consider a secant PP' which may be converted into a tangent TT' by letting the two points approach until they coincide. Draw the focal radii to P and P' and strike arcs with F and F' as centers. As $F'P+PF=F'P'+P'F=2a$, it follows that $NP=MP'$. Now consider the two triangles $PP'M$ and $P'PN$ nearly right-angled at M and N. The sides PP', PM, PN, $P'M$, $P'N$ are all infinitesimals of the same order and of the same order as the angles at F and F'. By proposition 4 of § 36



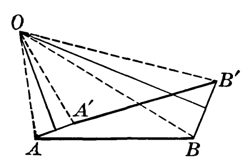
$$MP'=PP'\cos\angle PP'M+e_1, \qquad NP=PP'\cos\angle P'PN+e_2,$$

where e_1 and e_2 are infinitesimals relative to MP' and NP or PP'. Therefore

$$\lim\,[\cos\angle PP'M-\cos\angle P'PN]=\cos\angle TPF'-\cos\angle T'PF=\lim\frac{e_1-e_2}{PP'}=0,$$

and the two angles TPF' and $T'PF$ are proved to be equal as desired.

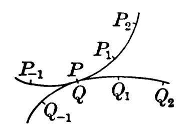
To prove the second theorem note first that if a body is rigid, its position is completely determined when the position AB of any rectilinear segment of the body is known. Let the points A and B of the body be describing curves AA' and BB' so that, in an infinitesimal interval of time, the line AB takes the neighboring position $A'B'$. Erect the perpendicular bisectors of the lines AA' and BB' and let them intersect at O. Then the triangles AOB and $A'OB'$ have the three sides of the one equal to the three sides of the other and are equal, and the second may be obtained from the first by a mere rotation about O through the angle $AOA'=BOB'$. Except for infinitesimals of higher order, the magnitude of the angle is AA'/OA or BB'/OB. Next let the interval of time approach 0 so that A' approaches A and B' approaches B. The perpendicular bisectors will approach



the normals to the arcs AA' and BB' at A and B, and the point O will approach the intersection of those normals.

The theorem may then be stated that: *At any instant of time the motion of a rigid body in a plane may be considered as a rotation through an infinitesimal angle about the intersection of the normals to the paths of any two of its points at that instant; the amount of the rotation will be the distance ds that any point moves divided by the distance of that point from the instantaneous center of rotation; the angular velocity about the instantaneous center will be this amount of rotation divided by the interval of time dt, that is, it will be* v/r, *where v is the velocity of any point of the body and r is its distance from the instantaneous center of rotation.* It is therefore seen that not only is the desired theorem proved, but numerous other details are found. As has been stated, the point about which the body is rotating at a given instant is called the *instantaneous center* for that instant.

As time goes on, the position of the instantaneous center will generally change. If at each instant of time the position of the center is marked on the moving plane or body, there results a locus which is called the *moving centrode* or *body centrode;* if at each instant the position of the center is also marked on a fixed plane over which the moving plane may be considered to glide, there results another locus which is called the *fixed centrode* or the *space centrode.* From these definitions it follows that at each instant of time the body centrode and the space centrode intersect at the instantaneous center for that instant. Consider a series of positions of the instantaneous center as $P_{-2}P_{-1}PP_1P_2$ marked in space and $Q_{-2}Q_{-1}QQ_1Q_2$ marked in the body. At a given instant two of the points, say P and Q, coincide; an instant later the body will have moved so as to bring Q_1 into coincidence with P_1; at an earlier instant Q_{-1} was coincident with P_{-1}. Now as the motion at the instant when P and Q are together is one of rotation through an infinitesimal angle about that point, the angle between PP_1 and QQ_1 is infinitesimal and the lengths PP_1 and QQ_1 are equal; for it is by the rotation about P and Q that Q_1 is to be brought into coincidence with P_1. Hence it follows 1° that the two centrodes are tangent and 2° that the distances $PP_1 = QQ_1$ which the point of contact moves along the two curves during an infinitesimal interval of time are the same, and this means that the two curves roll on one another without slipping — because the very idea of slipping implies that the point of contact of the two curves should move by different amounts along the two curves, the difference in the amounts being the amount of the slip. The third theorem is therefore proved.



EXERCISES

1. If a finite parallelogram is nearly rectangled, what is the order of infinitesimals neglected by taking the area as the product of the two sides? What if the figure were an isosceles trapezoid? What if it were any rectilinear quadrilateral all of whose angles differ from right angles by infinitesimals of the same order?

2. On a sphere of radius r the area of the zone between the parallels of latitude λ and $\lambda + d\lambda$ is taken as $2\pi r \cos\lambda \cdot r d\lambda$, the perimeter of the base times the slant height. Of what order relative to $d\lambda$ is the infinitesimal neglected? What if the perimeter of the middle latitude were taken so that $2\pi r^2 \cos(\lambda + \frac{1}{2}d\lambda)\,d\lambda$ were assumed?

3. What is the order of the infinitesimal neglected in taking $4\pi r^2 dr$ as the volume of a hollow sphere of interior radius r and thickness dr? What if the mean radius were taken instead of the interior radius? Would any particular radius be best?

4. Discuss the length of a space curve $y = f(x)$, $z = g(x)$ analytically as the length of the plane curve was discussed in the text.

5. Discuss proposition 2, p. 68, by Maclaurin's Formula and in particular show that if the second derivative is continuous at the point of tangency, the infinitesimal in question is of the second order at least. How about the case of the tractrix

$$y = \frac{a}{2}\log\frac{a - \sqrt{a^2 - x^2}}{a + \sqrt{a^2 - x^2}} + \sqrt{a^2 - x^2},$$

and its tangent at the vertex $x = a$? How about $s(x) - c(x)$ of § 37?

6. Show that if two curves have contact of order $n - 1$, their derivatives will have contact of order $n - 2$. What is the order of contact of the kth derivatives $k < n - 1$?

7. State the conditions for maxima, minima, and points of inflection in the neighborhood of a point where $f^{(n)}(a)$ is the first derivative that does not vanish.

8. Determine the order of contact of these curves at their intersections:

$$(\alpha)\ \begin{matrix}\sqrt{2}(x^2 + y^2 + z) = 3(x + y)\\ 5x^2 - 6xy + 5y^2 = 8,\end{matrix} \qquad (\beta)\ \begin{matrix}r^2 = a^2\cos 2\phi\\ y^2 = \frac{2}{3}a(a - x),\end{matrix} \qquad (\gamma)\ \begin{matrix}x^2 + y^2 = y\\ x^3 + y^3 = xy.\end{matrix}$$

9. Show that at points where the radius of curvature is a maximum or minimum the contact of the osculating circle with the curve must be of at least the third order and must always be of odd order.

10. Let PN be a normal to a curve and $P'N$ a neighboring normal. If O is the center of the osculating circle at P, show with the aid of Ex. 6 that ordinarily the perpendicular from O to $P'N$ is of the second order relative to the arc PP' and that the distance ON is of the first order. Hence interpret the statement: Consecutive normals to a curve meet at the center of the osculating circle.

11. Does the osculating circle cross the curve at the point of osculation? Will the osculating circles at neighboring points of the curve intersect in real points?

12. In the hyperbola the focal radii drawn to any point make equal angles with the tangent. Prove this and state and prove the corresponding theorem for the parabola.

13. Given an infinitesimal arc AB cut at C by the perpendicular bisector of its chord AB. What is the order of the difference $AC - BC$?

14. Of what order is the area of the segment included between an infinitesimal arc and its chord compared with the square on the chord?

15. Two sides AB, AC of a triangle are finite and differ infinitesimally; the angle θ at A is an infinitesimal of the same order and the side BC is either rectilinear or curvilinear. What is the order of the neglected infinitesimal if the area is assumed as $\frac{1}{2}\overline{AB}^2\theta$? What if the assumption is $\frac{1}{2}AB\cdot AC\cdot\theta$?

16. A cycloid is the locus of a fixed point upon a circumference which rolls on a straight line. Show that the tangent and normal to the cycloid pass through the highest and lowest points of the rolling circle at each of its instantaneous positions.

17. Show that the increment of arc Δs in the cycloid differs from $2a\sin\frac{1}{2}\theta d\theta$ by an infinitesimal of higher order and that the increment of area (between two consecutive normals) differs from $3a^2\sin^2\frac{1}{2}\theta d\theta$ by an infinitesimal of higher order. Hence show that the total length and area are $8a$ and $3\pi a^2$. Here a is the radius of the generating circle and θ is the angle subtended at the center by the lowest point and the fixed point which traces the cycloid.

18. Show that the radius of curvature of the cycloid is bisected at the lowest point of the generating circle and hence is $4a\sin\frac{1}{2}\theta$.

19. A triangle ABC is circumscribed about any oval curve. Show that if the side BC is bisected at the point of contact, the area of the triangle will be changed by an infinitesimal of the second order when BC is replaced by a neighboring tangent $B'C'$, but that if BC be not bisected, the change will be of the first order. Hence infer that the minimum triangle circumscribed about an oval will have its three sides bisected at the points of contact.

20. If a string is wrapped about a circle of radius a and then unwound so that its end describes a curve, show that the length of the curve and the area between the curve, the circle, and the string are

$$s = \int_0^\theta a\theta d\theta, \qquad A = \int_0^\theta \tfrac{1}{2} a^2\theta^2 d\theta,$$

where θ is the angle that the unwinding string has turned through.

21. Show that the motion in space of a rigid body one point of which is fixed may be regarded as an instantaneous rotation about some axis through the given point. To do this examine the displacements of a unit sphere surrounding the fixed point as center.

22. Suppose a fluid of variable density $D(x)$ is flowing at a given instant through a tube surrounding the x-axis. Let the velocity of the fluid be a function $v(x)$ of x. Show that during the infinitesimal time δt the diminution of the amount of the fluid which lies between $x = a$ and $x = a + h$ is

$$S\,[v(a+h)\,D(a+h)\,\delta t - v(a)\,D(a)\,\delta t],$$

where S is the cross section of the tube. Hence show that $D(x)\,v(x) = \text{const.}$ is the condition that the flow of the fluid shall not change the density at any point.

23. Consider the curve $y = f(x)$ and three equally spaced ordinates at $x = a - \delta$, $x = a$, $x = a + \delta$. Inscribe a trapezoid by joining the ends of the ordinates at $x = a \pm \delta$ and circumscribe a trapezoid by drawing the tangent at the end of the ordinate at $x = a$ and producing to meet the other ordinates. Show that

$$S_0 = 2\,\delta f(a), \qquad S = 2\,\delta\left[f(a) + \frac{\delta^2}{6}f''(a) + \frac{\delta^4}{120}f^{(\text{iv})}(\xi)\right],$$

$$S_1 = 2\,\delta\left[f(a) + \frac{\delta^2}{2}f''(a) + \frac{\delta^4}{24}f^{(\text{iv})}(\xi_1)\right]$$

are the areas of the circumscribed trapezoid, the curve, the inscribed trapezoid. Hence infer that to compute the area under the curve from the inscribed or circumscribed trapezoids introduces a relative error of the order δ^2, but that to compute from the relation $S = \frac{1}{3}(2S_0 + S_1)$ introduces an error of only the order of δ^4.

24. Let the interval from a to b be divided into an even number $2n$ of equal parts δ and let the $2n+1$ ordinates $y_0, y_1, \cdots, y_{2n}$ at the extremities of the intervals be drawn to the curve $y = f(x)$. Inscribe trapezoids by joining the ends of every other ordinate beginning with y_0, y_2, and going to y_{2n}. Circumscribe trapezoids by drawing tangents at the ends of every other ordinate $y_1, y_3, \cdots, y_{2n-1}$. Compute the area under the curve as

$$S = \int_a^b f(x)\,dx = \frac{b-a}{6n}[4(y_1 + y_3 + \cdots + y_{2n-1}) + 2(y_0 + y_2 + \cdots + y_{2n}] - y_0 - y_{2n}] + R$$

by using the work of Ex. 23 and infer that the error R is less than $(b-a)\delta^4 f^{(\text{iv})}(\xi)/45$. This method of computation is known as *Simpson's Rule*. It usually gives accuracy sufficient for work to four or even five figures when $\delta = 0.1$ and $b - a = 1$; for $f^{(\text{iv})}(x)$ usually is small.

25. Compute these integrals by Simpson's Rule. Take $2n = 10$ equal intervals. Carry numerical work to six figures except where tables must be used to find $f(x)$:

$(\alpha)\ \int_1^2 \frac{dx}{x} = \log 2 = 0.69315,$ $(\beta)\ \int_0^1 \frac{dx}{1+x^2} = \tan^{-1} 1 = \frac{1}{4}\pi = 0.78535,$

$(\gamma)\ \int_0^{\frac{1}{2}\pi} \sin x dx = 1.00000,$ $(\delta)\ \int_1^2 \log_{10} x dx = 2\log_{10} x - M = 0.16776,$

$(\epsilon)\ \int_0^1 \frac{\log(1+x)}{1+x^2} dx = 0.27220,$ $(\zeta)\ \int_0^1 \frac{\log(1+x)}{x} dx = 0.82247.$

The answers here given are the true values of the integrals to five places.

26. Show that the quadrant of the ellipse $x = a\sin\phi$, $y = b\cos\phi$ is

$$s = a\int_0^{\frac{1}{2}\pi} \sqrt{1 - e^2\sin^2\phi}\,d\phi = \tfrac{1}{2}\pi a \int_0^1 \sqrt{\tfrac{1}{2}(2 - e^2) + \tfrac{1}{2}e^2\cos\pi u}\,du.$$

Compute to four figures by Simpson's Rule with six divisions the quadrants of the ellipses:

$(\alpha)\ e = \frac{1}{2}\sqrt{3},\quad s = 1.211\,a,$ $(\beta)\ e = \frac{1}{2}\sqrt{2},\quad s = 1.351\,a.$

27. Expand s in Ex. 26 into a series and discuss the remainder.

$$s = \frac{1}{2}\pi a\left[1 - \left(\frac{1}{2}\right)^2 e^2 - \left(\frac{1\cdot 3}{2\cdot 4}\right)^2\frac{e^4}{3} - \left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)^2\frac{e^6}{5} - \cdots \left(\frac{1\cdot 3\cdots(2n-1)}{2\cdot 4\cdots 2n}\right)^2\frac{e^{2n}}{2n-1} - R_n\right]$$

$$R_n < \frac{1}{1-e^2}\left(\frac{1\cdot 3\cdots(2n+1)}{2\cdot 4\cdots(2n+2)}\right)^2\frac{e^{2n+2}}{2n+1}$$ See Ex. 18, p. 60, and Peirce's "Tables," p. 62.

Estimate the number of terms necessary to compute Ex. 26 (β) with an error not greater than 2 in the last place and compare the labor with that of Simpson's Rule.

28. If the eccentricity of an ellipse is $\frac{1}{60}$, find to five decimals the percentage error made in taking $2\pi a$ as the perimeter. *Ans.* 0.00694%

29. If the catenary $y = c \cosh(x/c)$ gives the shape of a wire of length L suspended between two points at the same level and at a distance l nearly equal to L, find the first approximation connecting L, l, and d, where d is the dip of the wire at its lowest point below the level of support.

30. At its middle point the parabolic cable of a suspension bridge 1000 ft. long between the supports sags 50 ft. below the level of the ends. Find the length of the cable correct to inches.

40. Some differential geometry. Suppose that between the increments of a set of variables all of which depend on a single variable t there exists an equation which is true except for infinitesimals of higher order than $\Delta t = dt$, then the equation will be exactly true for the differentials of the variables. Thus if

$$f\Delta x + g\Delta y + h\Delta z + l\Delta t + \cdots + e_1 + e_2 + \cdots = 0$$

is an equation of the sort mentioned and if the coefficients are any functions of the variables and if $e_1, e_2, \cdots$ are infinitesimals of higher order than dt, the limit of

$$f\frac{\Delta x}{\Delta t} + g\frac{\Delta y}{\Delta t} + h\frac{\Delta z}{\Delta t} + l\frac{\Delta t}{\Delta t} + \cdots + \frac{e_1}{\Delta t} + \frac{e_2}{\Delta t} = 0$$

is
$$f\frac{dx}{dt} + g\frac{dy}{dt} + h\frac{dz}{dt} + l = 0,$$

or
$$fdx + gdy + hdz + ldt = 0;$$

and the statement is proved. This result is very useful in writing down various differential formulas of geometry where the approximate relation between the increments is obvious and where the true relation between the differentials can therefore be found.

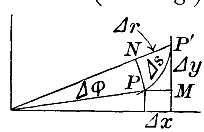
For instance in the case of the differential of arc in rectangular coördinates, if the increment of arc is known to differ from its chord by an infinitesimal of higher order, the Pythagorean theorem shows that the equation

$$\Delta s^2 = \Delta x^2 + \Delta y^2 \quad \text{or} \quad \Delta s^2 = \Delta x^2 + \Delta y^2 + \Delta z^2 \tag{7}$$

is true except for infinitesimals of higher order; and hence

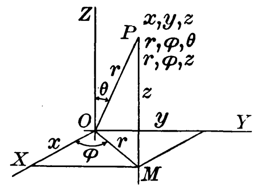
$$ds^2 = dx^2 + dy^2 \quad \text{or} \quad ds^2 = dx^2 + dy^2 + dz^2. \tag{7'}$$

In the case of plane polar coördinates, the triangle $PP'N$ (see Fig.) has two curvilinear sides PP' and PN and is right-angled at N. The Pythagorean theorem may be applied to a curvilinear triangle, or the triangle may be replaced by the rectilinear triangle $PP'N$ with the angle at N no longer a right angle but nearly so. In either way of looking at the figure, it is easily seen that the equation $\Delta s^2 = \Delta r^2 + r^2\Delta\phi^2$



which the figure suggests differs from a true equation by an infinitesimal of higher order; and hence the inference that in polar coördinates $ds^2 = dr^2 + r^2 d\phi^2$.

The two most used systems of coördinates other than rectangular in space are the *polar or spherical* and the *cylindrical*. In the first the distance $r = OP$ from the pole or center, the longitude or meridional angle ϕ, and the colatitude or polar angle θ are chosen as coördinates; in the second, ordinary polar coördinates $r = OM$ and ϕ in the xy-plane are combined with the ordinary rectangular z for distance from that plane. The formulas of transformation are

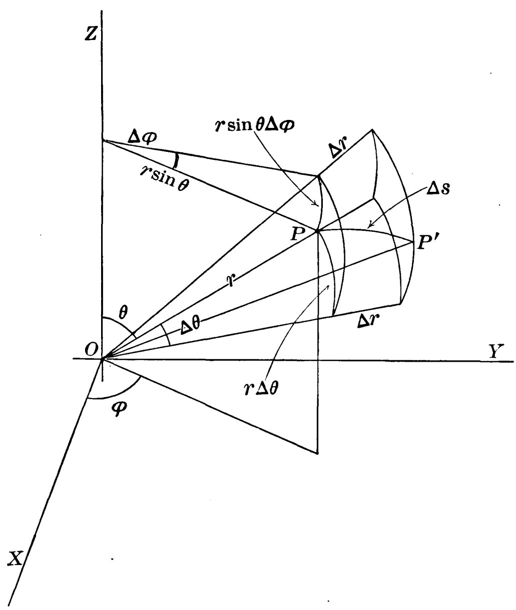


$$z = r\cos\theta, \qquad r = \sqrt{x^2 + y^2 + z^2},$$
$$y = r\sin\theta\sin\phi, \qquad \theta = \cos^{-1}\frac{z}{\sqrt{x^2+y^2+z^2}}, \tag{8}$$
$$x = r\sin\theta\cos\phi, \qquad \phi = \tan^{-1}\frac{y}{x},$$

for polar coördinates, and for cylindrical coördinates they are

$$z = z, \quad y = r\sin\phi, \quad x = r\sin\phi, \quad r = \sqrt{x^2 + y^2}, \quad \phi = \tan\frac{y}{x}. \tag{9}$$

Formulas such as that for the differential of arc may be obtained for these new coördinates by mere transformation of (7′) according to the rules for change of variable.



In both these cases, however, the value of ds may be found readily by direct inspection of the figure. The small parallelepiped (figure for polar case) of which Δs is the diagonal has some of its edges and faces curved instead of straight; all the angles, however, are right angles, and as the edges are infinitesimal, the equations certainly suggested as holding except for infinitesimals of higher order are

$$\Delta s^2 = \Delta r^2 + r^2 \sin^2\theta\Delta\phi^2 + r^2\Delta\theta^2 \quad \text{and} \quad \Delta s^2 = \Delta r^2 + r^2\Delta\phi^2 + \Delta z^2 \quad (10)$$

$$\text{or} \quad ds^2 = dr^2 + r^2 \sin^2\theta d\phi^2 + r^2 d\theta^2 \quad \text{and} \quad ds^2 = dr^2 + r^2 d\phi^2 + dz^2. \quad (10')$$

To make the proof complete, it would be necessary to show that nothing but infinitesimals of higher order have been neglected and it might actually be easier to transform $\sqrt{dx^2 + dy^2 + dz^2}$ rather than give a rigorous demonstration of this fact. Indeed the infinitesimal method is seldom used rigorously; its great use is to make the facts so clear to the rapid worker that he is willing to take the evidence and omit the proof.

In the plane for rectangular coördinates with rulings parallel to the y-axis and for polar coördinates with rulings issuing from the pole the increments of area differ from

$$dA = ydx \quad \text{and} \quad dA = \tfrac{1}{2} r^2 d\phi \qquad (11)$$

respectively by infinitesimals of higher order, and

$$A = \int_{x_0}^{x_1} ydx \quad \text{and} \quad A = \tfrac{1}{2}\int_{\phi_0}^{\phi_1} r^2 d\phi \qquad (11')$$

are therefore the formulas for the area under a curve and between two ordinates, and for the area between the curve and two radii. If the plane is ruled by lines parallel to both axes or by lines issuing from the pole and by circles concentric with the pole, as is customary for double integration (§§ 131, 134), the increments of area differ respectively by infinitesimals of higher order from

$$dA = dxdy \quad \text{and} \quad dA = rdrd\phi, \qquad (12)$$

and the formulas for the area in the two cases are

$$A = \lim \sum \Delta A = \iint dA = \iint dxdy, \qquad (12')$$

$$A = \lim \sum \Delta A = \iint dA = \iint rdrd\phi,$$

where the double integrals are extended over the area desired.

The elements of volume which are required for triple integration (§§ 133, 134) over a volume in space may readily be written down for the three cases of rectangular, polar, and cylindrical coördinates. In the first case space is supposed to be divided up by planes $x = a$, $y = b$, $z = c$ perpendicular to the axes and spaced at infinitesimal intervals; in the second case the division is made by the spheres $r = a$ concentric with the pole, the planes $\phi = b$ through the polar axis, and the cones $\theta = c$ of revolution about the polar axis; in the third case by the cylinders $r = a$, the planes $\phi = b$, and the planes $z = c$. The infinitesimal volumes into which space is divided then differ from

$$dv = dxdydz, \qquad dv = r^2 \sin\theta dr d\phi d\theta, \qquad dv = r dr d\phi dz \tag{13}$$

respectively by infinitesimals of higher order, and

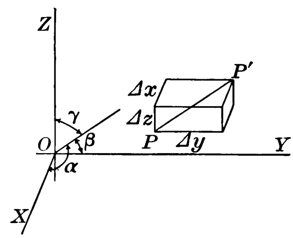
$$\iiint dxdydz, \qquad \iiint r^2 \sin\theta dr d\phi d\theta, \qquad \iiint r dr d\phi dz \tag{13'}$$

are the formulas for the volumes.

41. The direction of a line in space is represented by the three angles which the line makes with the positive directions of the axes or by the cosines of those angles, the direction cosines of the line. From the definition and figure it appears that

$$l = \cos\alpha = \frac{dx}{ds}, \qquad m = \cos\beta = \frac{dy}{ds}, \qquad n = \cos\gamma = \frac{dz}{ds} \tag{14}$$

are the direction cosines of the tangent to the arc at the point; of the tangent and not of the chord for the reason that the increments are replaced by the differentials. Hence it is seen that for the *direction cosines of the tangent* the proportion



$$l : m : n = dx : dy : dz \tag{14'}$$

holds. The equations of a space curve are

$$x = f(t), \qquad y = g(t), \qquad z = h(t)$$

in terms of a variable parameter t.* At the point (x_0, y_0, z_0) where $t = t_0$ the *equations of the tangent lines* would then be

$$\frac{x - x_0}{(dx)_0} = \frac{y - y_0}{(dy)_0} = \frac{z - z_0}{(dz)_0} \quad \text{or} \quad \frac{x - x_0}{f'(t_0)} = \frac{y - y_0}{g'(t_0)} = \frac{z - z_0}{h'(t_0)}. \tag{15}$$

As the cosine of the angle θ between the two directions given by the direction cosines l, m, n and l', m', n' is

$$\cos\theta = ll' + mm' + nn', \quad \text{so} \quad ll' + mm' + nn' = 0 \tag{16}$$

is the condition for the perpendicularity of the lines. Now if (x, y, z) lies in the plane normal to the curve at x_0, y_0, z_0, the lines determined by the ratios $x - x_0 : y - y_0 : z - z_0$ and $(dx)_0 : (dy)_0 : (dz)_0$ will be perpendicular. Hence the *equation of the normal plane* is

$$(x - x_0)(dx)_0 + (y - y_0)(dy)_0 + (z - z_0)(dz)_0 = 0$$

or

$$f'(t_0)(x - x_0) + g'(t_0)(y - y_0) + h'(t_0)(z - z_0) = 0. \tag{17}$$

* For the sake of generality the parametric form in t is assumed; in a particular case a simplification might be made by taking one of the variables as t and one of the functions f', g', h' would then be 1. Thus in Ex. 8 (ϵ), y should be taken as t.

The *tangent plane* to the curve is not determinate; any plane through the tangent line will be tangent to the curve. If λ be a parameter, the pencil of tangent planes is

$$\frac{x-x_0}{f'(t_0)}+\lambda\frac{y-y_0}{g'(t_0)}-(1+\lambda)\frac{z-z_0}{h'(t_0)}=0.$$

There is one particular tangent plane, called *the osculating plane*, which is of especial importance. Let

$$x-x_0=f'(t_0)\tau+\tfrac{1}{2}f''(t_0)\tau^2+\tfrac{1}{6}f'''(\xi)\tau^3,\quad \tau=t-t_0,\quad \tau_0<\xi<t,$$

with similar expansions for y and z, be the Taylor developments of x, y, z about the point of tangency. When these are substituted in the equation of the plane, the result is

$$\frac{1}{2}\tau^2\left[\frac{f''(t_0)}{f'(t_0)}+\lambda\frac{g''(t_0)}{g'(t_0)}-(1+\lambda)\frac{h''(t_0)}{h'(t_0)}\right]$$
$$+\frac{1}{6}\tau^3\left[\frac{f'''(\xi)}{f'(t_0)}+\lambda\frac{g'''(\eta)}{g'(t_0)}-(1+\lambda)\frac{h'''(\zeta)}{h'(t_0)}\right].$$

This expression is of course proportional to the distance from any point x, y, z of the curve to the tangent plane and is seen to be in general of the second order with respect to τ or ds. It is, however, possible to choose for λ that value which makes the first bracket vanish. The tangent plane thus selected has the property that *the distance of the curve from it in the neighborhood of the point of tangency is of the third order and is called the osculating plane.* The substitution of the value of λ gives

$$\begin{vmatrix} x-x_0 & y-y_0 & z-z_0 \\ f'(t_0) & g'(t_0) & h'(t_0) \\ f''(t_0) & g''(t_0) & h''(t_0) \end{vmatrix}=0 \quad\text{or}\quad \begin{vmatrix} x-x_0 & y-y_0 & z-z_0 \\ (dx)_0 & (dy)_0 & (dz)_0 \\ (d^2x)_0 & (d^2y)_0 & (d^2z)_0 \end{vmatrix}=0 \qquad (18)$$

or
$$(dyd^2z-dzd^2y)_0(x-x_0)+(dzd^2x-dxd^2z)_0(y-y_0)$$
$$+(dxd^2y-dyd^2x)_0(z-z_0)=0$$

as the equation of the osculating plane. In case $f''(t_0)=g''(t_0)=h''(t_0)=0$, this equation of the osculating plane vanishes identically and it is necessary to push the development further (Ex. 11).

42. For the case of plane curves the *curvature* is defined as the rate at which the tangent turns compared with the description of arc, that is, as $d\phi/ds$ if $d\phi$ denotes the differential of the angle through which the tangent turns when the point of tangency advances along the curve by ds. The radius of curvature R is the reciprocal of the curvature, that is, it is $ds/d\phi$. Then

$$d\phi=d\tan^{-1}\frac{dy}{dx},\qquad \frac{d\phi}{ds}=\frac{d\phi}{dx}\frac{dx}{ds}=\frac{y''}{[1+y'^2]^{\frac{3}{2}}},\qquad R=\frac{[1+y'^2]^{\frac{3}{2}}}{y''},\quad (19)$$

where accents denote differentiation with respect to x. For space curves the same definitions are given. If l, m, n and $l + dl$, $m + dm$, $n + dn$ are the direction cosines of two successive tangents,

$$\cos d\phi = l(l + dl) + m(m + dm) + n(n + dn).$$

But $\quad l^2 + m^2 + n^2 = 1 \quad$ and $\quad (l + dl)^2 + (m + dm)^2 + (n + dn)^2 = 1.$

Hence $\quad dl^2 + dm^2 + dn^2 = 2 - 2 \cos \phi = (2 \sin \frac{1}{2} \phi)^2,$

$$\frac{1}{R^2} = \left(\frac{d\phi}{ds}\right)^2 = \left[\frac{d(2 \sin \frac{1}{2} \phi)}{ds}\right]^2 = \frac{dl^2 + dm^2 + dn^2}{ds^2} = l'^2 + m'^2 + n'^2, \quad (19')$$

where accents denote differentiation with respect to s.

The *torsion* of a space curve is defined as the rate of turning of the osculating plane compared with the increase of arc (that is, $d\psi/ds$, where $d\psi$ is the differential angle the normal to the osculating plane turns through), and may clearly be calculated by the same formula as the curvature provided the direction cosines L, M, N of the normal to the plane take the places of the direction cosines l, m, n of the tangent line. Hence the torsion is

$$\frac{1}{\mathsf{R}^2} = \left(\frac{d\psi}{ds}\right)^2 = \frac{dL^2 + dM^2 + dN^2}{ds^2} = L'^2 + M'^2 + N'^2; \quad (20)$$

and the radius of torsion R is defined as the reciprocal of the torsion, where from the equation of the osculating plane

$$\frac{L}{dyd^2z - dzd^2y} = \frac{M}{dzd^2x - dxd^2z} = \frac{N}{dxd^2y - dyd^2x}$$
$$= \frac{1}{\sqrt{\text{sum of squares}}}. \quad (20')$$

The actual computation of these quantities is somewhat tedious.

The vectorial discussion of curvature and torsion (§ 77) gives a better insight into the principal directions connected with a space curve. These are the direction of the *tangent*, that of the normal in the osculating plane and directed towards the concave side of the curve and called the *principal normal*, and that of the normal to the osculating plane drawn upon that side which makes the three directions form a right-handed system and called the *binormal*. In the notations there given, combined with those above,

$$\mathbf{r} = x\mathbf{i} + y\mathbf{i} + z\mathbf{k}, \quad \mathbf{t} = l\mathbf{i} + m\mathbf{j} + n\mathbf{k}, \quad \mathbf{c} = \lambda\mathbf{i} + \mu\mathbf{j} + \nu\mathbf{k}, \quad \mathbf{n} = L\mathbf{i} + M\mathbf{j} + N\mathbf{k},$$

where λ, μ, ν are taken as the direction cosines of the principal normal. Now $d\mathbf{t}$ is parallel to $\mathbf{c}$ and $d\mathbf{n}$ is parallel to $-\mathbf{c}$. Hence the results

$$\frac{dl}{\lambda} = \frac{dm}{\mu} = \frac{dn}{\nu} = \frac{ds}{R} \quad \text{and} \quad \frac{dL}{\lambda} = \frac{dM}{\mu} = \frac{dN}{\nu} = -\frac{ds}{\mathsf{R}} \quad (21)$$

follow from $d\mathbf{c}/ds = \mathbf{C}$ and $d\mathbf{n}/ds = \mathbf{T}$. Now $d\mathbf{c}$ is perpendicular to $\mathbf{c}$ and hence in the plane of $\mathbf{t}$ and $\mathbf{n}$; it may be written as $d\mathbf{c} = (\mathbf{t}\cdot d\mathbf{c})\mathbf{t} + (\mathbf{n}\cdot d\mathbf{c})\mathbf{n}$. But as $\mathbf{t}\cdot\mathbf{c} = \mathbf{n}\cdot\mathbf{c} = 0$, $\mathbf{t}\cdot d\mathbf{c} = -\mathbf{c}\cdot d\mathbf{t}$ and $\mathbf{n}\cdot d\mathbf{c} = -\mathbf{c}\cdot d\mathbf{n}$. Hence

$$d\mathbf{c} = -(\mathbf{c}\cdot d\mathbf{t})\mathbf{t} - (\mathbf{c}\cdot d\mathbf{n})\mathbf{n} = -C\mathbf{t}ds + T\mathbf{n}ds = -\frac{\mathbf{t}}{R}ds + \frac{\mathbf{n}}{\mathsf{R}}ds.$$

Hence
$$\frac{d\lambda}{ds} = -\frac{l}{R} + \frac{L}{\mathsf{R}}, \quad \frac{d\mu}{ds} = -\frac{m}{R} + \frac{M}{\mathsf{R}}, \quad \frac{d\nu}{ds} = -\frac{n}{R} + \frac{N}{\mathsf{R}}. \tag{22}$$

Formulas (22) are known as *Frenet's Formulas;* they are usually written with $-\mathsf{R}$ in the place of R because a left-handed system of axes is used and the torsion, being an odd function, changes its sign when all the axes are reversed. If accents denote differentiation by s,

$$\text{above formulas, right-handed} \quad \frac{1}{\mathsf{R}} = \frac{\begin{vmatrix} x' & y' & z' \\ x'' & y'' & z'' \\ x''' & y''' & z''' \end{vmatrix}}{x''^2 + y''^2 + z''^2}, \quad \text{usual formulas, left-handed} \quad \frac{1}{\mathsf{R}} = -\frac{\begin{vmatrix} x' & y' & z' \\ x'' & y'' & z'' \\ x''' & y''' & z''' \end{vmatrix}}{x''^2 + y''^2 + z''^2}. \tag{23}$$

EXERCISES

1. Show that in polar coördinates in the plane, the tangent of the inclination of the curve to the radius vector is $rd\phi/dr$.

2. Verify (10), (10′) by direct transformation of coördinates.

3. Fill in the steps omitted in the text in regard to the proof of (10), (10′) by the method of infinitesimal analysis.

4. A rhumb line on a sphere is a line which cuts all the meridians at a constant angle, say α. Show that for a rhumb line $\sin\theta d\phi = \tan\alpha d\theta$ and $ds = r\sin\alpha d\theta$. Hence find the equation of the line, show that it coils indefinitely around the poles of the sphere, and that its total length is $\pi r\sec\alpha$.

5. Show that the surfaces represented by $F(\phi, \theta) = 0$ and $F(r, \theta) = 0$ in polar coördinates in space are respectively cones and surfaces of revolution about the polar axis. What sort of surface would the equation $F(r, \phi) = 0$ represent?

6. Show accurately that the expression given for the differential of area in polar coördinates in the plane and for the differentials of volume in polar and cylindrical coördinates in space differ from the corresponding increments by infinitesimals of higher order.

7. Show that $\frac{dr}{ds}$, $r\frac{d\theta}{ds}$, $r\sin\theta\frac{d\phi}{ds}$ are the direction cosines of the tangent to a space curve relative to the radius, meridian, and parallel of latitude.

8. Find the tangent line and normal plane of these curves.

(α) $xyz = 1,\ y^2 = x$ at $(1, 1, 1)$, (β) $x = \cos t,\ y = \sin t,\ z = kt$,
(γ) $2ay = x^2,\ 6a^2z = x^3$, (δ) $x = t\cos t,\ y = t\sin t,\ z = kt$,
(ϵ) $y = x^2,\ z^2 = 1 - y$, (ζ) $x^2 + y^2 + z^2 = a^2,\ x^2 + y^2 + 2ax = 0$.

9. Find the equation of the osculating plane in the examples of Ex. 8. Note that if x is the independent variable, the equation of the plane is

$$\left(\frac{dy}{dx}\frac{d^2z}{dx^2} - \frac{dz}{dx}\frac{d^2y}{dx^2}\right)_0 (x - x_0) - \left(\frac{d^2z}{dx^2}\right)_0 (y - y_0) + \left(\frac{d^2y}{dx^2}\right)_0 (z - z_0) = 0.$$

10. A space curve passes through the origin, is tangent to the x-axis, and has $z = 0$ as its osculating plane at the origin. Show that

$$x = tf'(0) + \tfrac{1}{2}t^2f''(0) + \cdots, \qquad y = \tfrac{1}{2}t^2g''(0) + \cdots, \qquad z = \tfrac{1}{6}t^3h'''(0) + \cdots$$

will be the form of its Maclaurin development if $t = 0$ gives $x = y = z = 0$.

11. If the 2d, 3d, $\cdots$, $(n-1)$st derivatives of f, g, h vanish for $t = t_0$ but not all the nth derivatives vanish, show that there is a plane from which the curve departs by an infinitesimal of the $(n+1)$st order and with which it therefore has contact of order n. Such a plane is called a hyperosculating plane. Find its equation.

12. At what points if any do the curves (β), (γ), (ϵ), (ζ), Ex. 8 have hyperosculating planes and what is the degree of contact in each case?

13. Show that the expression for the radius of curvature is

$$\frac{1}{R} = \sqrt{x''^2 + y''^2 + z''^2} = \frac{[(g'h'' - h'g'')^2 + (h'f'' - f'h'')^2 + (f'g'' - g'f'')^2]^{\frac{1}{2}}}{[f'^2 + g'^2 + h'^2]^{\frac{3}{2}}},$$

where in the first case accents denote differentiation by s, in the second by t.

14. Show that the radius of curvature of a space curve is the radius of curvature of its projection on the osculating plane at the point in question.

15. From Frenet's Formulas show that the successive derivatives of x are

$$x' = l, \qquad x'' = l' = \frac{\lambda}{R}, \qquad x''' = \frac{\lambda'}{R} - \frac{\lambda R'}{R^2} = -\frac{l}{R^2} - \lambda\frac{R'}{R^2} + \frac{L}{R\mathsf{R}},$$

where accents denote differentiation by s. Show that the results for y and z are the same except that m, μ, M or n, ν, N take the places of l, λ, L. Hence infer that for the nth derivatives the results are

$$x^{(n)} = lP_1 + \lambda P_2 + LP_3, \qquad y^{(n)} = mP_1 + \mu P_2 + MP_3, \qquad z^{(n)} = nP_1 + \nu P_2 + NP_3,$$

where P_1, P_2, P_3 are rational functions of R and R and their derivatives by s.

16. Apply the foregoing to the expansion of Ex. 10 to show that

$$x = s - \frac{1}{6R^2}s^3 + \cdots, \qquad y = \frac{s^2}{2R} - \frac{R'}{6R^2}s^3 + \cdots, \qquad z = \frac{s^3}{6R\mathsf{R}} + \cdots,$$

where R and R are the values at the origin where $s = 0$, $l = \mu = N = 1$, and the other six direction cosines m, n, λ, ν, L, M vanish. Find s and write the expansion of the curve of Ex. 8 (γ) in this form.

17. Note that the distance of a point on the curve as expanded in Ex. 16 from the sphere through the origin and with center at the point $(0, R, R'\mathsf{R})$ is

$$\sqrt{x^2 + (y-R)^2 + (z - R'\mathsf{R})^2} - \sqrt{R^2 + R'^2\mathsf{R}^2} = \frac{(x^2 + y^2 - 2Ry + z^2 - 2R'\mathsf{R}z)}{\sqrt{x^2 + (y-R)^2 + (z - R'\mathsf{R})^2} + \sqrt{R^2 + R'^2\mathsf{R}^2}},$$

and consequently is of the fourth order. The curve therefore has contact of the third order with this sphere. Can the equation of this sphere be derived by a limiting process like that of Ex. 18 as applied to the osculating plane?

18. The osculating plane may be regarded as the plane passed through three consecutive points of the curve; in fact it is easily shown that

$$\lim_{\substack{\delta x,\, \delta y,\, \delta z \\ \Delta x,\, \Delta y,\, \Delta z \\ \text{approach } 0}} \begin{vmatrix} x & y & z & 1 \\ x_0 & y_0 & z_0 & 1 \\ x_0 + \delta x & y_0 + \delta y & z_0 + \delta z & 1 \\ x_0 + \Delta x & y_0 + \Delta y & z_0 + \Delta z & 1 \end{vmatrix} = \frac{1}{2} \begin{vmatrix} x - x_0 & y - y_0 & z - z_0 \\ (dx)_0 & (dy)_0 & (dz)_0 \\ (d^2x)_0 & (d^2y)_0 & (d^2z)_0 \end{vmatrix} = 0.$$

19. Express the radius of torsion in terms of the derivatives of x, y, z by t (Ex. 10, p. 67).

20. Find the direction, curvature, osculating plane, torsion, and osculating sphere (Ex. 17) of the conical helix $x = t \cos t$, $y = t \sin t$, $z = kt$ at $t = 2\pi$.

21. Upon a plane diagram which shows Δs, Δx, Δy, exhibit the lines which represent ds, dx, dy under the different hypotheses that x, y, or s is the independent variable.