I. THE RESULTANT

1. The Resultant is defined in the first instance with respect to n *homogeneous* polynomials $F_1, F_2, \ldots, F_n$ in n variables, of degrees $l_1, l_2, \ldots, l_n$, each polynomial being complete in all its terms with literal coefficients, all different. The resultant of any n given homogeneous polynomials in n variables is the value which the resultant in the general case assumes for the given case. The resultant of n given non-homogeneous polynomials in $n-1$ variables is the resultant of the corresponding homogeneous polynomials of the same degrees obtained by introducing a variable x_0 of homogeneity.

Definitions. An *elementary member* of the module $(F_1, F_2, \ldots, F_n)$ is any member of the type ωF_i $(i = 1, 2, \ldots, n)$, where ω is any power product of $x_1, x_2, \ldots, x_n$. What is and what is not an elementary member depends on the basis chosen for the module.

The total number of elementary members of an assigned degree is evidently finite.

The diagram below represents the array of the coefficients of all elementary members of $(F_1, F_2, \ldots, F_n)$ of degree t, arranged under the power products $\omega_1^{(t)}, \omega_2^{(t)}, \ldots, \omega_\mu^{(t)}$ of degree t $\left(\mu = \dfrac{(t+n-1)!}{t!\,(n-1)!}\right)$:

$$\begin{array}{c|cccc|}
 & \omega_1^{(t)} & \omega_2^{(t)} & \ldots\ldots & \omega_\mu^{(t)} \\
\hline
\lambda_1 & a_1 & b_1 & \ldots\ldots & k_1 \\
\lambda_2 & a_2 & b_2 & \ldots\ldots & k_2 \\
 & \ldots & \ldots & \ldots\ldots & \ldots \\
\lambda_\rho & a_\rho & b_\rho & \ldots\ldots & k_\rho
\end{array}$$

Each row of the array, in association with $\omega_1^{(t)}, \omega_2^{(t)}, \ldots, \omega_\mu^{(t)}$, represents an elementary member of degree t; and the rows of the array corresponding to F_i all consist of the same elements (the coefficients of F_i and zeros) but in different columns.

Any member $F = X_1 F_1 + X_2 F_2 + \ldots + X_n F_n$ of degree t is evidently a linear combination $\lambda_1\omega_1 F_1 + \lambda_2\omega_2 F_1 + \ldots + \lambda_p\omega_p F_i + \ldots + \lambda_\rho\omega_\rho F_n$ of elementary members of degree t, and is represented by the above array when bordered by $\lambda_1, \lambda_2, \ldots, \lambda_\rho$ on the left, where $\lambda_1, \lambda_2, \ldots, \lambda_\rho$ are the coefficients of $X_1, X_2, \ldots, X_n$, some of which may be zeros.

This bordered array also shows in a convenient way the whole coefficients of the terms of F, viz. $\Sigma\lambda a, \Sigma\lambda b, \ldots, \Sigma\lambda k$.

These remarks and definitions are equally applicable to any module $(F_1, F_2, \ldots, F_k)$ of homogeneous or non-homogeneous polynomials; but the following definition applies only to the particular module $(F_1, F_2, \ldots, F_n)$.

The *resultant* R of $F_1, F_2, \ldots, F_n$ is the H.C.F. of the determinants of the above array for degree $t = l + 1$, where $l = l_1 + l_2 + \ldots + l_n - n$. It will be shown (§ 7) that R is homogeneous and of degree $l_1 l_2 \ldots l_n / l_i$ in the coefficients of F_i $(i = 1, 2, \ldots, n)$.

2. Resultant of two homogeneous polynomials in two variables.

Let
$$F_1 = a_1 x_1^{l_1} + b_1 x_1^{l_1 - 1} x_2 + \ldots + k_1 x_2^{l_1},$$
$$F_2 = k_2 x_1^{l_2} + \ldots + a_2 x_2^{l_2},$$
$$l = l_1 + l_2 - 2.$$

The array of the coefficients of all elementary members of (F_1, F_2) of degree $l + 1$, viz. $x_1^{l_2 - 1} F_1, x_1^{l_2 - 2} x_2 F_1, \ldots x_2^{l_2 - 1} F_1, x_1^{l_1 - 1} F_2, \ldots, x_2^{l_1 - 1} F_2$, has l_2 rows corresponding to F_1 and l_1 rows corresponding to F_2, and the same number $l_1 + l_2$ of rows in all as columns. The resultant R is therefore the determinant of this array. The array is

$$\begin{array}{c|cccccc|l}
 & \omega_1^{(l+1)} & \omega_2^{(l+1)} & \ldots\ldots & \omega_{l_1+1}^{(l+1)} & \ldots\ldots & \omega_{l+2}^{(l+1)} & \\
\hline
\lambda_1 & a_1 & b_1 \ldots\ldots & k_1 & . & . & . & = x_1^{l_2-1} F_1 \\
\lambda_2 & . & a_1 & b_1 \ldots\ldots & k_1 & . & . & = x_1^{l_2-2} x_2 F_1 \\
 & & & & & & & \\
\lambda_{l_2} & . & . & a_1 & b_1 \ldots\ldots & \ldots\ldots & k_1 & = x_2^{l_2-1} F_1 \\
\lambda_{l_2+1} & k_2 \ldots\ldots & \ldots\ldots & a_2 & . & . & . & = x_1^{l_1-1} F_2 \\
 & . & k_2 \ldots\ldots & \ldots\ldots & a_2 & . & . & = x_1^{l_1-2} x_2 F_2 \\
 & & & & & & & \\
\lambda_{l+2} & . & . & k_2 & \ldots\ldots & \ldots\ldots & a_2 & = x_2^{l_1-1} F_2
\end{array}$$

On the right are written the elementary members which the rows represent. Thus, neglecting the left hand border, we may regard the diagram as a set of $l + 2$ identical equations for

$$\omega_1^{(l+1)}, \quad \omega_2^{(l+1)}, \ldots, \omega_{l+2}^{(l+1)}.$$

Solving them we have

$$R\omega_i^{(l+1)} = A_{i1} F_1 + A_{i2} F_2 \quad (i = 1, 2, \ldots, l + 2),$$

where A_{i1}, A_{i2} are polynomials whose coefficients are whole functions of the coefficients of F_1, F_2. Hence

$$R\omega^{(l+1)} \equiv 0 \bmod (F_1, F_2),$$

where $\omega^{(l+1)}$ is any power product of x_1, x_2 of degree $l+1$. This expresses the first important property of R.

3. Irreducibility of R. The general expression for the resultant R is irreducible in the sense that it cannot be resolved into two factors each of which is a whole function of the coefficients of F_1, F_2. When this has been proved it follows that any whole function of the coefficients of F_1, F_2 which vanishes as a consequence of R vanishing must be divisible by R.

R has a term $a_1^{l_2} a_2^{l_1}$ obtained from the diagonal of the determinant, and this is the only term of R containing $a_2^{l_1}$. Also, when $a_1 = 0$, R has a term $(-1)^{l_2} k_2 b_1^{l_2} a_2^{l_1-1}$, and this is the only term of R containing $a_2^{l_1-1}$ when $a_1 = 0$. Hence, when R is expanded in powers of a_2 to two terms, we have

$$R = a_1^{l_2} a_2^{l_1} + b a_2^{l_1-1} + \dots,$$

where
$$b \equiv (-1)^{l_2} k_2 b_1^{l_2} \bmod a_1.$$

Hence if R can be written as a product of two factors, we have

$$R = (a_1^{p_1} a_2^{p_2} + \dots)(a_1^{q_1} a_2^{q_2} + \dots),$$

where $p_1 + q_1 = l_2$ and $p_2 + q_2 = l_1$, and either p_1 or q_1 is zero; for otherwise the coefficient b of $a_2^{l_1-1}$ would be zero or divisible by a_1, which is not the case. Hence one of the factors of R is independent of the coefficients of F_1, since both factors must be homogeneous in the coefficients of F_1. Similarly one of the factors must be independent of the coefficients of F_2, i.e.

$$R = (a_1^{l_2} + \dots)(a_2^{l_1} + \dots) = a_1^{l_2} a_2^{l_1},$$

since the whole coefficient of $a_1^{l_2}$ in R is $a_2^{l_1}$, and of $a_2^{l_1}$ is $a_1^{l_2}$. This is not true; hence R is irreducible.

4. *The necessary and sufficient condition that the equations $F_1 = F_2 = 0$ may have a proper solution (i.e. a solution other than $x_1 = x_2 = 0$) is the vanishing of R.*

This is the fundamental property of the resultant. If the equations $F_1 = F_2 = 0$ have a solution other than $x_1 = x_2 = 0$ it follows from

$$Rx_1^{l+1} \equiv 0 \bmod (F_1, F_2), \quad Rx_2^{l+1} \equiv 0 \bmod (F_1, F_2),$$

that $R = 0$, by giving to x_1, x_2 the values (not both zero) which satisfy the equations $F_1 = F_2 = 0$.

Conversely if $R = 0$ we can choose $\lambda_1, \lambda_2, \dots, \lambda_{l+2}$ so that the sum of their products with the elements in each column of the determinant R vanishes. Multiplying each sum by the power product corresponding to its column, and adding by rows, we have

$$(\lambda_1 x_1^{l_2-1} + \lambda_2 x_1^{l_2-2} x_2 + \ldots + \lambda_{l_2} x_2^{l_2-1}) F_1$$
$$+ (\lambda_{l_2+1} x_1^{l_1-1} + \ldots + \lambda_{l+2} x_2^{l_1-1}) F_2 = 0,$$

where $\lambda_1, \lambda_2, \ldots \lambda_{l+2}$ do not all vanish. Hence, since $\lambda_1 x_1^{l_2-1} + \ldots$ is of less degree than F_2, F_1 must have a factor in common with F_2, and the equations $F_1 = F_2 = 0$ have a proper solution.

In the following article another proof is given which can be extended more easily to any number of variables.

5. When $R \neq 0$ there are $l+2$ linearly independent members of (F_1, F_2) of degree $l+1$, and l of degree l. When $R = 0$ there are only $l+1$ linearly independent members of degree $l+1$ and still l of degree l, i.e. in each case 1 less than the number of terms in a polynomial of degree $l+1$ and l respectively. Hence there will be one and only one identical linear relation between the coefficients of the general member of (F_1, F_2) whether of degree $l+1$ or l. Let this identical relation for degree $l+1$ be

$$c_{l+1,0}\, z_{l+1,0} + c_{l,1}\, z_{l,1} + \ldots + c_{0,l+1}\, z_{0,l+1} = 0,$$

where $z_{i,j}$ denotes the coefficient of $x_1^i x_2^j$ in the general member of (F_1, F_2) of degree $i+j$, and the $c_{i,j}$ are constants. Then, if F is the general member

$$z_{l,0}\, x_1^l + z_{l-1,1}\, x_1^{l-1} x_2 + \ldots + z_{0,l}\, x_2^l$$

of (F_1, F_2) of degree l, $x_1 F$ is a member of degree $l+1$ whose coefficients must satisfy the relation above. Hence

$$c_{l+1,0}\, z_{l,0} + c_{l,1}\, z_{l-1,1} + \ldots + c_{1,l}\, z_{0,l} = 0.$$

Similarly $$c_{l,1}\, z_{l,0} + c_{l-1,2}\, z_{l-1,1} + \ldots + c_{0,l+1}\, z_{0,l} = 0,$$

since $x_2 F$ is a member of (F_1, F_2) of degree $l+1$. These two relations must be equivalent to one only, since only one identical relation exists for degree l. Hence we have

$$\frac{c_{l+1,0}}{c_{l,1}} = \frac{c_{l,1}}{c_{l-1,2}} = \ldots = \frac{c_{1,l}}{c_{0,l+1}} = \frac{\alpha_1}{\alpha_2} \text{ (say)},$$

i.e. $c_{l+1,0}, c_{l,1}, \ldots, c_{0,l+1}$ are proportional to $\alpha_1^{l+1}, \alpha_1^l \alpha_2, \ldots, \alpha_2^{l+1}$. Hence the original identical relation may be written

$$z_{l+1,0}\, \alpha_1^{l+1} + z_{l,1}\, \alpha_1^l \alpha_2 + \ldots + z_{0,l+1}\, \alpha_2^{l+1} = 0,$$

showing that the general member $z_{l+1,0} x_1^{l+1} + \ldots$ of (F_1, F_2) of degree $l+1$ vanishes when $x_1 = \alpha_1$, $x_2 = \alpha_2$, and that the equations $F_1 = F_2 = 0$ have the proper solution (α_1, α_2). The theorem being thus proved true in general is assumed to be true in particular.

6. Resultant of n homogeneous polynomials in n variables.

The general theory of the resultant to be now given is exactly parallel to that already given for two variables, although it involves points of much greater difficulty as might be expected. Another method of exposition depending on a different definition of the resultant is given in (K, p. 260 ff.).

Let $F_1, F_2, \ldots, F_n$ be n homogeneous polynomials of degrees $l_1, l_2, \ldots, l_n$ of which all the coefficients are different letters. In particular, let $a_1, a_2, \ldots, a_n$ be the coefficients of $x_1^{l_1}, x_2^{l_2}, \ldots, x_n^{l_n}$ in $F_1, F_2, \ldots, F_n$ respectively, and $c_1, c_2, \ldots, c_n$ the constant terms of $F_1, F_2, \ldots, F_n$ when x_n is put equal to 1, so that $c_n = a_n$. Let

$$l = l_1 + l_2 + \ldots + l_n - n,\quad L = l_1 l_2 \ldots l_n,\quad L_1 = L/l_1,\quad L_2 = L/l_2,\ \ldots\ L_n = L/l_n.$$

The resultant R of $F_1, F_2, \ldots, F_n$ has already been defined (§ 1) as the H.C.F. of the determinants of the array of the coefficients of all elementary members of $(F_1, F_2, \ldots, F_n)$ of degree $l+1$.

We shall first consider a particular determinant D of the array, viz. that representing (§ 1) the polynomial

$$X^{(0)} F_1 + X^{(1)} F_2 + \ldots + X^{(n-1)} F_n \text{ of degree } l+1,$$

where $X^{(i)}$ denotes a polynomial in which all terms divisible by $x_1^{l_1}$ or $x_2^{l_2} \ldots$ or $x_i^{l_i}$ are absent, which may be expressed by saying that $X^{(i)}$ is *reduced* in $x_1, x_2, \ldots, x_i$. The polynomial

$$X^{(0)} F_1 + X^{(1)} F_2 + \ldots + X^{(n-1)} F_n$$

is represented by the bordered array

$$\begin{array}{c|cccc|l} & \omega_1^{(l+1)} & \omega_2^{(l+1)} & \cdots\cdots & \omega_\mu^{(l+1)} & \\ \hline \lambda_1 & a_1 & b_1 & \cdots\cdots & k_1 & = \omega_1 F_1 \\ \lambda_2 & a_2 & b_2 & \cdots\cdots & k_2 & = \omega_2 F_1 \\ & \cdots & \cdots & \cdots\cdots & \cdots & \\ \lambda_\mu & a_\mu & b_\mu & \cdots\cdots & k_\mu & = \omega_\mu F_n \end{array}$$

where $\omega_1^{(l+1)}, \omega_2^{(l+1)}, \ldots, \omega_\mu^{(l+1)}$ are all the power products of $x_1, x_2, \ldots, x_n$ of degree $l+1$, and $\lambda_1, \lambda_2, \ldots, \lambda_\mu$ are the coefficients of $X^{(0)}, X^{(1)}, \ldots, X^{(n-1)}$. That this array has the same number μ of rows as columns is seen from the fact that *one and only one of the elements* $a_1, a_2, \ldots, a_n$* (*the coefficients of* $x_1^{l_1}, x_2^{l_2}, \ldots, x_n^{l_n}$ *in* $F_1, F_2, \ldots, F_n$) *occurs in each row and each column.* This is evident as regards the rows. To prove

* These are not the same as the $a_1, a_2, \ldots, a_n$ in the first column of the array. The latter should be represented by some other symbols.

that the same is true of the columns, we notice firstly that there is no power product $\omega^{(l+1)}$ of degree $l+1$ reduced in all the variables, for the highest power product of this kind is $x_1^{l_1-1} x_2^{l_2-1} \dots x_n^{l_n-1}$ which is of degree $l < l+1$; and secondly, if we put every coefficient of $F_1, F_2, \dots, F_n$, except only $a_1, a_2, \dots, a_n$, equal to zero, the diagram will represent the polynomial

$$X^{(0)} a_1 x_1^{l_1} + X^{(1)} a_2 x_2^{l_2} + \dots + X^{(n-1)} a_n x_n^{l_n},$$

in which each power product $\omega^{(l+1)}$ occurs once and once only, so that one and only one element $a_1, a_2, \dots, a_n$ occurs in each column of D.

Thus D when expanded has a term $\pm a_1^{\mu_1} a_2^{\mu_2} \dots a_n^{\mu_n}$, where μ_i is the number of terms in $X^{(i-1)}$, and by saying that the coefficient of this term in D is to be $+1$ we remove any ambiguity as to the sign of D. Also it is to be noted that *D vanishes when $c_1, c_2, \dots, c_n$ all vanish*, for the column of D corresponding to x_n^{l+1} contains no elements other than $c_1, c_2, \dots, c_n$ and zeros.

Regarding the diagram as giving μ identical equations for

$$\omega_1^{(l+1)}, \omega_2^{(l+1)}, \dots, \omega_\mu^{(l+1)},$$

and solving, we have

$$D\omega^{(l+1)} \equiv 0 \bmod (F_1, F_2, \dots, F_n),$$

where $\omega^{(l+1)}$ is any power product of $x_1, x_2, \dots, x_n$ of degree $l+1$. It can be proved that the factors of D other than R can be divided out of this congruence equation, so that

$$R\omega^{(l+1)} \equiv 0 \bmod (F_1, F_2, \dots, F_n);$$

but this will not be assumed in what follows*.

7. The number of rows in D corresponding to F_n is the number of terms in $X^{(n-1)}$. But $X^{(n-1)}$ is of degree $l+1-l_n$ or

$$(l_1 - 1) + (l_2 - 1) + \dots + (l_{n-1} - 1),$$

and its terms consist of all the power products in

$$(1 + x_1 + \dots + x_1^{l_1-1}) \dots (1 + x_{n-1} + \dots + x_{n-1}^{l_{n-1}-1})$$

each multiplied by a power of x_n; hence the number of the terms is $l_1 l_2 \dots l_{n-1} = L_n$. Thus D is homogeneous and of degree L_n in the

* No proof of this has been published so far as I know. It can be proved that if A is any whole function of the coefficients of $F_1, F_2, \dots, F_n$ not divisible by R, and $AF \equiv 0 \bmod (F_1, F_2, \dots, F_n)$, then $F \equiv 0 \bmod (F_1, F_2, \dots, F_n)$. Hence from $D\omega^{(l+1)} \equiv 0 \bmod (F_1, F_2, \dots, F_n)$ we have $R\omega^{(l+1)} \equiv 0 \bmod (F_1, F_2, \dots, F_n)$. The condition that A is not divisible by R is not needed if F is of degree $\leqslant l$.

coefficients of F_n, and homogeneous and of degree $> L_i$ in the coefficients of $F_i\,(i = 1, 2, \ldots, n-1)$. It follows that R, which is a factor of D, is at most of degree L_n in the coefficients of F_n. We shall prove that R is of this degree, and consequently of degree L_i in the coefficients of F_i.

Let D' be any other non-vanishing determinant of the array, viz.

$$\begin{array}{c|cccc|} & \omega_1^{(l+1)} & \omega_2^{(l+1)} & \ldots\ldots & \omega_\mu^{(l+1)} \\ \hline \alpha_1 & a_1' & b_1' & \ldots\ldots & k_1' \\ \alpha_2 & a_2' & b_2' & \ldots\ldots & k_2' \\ & \ldots & \ldots & \ldots & \ldots \\ \alpha_\mu & a_\mu' & b_\mu' & \ldots\ldots & k_\mu' \end{array}$$

This represents the polynomial $A_1 F_1 + A_2 F_2 + \ldots + A_n F_n$, in which $\alpha_1, \alpha_2, \ldots, \alpha_n$ are the (arbitrarily chosen) coefficients of $A_1, A_2, \ldots, A_n$ which are not zeros. Choose $\lambda_1, \lambda_2, \ldots, \lambda_\mu$ in the previous diagram so that we have identically

$$X^{(0)} F_1 + X^{(1)} F_2 + \ldots + X^{(n-1)} F_n = A_1 F_1 + A_2 F_2 + \ldots + A_n F_n.$$

This gives, by equating coefficients of power products on both sides,

$$\Sigma\lambda a = \Sigma\alpha a', \quad \Sigma\lambda b = \Sigma\alpha b', \ldots, \quad \Sigma\lambda k = \Sigma\alpha k'$$

as equations for $\lambda_1, \lambda_2, \ldots, \lambda_\mu$; and they have a unique solution, since D does not vanish.

Let $\begin{pmatrix}\lambda\\ \alpha\end{pmatrix}$ denote the determinant of the substitution corresponding to the solution of the above equations for $\lambda_1, \lambda_2, \ldots, \lambda_\mu$ as linear functions of $\alpha_1, \alpha_2, \ldots, \alpha_\mu$. Then if we put

$$\Sigma\lambda a = \Sigma\alpha a' = \lambda_1', \quad \Sigma\lambda b = \Sigma\alpha b' = \lambda_2', \quad \ldots, \quad \Sigma\lambda k = \Sigma\alpha k' = \lambda_\mu'$$

we have

$$\begin{pmatrix}\lambda'\\ \lambda\end{pmatrix} = D, \quad \begin{pmatrix}\lambda'\\ \alpha\end{pmatrix} = D', \quad \text{and} \quad \begin{pmatrix}\lambda'\\ \lambda\end{pmatrix}\begin{pmatrix}\lambda\\ \alpha\end{pmatrix} = \begin{pmatrix}\lambda'\\ \alpha\end{pmatrix}, \quad \text{i.e. } D\begin{pmatrix}\lambda\\ \alpha\end{pmatrix} = D',$$

by the rule of successive substitutions, or the rule for multiplying determinants. Hence

$$\frac{D'}{D} = \begin{pmatrix}\lambda\\ \alpha\end{pmatrix}.$$

Now we can find the solution for $\lambda_1, \lambda_2, \ldots, \lambda_\mu$, or the solution of

$$X^{(0)} F_1 + X^{(1)} F_2 + \ldots + X^{(n-1)} F_n = A_1 F_1 + A_2 F_2 + \ldots + A_n F_n,$$

in the following way. First solve the equation

$$Y^{(0)} F_1 + Y^{(1)} F_2 + \ldots + Y^{(n-2)} F_{n-1} + X^{(n-1)} = A_n$$

for the unknowns $Y^{(0)}, Y^{(1)}, \ldots, Y^{(n-2)}, X^{(n-1)}$. This equation has a unique solution, since the more particular equation

$$Y^{(0)} x_1^{l_1} + Y^{(1)} x_2^{l_2} + \ldots + Y^{(n-2)} x_{n-1}^{l_{n-1}} + X^{(n-1)} = A_n$$

has a unique solution (for any given polynomial A_n can be expressed in one and only one way in the form on the left) and shows that the number of the coefficients of $Y^{(0)}, Y^{(1)}, \ldots, Y^{(n-2)}, X^{(n-1)}$ is equal to the number of equations they have to satisfy.

Substituting the value thus found for $X^{(n-1)}$ in the equation

$$X^{(0)} F_1 + X^{(1)} F_2 + \ldots + X^{(n-1)} F_n = A_1 F_1 + A_2 F_2 + \ldots + A_n F_n,$$

it becomes

$$\begin{aligned} X^{(0)} F_1 + X^{(1)} F_2 + \ldots + X^{(n-2)} F_{n-1} \\ = (A_1 + Y^{(0)} F_n) F_1 + \ldots + (A_{n-1} + Y^{(n-2)} F_n) F_{n-1}, \end{aligned}$$

where $Y^{(0)}, Y^{(1)}, \ldots, Y^{(n-2)}$ have been found. Next solve the equation

$$Z^{(0)} F_1 + Z^{(1)} F_2 + \ldots + Z^{(n-3)} F_{n-2} + X^{(n-2)} = A_{n-1} + Y^{(n-2)} F_n,$$

which has a unique solution for $Z^{(0)}, Z^{(1)}, \ldots, Z^{(n-3)}, X^{(n-2)}$. We can proceed in this way till $X^{(0)}, X^{(1)}, \ldots, X^{(n-1)}$, i.e. $\lambda_1, \lambda_2, \ldots, \lambda_\mu$, have all been found.

In this method of solving the unknowns on the left are associated with $F_1, F_2, \ldots, F_{n-1}$ only and not with F_n. Hence $\binom{\lambda}{a}$ is a rational function of the coefficients of $F_1, F_2, \ldots, F_n$ whose denominator is independent of the coefficients of F_n, and the same is therefore true of $\frac{D'}{D} = \binom{\lambda}{a}$. Hence every determinant D' of the array has a factor in common with D which is of degree L_n in the coefficients of F_n. The resultant R, which is the H.C.F. of all the determinants D', is therefore of degree L_i in the coefficients of F_i $(i = 1, 2, \ldots, n)$.

If we put $D = AR$, A is called the *extraneous factor* of D. We have proved that A is independent of the coefficients of F_n; and it is proved at the end of § 8 that A depends only on the coefficients of $(F_1, F_2, \ldots, F_{n-1})_{x_n=0}$.

8. Properties of the Resultant. Since D has a term $a_1^{\mu_1} \ldots a_n^{\mu_n}$ (§ 6) *R has a term $a_1^{L_1} a_2^{L_2} \ldots a_n^{L_n}$.* This is called the *leading term* of R.

Since *D vanishes when $c_1, c_2, \ldots, c_n$ all vanish* (§ 6) *the same is true of R*; for $D = AR$ and A is independent of $c_1, c_2, \ldots, c_n$.

The extraneous factor A of D is a minor of D, viz. the minor obtained by omitting all the columns of D corresponding to power *products reduced in* $n-1$ *of the variables and the rows which contain the elements* $a_1, a_2, \ldots, a_n$ *in the omitted columns* (M_2, *p.* 14). *Thus* D/A, *where* A *is this minor of* D, *is an explicit expression for* R.

Each coefficient a of $F_1, F_2, \ldots, F_n$ is said to have a certain numerical *weight*, equal to the index of the power of one particular variable (say x_n) in the term of which a is the coefficient. In the case of non-homogeneous polynomials the variable chosen is generally the variable x_0 of homogeneity. Also the weight of a^p is defined as p times the weight of a, and the weight of $a^p b^q c^r \ldots$ as the sum of the weights of $a^p, b^q, c^r, \ldots$. A whole function of the coefficients is said to be *isobaric* when all its terms are of the same weight.

The resultant is isobaric and of weight L. Assign to $x_1, x_2, \ldots, x_n$ the weights $0, 0, \ldots, 0, 1$. Then the coefficients of $F_1, F_2, \ldots, F_n$ have the same weights as the power products of which they are the coefficients. The ith row of the determinant D represents the polynomial $\omega_i F_j = a_i \omega_1^{(l+1)} + b_i \omega_2^{(l+1)} + \ldots + k_i \omega_\mu^{(l+1)}$. Thus the weights of $a_i, b_i, \ldots, k_i$ are less than the weights of $\omega_1^{(l+1)}, \omega_2^{(l+1)}, \ldots, \omega_\mu^{(l+1)}$ respectively by the same amount, viz. the weight of ω_i. Hence, on expanding D, the weight of any term is less than the sum of the weights of $\omega_1^{(l+1)}, \omega_2^{(l+1)}, \ldots, \omega_\mu^{(l+1)}$ by the sum of the weights of $\omega_1, \omega_2, \ldots, \omega_\mu$; i.e. D is isobaric. Again, if in D each letter a is changed to au^q, where q is the weight of a, D becomes Du^w, where w is the weight of D; and consequently if D be expressed as a product of whole factors each factor must be isobaric. Thus R is isobaric and its weight is that of its leading term $a_1^{L_1} a_2^{L_2} \ldots a_n^{L_n}$, which is $l_n L_n = L$. The weight of D is the weight of $a_1^{\mu_1} a_2^{\mu_2} \ldots a_n^{\mu_n}$, which is also L, since $\mu_n = L_n$.

The whole coefficient of $a_1^{L_1} a_2^{L_2} \ldots a_{n-1}^{L_{n-1}}$ *in* R *is* $a_n^{L_n}$. For the coefficient must be a whole function of the coefficients of F_n only of degree L_n and weight $l_n L_n$, and a_n is the only coefficient of F_n of weight l_n.

A more general result (§ 9) is that *the whole coefficient of* $a_n^{L_n}$ *in* R *is* $R_n^{l_n}$ *where* R_n *is the resultant of* $(F_1, F_2, \ldots, F_{n-1})_{x_n=0}$. Hence also *the whole coefficient of* $a_r^{L_r} a_{r+1}^{L_{r+1}} \ldots a_n^{L_n}$ *is* $R_r^{l_r l_{r+1} \ldots l_n}$ *where* R_r *is the resultant of* $(F_1, F_2, \ldots, F_{r-1})_{x_r = \ldots = x_n = 0}$.

Since the weights of D and R are the same *the weight of the extraneous factor* A *of* D *is zero*. This, taken in conjunction with the fact that A is independent of the coefficients of F_n, shows that A *is a whole function of the coefficients of* $(F_1, F_2, \ldots, F_{n-1})_{x_n=0}$ *only*.

9. *The resultant of $F_1, F_2, \ldots, F_n$ is irreducible and invariant.* It has been proved that the resultant is irreducible when $n = 2$ (§ 3); and the proof can be extended to the general case by induction.

Let $R_n =$ the resultant of $(F_1, F_2, \ldots, F_{n-1})_{x_n=0}$;

$F_0 =$ the resultant of the homogeneous polynomials $F_1^{(0)}, F_2^{(0)}, \ldots, F_{n-1}^{(0)}$ in $x_1, x_2, \ldots, x_{n-2}, x_0$ obtained from $F_1, F_2, \ldots, F_{n-1}$ by changing x_{n-1}, x_n to $x_{n-1}x_0, x_nx_0$;

$F_n' = (F_n)_{x_1=\ldots=x_{n-2}=0} = k_n x_{n-1}^{l_n} + \ldots + a_n x_n^{l_n}$;

$R' =$ the resultant of $F_1, F_2, \ldots, F_{n-1}, F_n'$;

$R_0 =$ the resultant of F_0, F_n', two polynomials in x_{n-1}, x_n;

$L_1' l_1 = L_2' l_2 = \ldots = L'_{n-1} l_{n-1} = l_1 l_2 \ldots l_{n-1} = L_n$.

Finally let $a_1, a_2, \ldots, a_n, c_1, c_2, \ldots, c_{n-1}$ denote the same coefficients of $F_1, F_2, \ldots, F_n$ as in § 6. We assume R_n irreducible and have to prove that R is irreducible.

F_0 is of weight L_n in the coefficients of $F_1^{(0)}, F_2^{(0)}, \ldots, F_{n-1}^{(0)}$ and each coefficient is a homogeneous polynomial in x_{n-1}, x_n of degree equal to its weight in reference to x_0. Hence

$$F_0 = Ax_{n-1}^{L_n} + Bx_{n-1}^{L_n - 1} x_n + \ldots,$$

where $A, B, \ldots$ are whole functions of the coefficients of $F_1, F_2, \ldots, F_{n-1}$ of the same dimensions as R_n. When $x_n = 0$, F_0 becomes the resultant of $(F_1^{(0)}, F_2^{(0)}, \ldots, F_{n-1}^{(0)})_{x_n=0}$, viz. $R_n x_{n-1}^{L_n}$; hence $A = R_n$. Also the whole coefficient of $a_1^{L_1'} a_2^{L_2'} \ldots a_{n-2}^{L'_{n-2}}$ in F_0 is $a'^{L'_{n-1}}_{n-1}$, where a'_{n-1} is the coefficient of $x_0^{l_{n-1}}$ in $F_{n-1}^{(0)}$ (§ 8), viz.

$$a'_{n-1} = a_{n-1} x_{n-1}^{l_{n-1}} + bx_{n-1}^{l_{n-1}-1} x_n + \ldots + c_{n-1} x_n^{l_{n-1}}.$$

Hence B has a term $L'_{n-1} a_1^{L_1'} \ldots a_{n-2}^{L'_{n-2}} a_{n-1}^{L'_{n-1}-1} b$, and cannot be divisible by R_n, since R_n does not involve b. Hence we find that

$$F_0 = R_n x_{n-1}^{L_n} + Bx_{n-1}^{L_n - 1} x_n + \ldots$$

where B is neither zero nor divisible by R_n.

Now if R' vanishes one of the solutions of $F_n' = 0$ for $x_{n-1} : x_n$ will be the same as in one of the solutions of $F_1 = \ldots = F_{n-1} = 0$ (§ 10), and will therefore be a solution of $F_0 = 0$; i.e. $R' = 0$ requires $R_0 = 0$, and R_0 is divisible by each irreducible factor of R'. But (§ 3)

$$R_0 = R_n^{l_n} a_n^{L_n} + B' a_n^{L_n - 1} + \ldots, \text{ where } B' \equiv (-1)^{l_n} k_n B^{l_n} \bmod R_n,$$

so that B' is neither zero nor divisible by R_n. Hence, as in § 3, R_0 has an irreducible factor of the form $R_n^{l_n} a_n^{p} + \ldots$, and has no other factor involving the coefficients of $F_1, F_2, \ldots, F_{n-1}$. This must therefore be a factor of R'.

Again R' is what R becomes when all the coefficients of F_n other than those of F_n' are put equal to zero. Hence R has an irreducible factor of the form $R_n^{l_n} a_n^q + \ldots$, where $q \geqslant p$. The remaining factor of R is independent of the coefficients of $F_1, F_2, \ldots, F_{n-1}$, and therefore also of the coefficients of F_n when $n > 2$. Hence R is irreducible.

It easily follows that R is invariant for a homogeneous linear substitution whose determinant $\left(\frac{x}{x'}\right)$ does not vanish. Suppose that $R=0$ and that this is the only relation existing between the coefficients of $F_1, F_2, \ldots, F_n$. Then not more than one relation can exist between the coefficients of $F_1', F_2', \ldots, F_n'$, the polynomials into which $F_1, F_2, \ldots, F_n$ transform. Since $R=0$ there are less than μ linearly independent members of $(F_1, F_2, \ldots, F_n)$ of degree $l+1$ and therefore less than μ linearly independent members of $(F_1', F_2', \ldots, F_n')$ of degree $l+1$, and the only single relation between the coefficients of $F_1', F_2', \ldots, F_n'$ which will admit this is $R'=0$. Hence $R=0$ requires $R'=0$, and R' is divisible by R. The remaining factor of R' is independent of the coefficients of $F_1, F_2, \ldots, F_n$, and can be shown to be $\left(\frac{x}{x'}\right)^L$. A proof that R is invariant without assuming it irreducible is given in (E, p. 17).

10. *The necessary and sufficient condition that the equations* $F_1 = F_2 = \ldots = F_n = 0$ *may have a proper solution is the vanishing of* R.

In the general case, when the coefficients are letters,

$$ARx_n^{l+1} \equiv 0 \bmod (F_1, F_2, \ldots, F_n).$$

Put $x_n = 1$ and change* c_i to $c_i - F_i$ $(i = 1, 2, \ldots, n)$; then A does not change, being independent of $c_1, c_2, \ldots, c_n$ (§ 8); but R changes to $R - A_1F_1 - A_2F_2 - \ldots - A_nF_n$, and this must vanish; hence

$$R \equiv 0 \bmod (F_1, F_2, \ldots, F_n)_{x_n=1}.$$

Hence R vanishes if the equations $F_1 = F_2 = \ldots = F_n = 0$ have a solution in which $x_n = 1$, i.e. if they have a proper solution.

To prove that $R=0$ is a sufficient condition, we shall assume that $R=0$ is the only relation existing between the coefficients of $F_1, F_2, \ldots, F_n$. There are then less than μ linearly independent members of $(F_1, F_2, \ldots, F_n)$ of degree $l+1$. Hence the coefficients $z_{p_1, p_2, \ldots, p_n}$ of

* Called the Kronecker substitution.

the general member of degree $l+1$ must satisfy an identical linear relation

$$\Sigma c_{p_1, p_2, \ldots, p_n} z_{p_1, p_2, \ldots, p_n} = 0, \quad p_1 + p_2 + \ldots + p_n = l + 1.$$

The coefficients of the general member of degree l also satisfy one *and only one* identical linear relation, whether R vanishes or not. To prove this it has to be shown that the number N of linearly independent members of $(F_1, F_2, \ldots, F_n)$ of degree l is 1 less than the number ρ of power products of degree l. If no relation exists between the coefficients of $F_1, F_2, \ldots, F_n$ the equation

$$X^{(0)}F_1 + X^{(1)}F_2 + \ldots + X^{(n-1)}F_n = A_1F_1 + A_2F_2 + \ldots + A_nF_n$$

can always be solved by the method of § 7, where $A_1, A_2, \ldots, A_n$ are arbitrary given polynomials. Hence N is not greater than the number of coefficients in $X^{(0)}, X^{(1)}, \ldots, X^{(n-1)}$, or in

$$X^{(0)}x_1^{l_1} + X^{(1)}x_2^{l_2} + \ldots + X^{(n-1)}x_n^{l_n},$$

viz. $\rho - 1$, since, when this expression is of degree l, every power product except $x_1^{l_1-1}x_2^{l_2-1} \ldots x_n^{l_n-1}$ occurs once and only once in it. Hence $N \leqslant \rho - 1$.

Any particularity in $F_1, F_2, \ldots, F_n$ can only affect the value of N by diminishing it. Hence for the remainder of the proof it will be sufficient to show that $N = \rho - 1$ in a particular example in which $R = 0$. Let

$$F_1 = (x_1 - x_2)\, x_1^{l_1-1}, \quad F_2 = (x_2 - x_3)\, x_2^{l_2-1}, \; \ldots, \quad F_n = (x_n - x_1)\, x_n^{l_n-1}.$$

Then $R = 0$ since the equations $F_1 = F_2 = \ldots = F_n = 0$ have the proper solution $x_1 = x_2 = \ldots = x_n = 1$. Let $x_1^{p_1}x_2^{p_2} \ldots x_n^{p_n}$ be any power product of degree l. If $p_1 \geqslant l_1$ change $x_1^{p_1}x_2^{p_2}$ to $x_1^{l_1-1}x_2^{q_2}$ where $p_1 + p_2 = l_1 - 1 + q_2$; this is equivalent to changing $x_1^{p_1}x_2^{p_2} \ldots x_n^{p_n}$ to $x_1^{p_1}x_2^{p_2} \ldots x_n^{p_n} + A_1F_1$. Again if $q_2 \geqslant l_2$ change $x_2^{q_2}x_3^{p_3}$ to $x_2^{l_2-1}x_3^{q_3}$; and if $q_2 < l_2$ proceed to the first $p_r \geqslant l_r$ and change $x_r^{p_r}x_{r+1}^{p_{r+1}}$ to $x_r^{l_r-1}x_{r+1}^{q_{r+1}}$. If we continue this process, going round the cycle $x_1, x_2, \ldots, x_n$ as many times as is necessary, the power product $x_1^{p_1}x_2^{p_2} \ldots x_n^{p_n}$ will eventually become changed to $x_1^{l_1-1}x_2^{l_2-1} \ldots x_n^{l_n-1}$. Hence these two power products are congruent mod $(F_1, F_2, \ldots, F_n)$, while neither of them is a member of $(F_1, F_2, \ldots, F_n)$, since they do not vanish when $x_1 = \ldots = x_n = 1$. Hence $N = \rho - 1$.

Let $F = \Sigma z_{q_1, q_2, \ldots, q_n} x_1^{q_1}x_2^{q_2} \ldots x_n^{q_n}$ be the general member of $(F_1, F_2, \ldots, F_n)$ of degree l; then x_iF is a member of degree $l+1$ in which the coefficient of $x_1^{p_1}x_2^{p_2} \ldots x_n^{p_n}$ is $z_{p_1, p_2, \ldots, p_i-1, \ldots, p_n}$. Hence

$$\Sigma c_{p_1, p_2, \ldots, p_n} z_{p_1, \ldots, p_i-1, \ldots, p_n} = 0 \quad (i = 1, 2, \ldots, n),$$

or

$$\Sigma c_{q_1, q_2, \ldots, q_i+1, \ldots, q_n} z_{q_1, q_2, \ldots, q_n} = 0 \quad (i = 1, 2, \ldots, n).$$

These n equations in $z_{q_1, q_2, \ldots, q_n}$ are therefore equivalent to one only; and the continued ratio $c_{q_1+1, q_2, \ldots, q_n} : c_{q_1, q_2+1, \ldots, q_n} : \ldots : c_{q_1, q_2, \ldots, q_n+1}$ is the same for all sets of values of $q_1, q_2, \ldots, q_n$ whose sum is l. Equating to $\alpha_1 : \alpha_2 : \ldots : \alpha_n$, it follows that $c_{p_1, p_2, \ldots, p_n}$ is proportional to $\alpha_1^{p_1} \alpha_2^{p_2} \ldots \alpha_n^{p_n}$ $(p_1 + p_2 + \ldots + p_n = l + 1)$. Hence it follows that $(\alpha_1, \alpha_2, \ldots, \alpha_n)$ is a solution of the equations $F_1 = F_2 = \ldots = F_n = 0$.

11. The Product Theorem. *If F_n is the product of two polynomials F_n', F_n'', the resultant R of $F_1, F_2, \ldots, F_n$ is the product of the resultants R', R'' of $F_1, F_2, \ldots, F_n'$ and $F_1, F_2, \ldots, F_n''$.*

For in the general case R' and R'' are irreducible, and if either vanishes R vanishes. Hence R is divisible by $R'R''$. Also it can be easily verified that the leading terms of R and $R'R''$ are identical. Hence $R = R'R''$.

This result can easily be extended to the case in which any or all of $F_1, F_2, \ldots, F_n$ resolve into two or more factors.

If $F_1, F_2, \ldots, F_n$ are all members of the module $(F_1', F_2', \ldots, F_n')$ the resultant R of $F_1, F_2, \ldots, F_n$ is divisible by the resultant R' of $F_1', F_2', \ldots, F_n'$. For if $R' = 0$ then $R = 0$.

12. Solution of Equations by means of the Resultant. The method of the resultant for solving equations can only be applied in what is called the principal case, that is, the case in which the number r of the equations is not greater than the number n of the unknowns, and the resultant F_0 of the equations with respect to $x_1, x_2, \ldots, x_{r-1}$ (after a linear substitution of the unknowns) does not vanish identically. When F_0 vanishes identically the method of the resultant fails, but the equations can be solved by the method of the resolvent, due to Kronecker, as explained later. The method of the resolvent is also applicable to any number of equations whether greater or less than the number of unknowns.

Homogeneous Equations. Let the equations be $F_1 = F_2 = \ldots = F_r = 0$ of degrees $l_1, l_2, \ldots, l_r$, where $r \leqslant n$. We assume that their resultant F_0 with respect to $x_1, x_2, \ldots, x_{r-1}$ does not vanish. We regard $x_1, x_2, \ldots, x_r$ as the unknowns, the solutions being functions of $x_{r+1}, \ldots, x_n$. But instead of solving for one of the unknowns $x_1, x_2, \ldots, x_r$ we solve for a linear combination of them, viz. for $x = u_1 x_1 + u_2 x_2 + \ldots + u_r x_r$,* where $u_1, u_2, \ldots, u_r$ are undetermined quantities. Let F_u stand for $x - u_1 x_1 - u_2 x_2 - \ldots - u_r x_r$. Then we regard $F_1 = F_2 = \ldots = F_r = F_u = 0$

* Called the Liouville substitution.

as the given system of equations with $x_1, \dots, x_r, x$ as unknowns, and their resultant $F_0^{(u)}$ with respect to $x_1, x_2, \dots, x_r$ gives the equation $F_0^{(u)}=0$ for x.

Definition. $F_0^{(u)}$ is called the *u-resultant* of $(F_1, F_2, \dots, F_r)$.

Applying the reasoning of § 9 it is seen that F_0 is the resultant (with respect to $x_1, \dots, x_{r-1}, x_0$) of $F_1, F_2, \dots, F_r$ when $x_r, x_{r+1}, \dots, x_n$ are changed to $x_r x_0, x_{r+1}x_0, \dots, x_n x_0$, and is a homogeneous polynomial in $x_r, x_{r+1}, \dots, x_n$ of degree $L=l_1 l_2 \dots l_r$, viz.

$$F_0 = R_{r+1}\, x_r^L + \dots,$$

where R_{r+1} is the resultant of $(F_1, F_2, \dots, F_r)_{x_{r+1}=\dots=x_n=0}$, and does not vanish; for a homogeneous substitution beforehand between $x_r, x_{r+1}, \dots, x_n$ only would be carried through to F_0.

Similarly $F_0^{(u)}$ is the resultant (with respect to $x_1, \dots, x_r, x_0$) of $F_1, F_2, \dots, F_r, F_u$ when $x, x_{r+1}, \dots, x_n$ are changed to $x x_0, x_{r+1} x_0, \dots, x_n x_0$, and is a homogeneous polynomial $R'_{r+1}x^L + \dots$ in $x, x_{r+1}, \dots, x_n$ where R'_{r+1} is the resultant of $(F_1, F_2, \dots, F_r, F_u)_{x_{r+1}=\dots=x_n=0}$. It is easily seen* that $R'_{r+1}=R_{r+1}$. Hence

$$F_0^{(u)} = R_{r+1}\, x^L + \dots, \text{ where } R_{r+1} \neq 0.$$

To each solution $x_r = x_{ri}$ of $F_0=0$ corresponds a solution $(x_{1i}, x_{2i}, \dots, x_{ri})$ of the equations $F_1=F_2=\dots=F_r=0$ for $x_1, x_2, \dots, x_r$ (§ 10). There are therefore L solutions altogether, and they are all finite, since $R_{r+1}\neq 0$.

Similarly to each of the L solutions $x=x_i$ of $F_0^{(u)}=0$ there corresponds a solution $(x_{1i}, x_{2i}, \dots, x_{ri}, x_i)$ of $F_1=\dots=F_r=F_u=0$; and as regards $(x_{1i}, x_{2i}, \dots, x_{ri})$ the L solutions must be the same as those obtained by solving $F_0=0$. Hence it follows that

$$x_i = u_1 x_{1i} + u_2 x_{2i} + \dots + u_r x_{ri},$$

where $x_{1i}, x_{2i}, \dots, x_{ri}$ are independent of $u_1, u_2, \dots, u_r$. Hence

$$F_0^{(u)} = R_{r+1}\, \Pi\, (x - u_1 x_{1i} - \dots - u_r x_{ri}) \quad (i=1, 2, \dots, L).$$

Thus $F_0^{(u)}$ *is a product of* L *factors which are linear in* $x, u_1, u_2, \dots, u_r$, *and the coefficients of* $u_1, u_2, \dots, u_r$ *in each factor supply a solution of the equations* $F_1=F_2=\dots=F_r=0$.

Also the number of solutions is either $L=l_1 l_2 \dots l_r$ *or infinite, the latter being the case when* F_0 *vanishes identically.*

* By introducing a as coefficient of x in F_u it is seen that R'_{r+1} is divisible by a^L by considering weight with respect to x. Also the whole coefficient of a^L in R'_{r+1} is R_{r+1} (§ 8). Hence $R'_{r+1}=a^L R_{r+1}=R_{r+1}$.

If D_u is the determinant for $(F_1, F_2, \ldots, F_r, F_u)$, regarding $x_1, x_2, \ldots, x_r, x_0$ as the variables, like the D of § 6, we have $D_u = AF_0^{(u)}$. The extraneous factor A depends only on the coefficients of $(F_1, F_2, \ldots, F_r)_{x_0=0}$, that is, of $(F_1, F_2, \ldots, F_r)_{x_{r+1}=\ldots=x_n=0}$. *Hence A is a pure constant, independent of $x_{r+1}, \ldots, x_n$ and of $u_1, u_2, \ldots, u_r$, and we may take $D_u = 0$ as the equation for x.*

Definition. The number of times a linear factor $x - u_1x_{1i} - \ldots - u_rx_{ri}$ is repeated in $F_0^{(u)}$ or D_u is called the *multiplicity* of the solution $(x_{1i}, x_{2i}, \ldots, x_{ri})$. This term has a definite geometrical interpretation; it is the number of solutions or points, in the general case distinct, which come into coincidence with a particular solution or point in the particular example considered.

In the case of n homogeneous equations in n unknowns such that $R \neq 0$, the complete solution consists of the non-proper solution $(0, 0, \ldots, 0)$ with multiplicity $L = l_1 l_2 \ldots l_n$.

Non-homogeneous Equations. In the case of non-homogeneous equations a linear substitution beforehand affects only $x_1, x_2, \ldots, x_n$ and not the variable x_0 of homogeneity. Hence it is possible for R_{r+1} to vanish identically, while F_0 and $F_0^{(u)}$ do not, no matter what the original substitution may be. In this case there is a diminution in the number of finite solutions for x, but not in the number of linear factors of $F_0^{(u)}$. To a factor $u_1x_{1i} + u_2x_{2i} + \ldots + u_rx_{ri}$ of $F_0^{(u)}$ not involving x corresponds what is called an infinite solution of $F_1 = F_2 = \ldots = F_r = 0$ in the ratio $x_{1i} : x_{2i} : \ldots : x_{ri}$. Infinite solutions are however non-existent in the theory of modular systems (§ 42). An extreme case is that in which $F_0^{(u)}$ does not vanish identically, but is independent of x, when all the L solutions are at infinity.

It may happen that a system of non-homogeneous equations has only a finite number of finite solutions while the resultant F_0 vanishes identically. In such a case the method of the resultant fails to give the solutions.

Example. The equations $x_1^2 = x_2 + x_1x_3 = x_3 + x_1x_2 = 0$ have the finite solution $x_1 = x_2 = x_3 = 0$; but the resultant vanishes identically because the corresponding homogeneous equations

$$x_1^2 = x_0x_2 + x_1x_3 = x_0x_3 + x_1x_2 = 0$$

are satisfied by $x_0 = x_1 = 0$, a system of two independent equations only.