PART III

SYMBOLIC NOTATION

THE NOTATION AND ITS IMMEDIATE CONSEQUENCES, §§ 39–41

39. Introduction. The conditions that the binary cubic

(1) $$f = a_0x_1^3 + 3a_1x_1^2x_2 + 3a_2x_1x_2^2 + a_3x_2^3$$

shall be a perfect cube

(2) $$(\alpha_1x_1 + \alpha_2x_2)^3$$

are found by eliminating α_1 and α_2 between

(3) $$\alpha_1^3 = a_0, \qquad \alpha_1^2\alpha_2 = a_1, \qquad \alpha_1\alpha_2^2 = a_2, \qquad \alpha_2^3 = a_3,$$

and hence the conditions are

(4) $$a_0a_2 = a_1^2, \qquad a_1a_3 = a_2^2.$$

Thus only a very special form (1) is a perfect cube.

However, in a symbolic sense * any form (1) can be represented as a cube (2), in which α_1 and α_2 are now mere symbols such that

(3′) $$\alpha_1^3, \qquad \alpha_1^2\alpha_2, \qquad \alpha_1\alpha_2^2, \qquad \alpha_2^3$$

are given the interpretations (3), while any linear combination of these products, as $2\alpha_1^3 - 7\alpha_2^3$, is interpreted to be the corresponding combination of the a's, as $2a_0 - 7a_3$. But no interpretation is given to a polynomial in α_1, α_2, any one of whose terms is a product of more than three factors α, or fewer than three factors α. Thus the first relation (4) does not now follow from (3), since the expression $\alpha_1^4\alpha_2^2$ (formerly equal to both

* Due to Aronhold and Clebsch, but equivalent to the more complicated hyperdeterminants of Cayley.

a_0a_2 and a_1^2) is now excluded from consideration; likewise for $\alpha_1^2\alpha_2^4$ and the second relation (4).

In brief, the general binary cubic (1) may be represented in the symbolic form (2) since the products (3′) of the symbols α_1, α_2 are in effect independent quantities, in so far as we permit the use only of linear combinations of these products.

But we shall of course have need of other than linear functions of $a_0, \ldots, a_3$. To be able to express them symbolically, we represent f not merely by (2), but also in the symbolic forms

$$(5) \qquad (\beta_1x_1+\beta_2x_2)^3, \qquad (\gamma_1x_1+\gamma_2x_2)^3, \ldots,$$

so that

$$(6) \quad \beta_1^3=a_0, \quad \beta_1^2\beta_2=a_1, \quad \beta_1\beta_2^2=a_2, \quad \beta_2^3=a_3; \quad \gamma_1^3=a_0, \ldots .$$

Thus a_0a_2 is represented by either $\alpha_1^3\beta_1\beta_2^2$ or $\beta_1^3\alpha_1\alpha_2^2$, while neither of them is identical with the representation $\alpha_1^2\alpha_2\beta_1^2\beta_2$ of a_1^2. Hence

$$a_0a_2-a_1^2=\tfrac{1}{2}(\alpha_1^3\beta_1\beta_2^2+\beta_1^3\alpha_1\alpha_2^2-2\alpha_1^2\alpha_2\beta_1^2\beta_2)$$
$$=\tfrac{1}{2}\alpha_1\beta_1(\alpha_1\beta_2-\alpha_2\beta_1)^2.$$

We shall verify that this expression is a seminvariant of f. If

$$x_1=X_1+tX_2, \qquad x_2=X_2,$$

then f becomes $F=A_0X_1^3+\ldots$, where

$$A_0=a_0, \quad A_1=a_1+ta_0, \quad A_2=a_2+2ta_1+t^2a_0,$$
$$A_3=a_3+3ta_2+3t^2a_1+t^3a_0.$$

Hence, by (3),

$$F=(\alpha_1X_1+\alpha'_2X_2)^3, \qquad \alpha'_2=\alpha_2+t\alpha_1.$$

Similarly, the transform of (5_1) is

$$(\beta_1X_1+\beta'_2X_2)^3, \qquad \beta'_2=\beta_2+t\beta_1.$$

Hence we obtain the desired result

$$A_0A_2-A_1^2=\tfrac{1}{2}\alpha_1\beta_1(\alpha_1\beta'_2-\alpha'_2\beta_1)^2$$
$$=\tfrac{1}{2}\alpha_1\beta_1(\alpha_1\beta_2-\alpha_2\beta_1)^2=a_0a_2-a_1^2.$$

40. General Notations. The binary n-ic

$$f=a_0x_1^n+na_1x_1^{n-1}x_2+\dots+\binom{n}{k}a_kx_1^{n-k}x_2^k+\dots+a_nx_2^n$$

is represented symbolically as $\alpha_x^n=\beta_x^n=\dots$, where

$$\alpha_x=\alpha_1x_1+\alpha_2x_2, \qquad \beta_x=\beta_1x_1+\beta_2x_2, \dots,$$

$$\alpha_1^n=a_0, \quad \alpha_1^{n-1}\alpha_2=a_1, \dots, \quad \alpha_1^{n-k}\alpha_2^k=a_k, \quad \dots,$$

$$\alpha_2^n=a_n; \qquad \beta_1^n=a_0, \dots.$$

A product involving fewer than n or more than n factors α_1, α_2 is not employed except, of course, as a component of a product of n such factors.

The general binary linear transformation is denoted by

$$T: \qquad x_1=\xi_1X_1+\eta_1X_2, \qquad x_2=\xi_2X_1+\eta_2X_2, \qquad (\xi\eta)\neq 0,$$

where $(\xi\eta)=\xi_1\eta_2-\xi_2\eta_1$. It is an important principle of computation, verified for a special case at the end of § 39, that T transforms α_x^n into the nth power of the linear function

$$(\alpha_1\xi_1+\alpha_2\xi_2)X_1+(\alpha_1\eta_1+\alpha_2\eta_2)X_2=\alpha_\xi X_1+\alpha_\eta X_2,$$

which is the transform of α_x by T. Further,

$$(1) \qquad \begin{vmatrix}\alpha_\xi & \alpha_\eta\\ \beta_\xi & \beta_\eta\end{vmatrix}=\begin{vmatrix}\alpha_1 & \alpha_2\\ \beta_1 & \beta_2\end{vmatrix}\cdot\begin{vmatrix}\xi_1 & \eta_1\\ \xi_2 & \eta_2\end{vmatrix}=(\alpha\beta)(\xi\eta),$$

where $(\alpha\beta)=\alpha_1\beta_2-\alpha_2\beta_1=-(\beta\alpha)$. Thus

$$(\alpha_\xi\beta_\eta-\alpha_\eta\beta_\xi)^n=(\xi\eta)^n(\alpha\beta)^n,$$

so that $(\alpha\beta)^n$ is an invariant of $\alpha_x^n=\beta_x^n$ of index n. Since $(\beta\alpha)^n$ represents the same invariant, the invariant is identically zero if n is odd.

EXERCISES

1. $(\alpha\beta)^2$ is the invariant $2(a_0a_2-a_1^2)$ of $\alpha_x^2=\beta_x^2$.
2. $(\alpha\beta)^4$ is the invariant $2I$ of $\alpha_x^4=\beta_x^4$ (§ 31).
3. $(\alpha\beta)^2(\beta\gamma)^2(\gamma\alpha)^2$ is the invariant $6J$ of $\alpha_x^4=\beta_x^4=\gamma_x^4$ (§ 31).
4. The Jacobian of α_x^m and β_x^n is

$$\begin{vmatrix}m\alpha_x^{m-1}\alpha_1 & m\alpha_x^{m-1}\alpha_2\\ n\beta_x^{n-1}\beta_1 & n\beta_x^{n-1}\beta_2\end{vmatrix}=mn(\alpha\beta)\alpha_x^{m-1}\beta_x^{n-1}.$$

5. The quotient of the Hessian of $\alpha_x^n=\beta_x^n$ by $n^2(n-1)^2$ equals

$$\begin{vmatrix} \alpha_x^{n-2}\alpha_1^2 & \alpha_x^{n-2}\alpha_1\alpha_2 \\ \beta_x^{n-2}\beta_1\beta_2 & \beta_x^{n-2}\beta_2^2 \end{vmatrix} = \begin{vmatrix} \beta_x^{n-2}\beta_1^2 & \beta_x^{n-2}\beta_1\beta_2 \\ \alpha_x^{n-2}\alpha_1\alpha_2 & \alpha_x^{n-2}\alpha_2^2 \end{vmatrix},$$

one-half of the sum of which equals $\frac{1}{2}\,\alpha_x^{n-2}\beta_x^{n-2}(\alpha\beta)^2$.

6. $$\begin{vmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_x & \beta_x & \gamma_x \end{vmatrix} = (\alpha\beta)\gamma_x+(\beta\gamma)\alpha_x+(\gamma\alpha)\beta_x\equiv 0.$$

41. Evident Covariants. We obtain a covariant K of

$$f=\alpha_x^n=\beta_x^n=\ldots$$

by taking a product of ω factors of type α_x and λ factors of type $(\alpha\beta)$, such that α occurs in exactly n factors, β in exactly n factors, etc. On the one hand, the product can be interpreted as a polynomial in $a_0, \ldots, a_n, x_1, x_2$. On the other hand, the product is a covariant of index λ of f, since, by (1), § 40,

$$(AB)^r(AC)^s(BC)^t\ldots A_X^aB_X^bC_X^c\ldots$$
$$=(\xi\eta)^\lambda(\alpha\beta)^r(\alpha\gamma)^s(\beta\gamma)^t\ldots\alpha_x^a\beta_x^b\gamma_x^c\ldots,$$

if $\lambda=r+s+t+\ldots$ and

$$A_X=A_1X_1+A_2X_2,\quad A_1=\alpha_\xi,\quad A_2=\alpha_\eta,\quad (AB)=A_1B_2-A_2B_1,$$

etc. The total degree of the right member in the α's, β's, ... is $2\lambda+\omega=nd$, if d is the number of distinct pairs of symbols α_1, α_2; β_1, β_2; ... in the product. Evidently d is the degree of K in $a_0, a_1, \ldots$, and ω is its order in x_1, x_2.

Any linear combination of such products with the same ω and λ, and hence same d, is a covariant of order ω, index λ and degree d of f.

EXERCISES

1. $(\alpha\beta)(\alpha\gamma)\alpha_x^3\beta_x^4\gamma_x^4$ and $(\alpha\beta)^2(\alpha\gamma)\alpha_x^2\beta_x^3\gamma^4_x$ are covariants of $\alpha_x^5=\beta_x^5=\gamma_x^5$.
2. $(\alpha\beta)^r\alpha_x^{n-r}\beta_x^{m-r}$ is a covariant of α_x^n, β_x^m.
3. If $m=n$, $\beta_x^n=\alpha_x^n$ and r is odd, the last covariant is identically zero.
4. $a_0x_1^2+2a_1x_1x^2+a_2x_2^2$ and $b_0x_1^2+2b_1x_1x_2+b_2x_2^2$ have the invariant

$$(\alpha\beta)^2=a_0b_2-2a_1b_1+a_2b_0.$$

COVARIANTS AS FUNCTIONS OF TWO SYMBOLIC TYPES, §§ 42–45

42. Any Covariant is a Polynomial in the α_x, $(\alpha\beta)$. This fundamental theorem, due to Clebsch, justifies the symbolic notation. It shows that any covariant can be expressed in a simple notation which reveals at sight the covariant property.

While a similar result was accomplished by expressing covariants in terms of the roots (§ 36), manipulations with symmetric functions of the roots are usually far more complex than those with our symbolic expressions.

The nature of the proof will be clearer if first made for a special case. The binary quadratic α_x^2 has the invariant

$$K = a_0 a_2 - a_1^2$$

of index 2. Under transformation T of § 40, α_x^2 becomes

$$(\alpha_\xi X_1 + \alpha_\eta X_2)^2 = A_0 X_1^2 + \ldots, \quad A_0 = \alpha_\xi^2, \quad A_1 = \alpha_\xi \alpha_\eta, \quad A_2 = \alpha_\eta^2.$$

Hence $A_0 A_2 - A_1^2$ equals

$$\alpha_\xi^2 \beta_\eta^2 - \alpha_\xi \beta_\xi \alpha_\eta \beta_\eta = (\xi\eta)^2 K.$$

We operate on each member twice with

(1) $$V = \frac{\partial^2}{\partial \xi_1 \partial \eta_2} - \frac{\partial^2}{\partial \xi_2 \partial \eta_1},$$

and prove that we get $6(\alpha\beta)^2 = 12K$, so that K is expressed in the desired symbolic form. We have

$$(\xi\eta) = \xi_1 \eta_2 - \xi_2 \eta_1,$$

$$\frac{\partial}{\partial \eta_2}(\xi\eta)^2 = 2(\xi\eta)\xi_1, \qquad \frac{\partial^2}{\partial \xi_1 \partial \eta_2}(\xi\eta)^2 = 2(\xi\eta) + 2\eta_2 \xi_1,$$

$$\frac{\partial}{\partial \eta_1}(\xi\eta)^2 = -2(\xi\eta)\xi_2, \quad \frac{\partial^2}{\partial \xi_2 \partial \eta_1}(\xi\eta)^2 = -2(\xi\eta) + 2\eta_1 \xi_2,$$

$$V(\xi\eta)^2 = 6(\xi\eta), \qquad V^2(\xi\eta)^2 = 12,$$

since $V(\xi\eta) = 2$, by inspection. Next

(2) $$V\alpha_\xi \beta_\eta = V(\alpha_1 \xi_1 + \alpha_2 \xi_2)(\beta_1 \eta_1 + \beta_2 \eta_2) = \alpha_1 \beta_2 - \alpha_2 \beta_1 = (\alpha\beta).$$

Hence

$$V\alpha_\xi^2\beta_\eta^2=4\alpha_\xi\beta_\eta(\alpha\beta),\quad V^2\alpha_\xi^2\beta_\eta^2=4(\alpha\beta)^2,$$

$$V\alpha_\xi\beta_\xi\alpha_\eta\beta_\eta=\beta_\xi\alpha_\eta\cdot V\alpha_\xi\beta_\eta+\alpha_\xi\beta_\eta\cdot V\beta_\xi\alpha_\eta$$

$$=\beta_\xi\alpha_\eta(\alpha\beta)+\alpha_\xi\beta_\eta(\beta\alpha),$$

$$V^2\alpha_\xi\beta_\xi\alpha_\eta\beta_\eta=(\beta\alpha)(\alpha\beta)+(\alpha\beta)(\beta\alpha)=-2(\alpha\beta)^2.$$

The difference of the expressions involving V^2 is $6(\alpha\beta)^2$. Hence if (1) operates twice on the equation preceding it, the result is

$$6(\alpha\beta)^2=12K,\qquad K=\tfrac{1}{2}(\alpha\beta)^2.$$

43. Lemma. $V^n(\xi\eta)^n=(n+1)(n!)^2.$

We have proved this for $n=1$ and $n=2$. If $n\geq 2$,

$$\frac{\partial}{\partial\eta_2}(\xi\eta)^n=n(\xi\eta)^{n-1}\xi_1,$$

$$\frac{\partial^2}{\partial\xi_1\partial\eta_2}(\xi\eta)^n=n(\xi\eta)^{n-1}+n(n-1)(\xi\eta)^{n-2}\eta_2\xi_1.$$

Similarly, or by interchanging subscripts 1 and 2, we get

$$\frac{\partial^2}{\partial\xi_2\partial\eta_1}(\xi\eta)^n=-n(\xi\eta)^{n-1}+n(n-1)(\xi\eta)^{n-2}\eta_1\xi_2.$$

Subtracting, we get

$$V(\xi\eta)^n=\{2n+n(n-1)\}(\xi\eta)^{n-1}=n(n+1)(\xi\eta)^{n-1}.$$

It follows by induction that, if r is a positive integer,

$$V^r(\xi\eta)^n=(n+1)\{n(n-1)\ \ldots\ (n-r+2)\}^2(n-r+1)(\xi\eta)^{n-r}.$$

The case $r=n$ yields the Lemma.

44. Lemma. *If the operator V is applied r times to a product of k factors of the type α_ξ and l factors of the type β_η, there results a sum of terms each containing $k-r$ factors α_ξ, $l-r$ factors β_η, and r factors $(\alpha\beta)$.*

The Lemma is a generalization of (2), § 42. To prove it, set

$$A=\alpha_\xi^{(1)}\alpha_\xi^{(2)}\ \ldots\ \alpha_\xi^{(k)},\qquad B=\beta_\eta^{(1)}\beta_\eta^{(2)}\ \ldots\ \beta_\eta^{(l)}.$$

Then

$$\frac{\partial^2 AB}{\partial \xi_1 \partial \eta_2} = \sum_{s=1}^{k} \sum_{t=1}^{l} \alpha_1^{(s)} \beta_2^{(t)} \frac{A}{\alpha_\xi^{(s)}} \frac{B}{\beta_\eta^{(t)}},$$

$$\frac{\partial^2 AB}{\partial \xi_2 \partial \eta_1} = \sum_{s=1}^{k} \sum_{t=1}^{l} \alpha_2^{(s)} \beta_1^{(t)} \frac{A}{\alpha_\xi^{(s)}} \frac{B}{\beta_\eta^{(t)}}.$$

Subtracting, we get

$$VAB = \sum_{s=1}^{k} \sum_{t=1}^{l} (\alpha^{(s)} \beta^{(t)}) \frac{A}{\alpha_\xi^{(s)}} \frac{B}{\beta_\eta^{(t)}}.$$

Hence the lemma is true when $r=1$. It now follows at once by induction that

(1) $\quad V^r AB$

$$= \Sigma\Sigma (\alpha^{(s_1)} \beta^{(t_1)}) \ldots (\alpha^{(s_r)} \beta^{(t_r)}) \frac{A}{\alpha_\xi^{(s_1)} \ldots \alpha_\xi^{(s_r)}} \frac{B}{\beta_\eta^{(t_1)} \ldots \beta_\eta^{(t_r)}},$$

where the first summation extends over all of the $k(k-1) \ldots (k-r+1)$ permutations $s_1, \ldots, s_r$ of $1, \ldots, k$ taken r at a time, and the second summation extends over all of the $l(l-1) \ldots (l-r+1)$ permutations $t_1, \ldots, t_r$ of $1, \ldots, l$ taken r at a time.

Corollary. The terms of (1) coincide in sets of $r!$ and the number of formally distinct terms is

$$\frac{k!}{(k-r)!} \cdot \frac{l!}{(l-r)!} \cdot \frac{1}{r!} = \binom{k}{r} \binom{l}{r} \cdot r!.$$

For, we obtain the same product of determinantal factors if we rearrange $s_1, \ldots, s_r$ and make the same rearrangement of $t_1, \ldots, t_r$.

45. Proof of the Fundamental Theorem in § 42. Let K be a homogeneous covariant of order ω and index λ of the binary form f in § 40. By § 40, the general linear transformation replaces $f = \alpha_x^n$ by

$$\sum_{k=0}^{n} \binom{n}{k} A_k X_1^{n-k} X_2^k = (\alpha_\xi X_1 + \alpha_\eta X_2)^n.$$

Hence

(1) $$A_k = \alpha_\xi^{n-k} \alpha_\eta^k \qquad (k=0, 1, \ldots, n).$$

By the covariance of K,

(2) $K(A_0, \ldots, A_n;\ X_1, X_2) = (\xi\eta)^\lambda K(a_0, \ldots, a_n;\ x_1, x_2).$

By (1) the left member equals

$$\sum_{i=0}^{\omega} \Sigma ABX_1^{\omega-i}X_2^i,$$

in which the inner summation extends over various products AB, where A is a product of a constant and factors of type α_ξ, and B is a product of a constant and factors of type α_η. Let $x_1 = y_2$, and $x_2 = -y_1$. Then, by solving the equations of T, § 40,

$$X_1 = y_\eta/(\xi\eta), \qquad X_2 = -y_\xi/(\xi\eta).$$

Hence the equation (2) becomes

$$\sum_{i=0}^{\omega} \Sigma(-1)^i ABy_\eta^{\omega-i}y_\xi^i = (\xi\eta)^{\lambda+\omega}K.$$

Since the right member is of degree $\lambda+\omega$ in ξ_1, ξ_2, and of degree $\lambda+\omega$ in η_1, η_2, we infer that each term of the left member involves exactly $\lambda+\omega$ factors with subscript ξ and $\lambda+\omega$ factors with subscript η.

Operate with $V^{\lambda+\omega}$ on each member. By § 43, the right member becomes cK, where c is a numerical constant $\neq 0$. By § 44, the left member becomes a sum of products each of $\lambda+\omega$ determinantal factors of which ω are of type $(\alpha y) \equiv \alpha_x$, and hence λ of type $(\alpha\beta)$. The last is true also by the definiton of the index λ of K. Hence K equals a polynomial in the symbols of the types α_x, $(\alpha\beta)$.

To extend the proof to covariants of several binary forms α_x^n, γ_x^m, . . . , we employ, in addition to (1), $C_k = \gamma_\xi^{m-k}\gamma_\eta^k$, . . . and read α_ξ, γ_ξ, . . . for α_ξ in the above proof.

Finiteness of a Fundamental System of Covariants, §§ 46–51

46. Remarks on the Problem. It was shown in §§28–31 that a binary form f of order <5 has a finite fundamental system of rational integral covariants K_1, . . . , K_s, such therefore that any rational integral covariant of f is a polynomial in $K_1, \ldots, K_s$ with numerical coefficients. We shall now prove a like theorem for the covariants of any system of binary forms of any orders. The first proof was that by Gordan; it was based upon the symbolic notation and gave the means of actually constructing a fundamental system. Cayley had earlier come to the conclusion that the fundamental system for a binary quintic is infinite, after making a false assumption on the independence of the syzygies between the covariants. The proof reproduced here is one of those by Hilbert; it is merely an existence proof, giving no clue as to the actual covariants in a fundamental system.

47. Reduction of the Problem on Covariants to one on Invariants. We shall prove that the set of all covariants of the binary forms $f_1, \ldots, f_k$ is identical with the set of forms derived from the invariants I of $f_1, \ldots, f_k$ and $l \equiv xy' - x'y$ by replacing x' by x and y' by y in each I. It is here assumed (§ 15) that I is homogeneous in the coefficients of l and that the covariants are homogeneous in the variables.

Let the coefficients of the f's be $a, b, \ldots$, arranged in any sequence. Let $A, B, \ldots$ be the corresponding coefficients of the forms obtained by applying the transformation in § 5. The latter replaces l by $\xi\eta' - \xi'\eta$, where

$$\eta' = \alpha y' - \gamma x', \qquad \xi' = \delta x' - \beta y'.$$

Solving these, we get

$$\Delta x' = \alpha \xi' + \beta \eta', \qquad \Delta y' = \gamma \xi' + \delta \eta'.$$

Let $I(a, b, \ldots; x', y')$ be an invariant of l and the f's. Then

$$I(A, B, \ldots; \xi', \eta') = \Delta^\lambda I(a, b, \ldots; x', y').$$

Since I is homogeneous, of order ω, in x', y', the right member equals

$$\Delta^{\lambda - \omega} I(a, b, \ldots; \Delta x', \Delta y').$$

Hence we have the identity in ξ', η':

$$I(A, B, \ldots; \xi', \eta') \equiv \Delta^{\lambda - \omega} I(a, b, \ldots; \alpha\xi' + \beta\eta', \quad \gamma\xi' + \delta\eta').$$

Thus we may remove the accents on ξ', η'. Then, by our transformation,

$$I(A, B, \ldots; \xi, \eta) = \Delta^{\lambda-\omega} I(a, b, \ldots; x, y).$$

Hence $I(a, b, \ldots; x, y)$ is a covariant of $f_1, \ldots, f_k$ of order ω and index $\lambda - \omega$.

The argument can be reversed. Note that the sum of the order and the index of a covariant is its weight (§ 22) and hence is not negative.

COROLLARY. A covariant of the binary form f has the annhilators in § 23.

For, an invariant of f and $xy' - x'y$ has the annihilators

$$\Omega - y'\frac{\partial}{\partial x'}, \qquad O - x'\frac{\partial}{\partial y'}.$$

48. Hilbert's Theorem. *Any set S of forms in $x_1, \ldots, x_n$ contains a finite number of forms $F_1, \ldots, F_k$ such that any form F of the set can be expressed as $F \equiv f_1F_1 + \ldots + f_kF_k$, where $f_1, \ldots, f_k$ are forms in $x_1, \ldots, x_n$, but not necessarily in the set S.*

For $n=1$, S is composed of certain forms $c_1x^{e_1}$, $c_2x^{e_2}$, ... Let e_s be the least of the e's, and set $F_1 = c_sx^{e_s}$. Then each form in S is the product of F_1 by a factor of the form cx^e, $e \geqq 0$. Thus the theorem holds when $n=1$.

To proceed by induction, let the theorem hold for every set of forms in $n-1$ variables. To prove it for the system S, we may assume, without real loss of generality,* that S contains a form F_0 of total order r in which the coefficient of x_n^r is not zero. Let F be any form of the set S. By division we have $F = F_0P + R$, where R is a form whose order in x_n

* Let F be a form in S not identically zero and let the linear transformation

$$x_i = c_{i1}y_1 + c_{i2}y_2 + \ldots + c_{in}y_n \qquad (i=1, \ldots, n)$$

replace $F(x_1, \ldots, x_n)$ by $K(y_1, \ldots, y_n)$. In the latter the coefficient of the term involving only y_n is obtained from F by setting $x_i = c_{in}$ and hence is $F(c_{1n}, c_{2n}, \ldots, c_{nn})$, which is not zero for suitably chosen c's (Weber's *Algebra*, vol. I, p. 457; second edition, p. 147). But our theorem will be true for S if proved true for the set of forms K.

is $<r$. In R we segregate the terms whose order in x_n is exactly $r-1$, and have

$$F=F_0P+Mx_n^{r-1}+N,$$

where M is a form in $x_1, \ldots, x_{n-1}$, while N is a form in $x_1, \ldots, x_n$ whose order in x_n is $\leqq r-2$. Each F uniquely determines an M.

For the definite set of forms M in $n-1$ variables the theorem is true by hypothesis. Hence there exists a finite number of the M's, say $M_1, \ldots, M_l$ (derived from $F_1, \ldots, F_l$), such that any M can be expressed as

$$M=f_1M_1+\ldots+f_lM_l,$$

where the f's are forms in $x_1, \ldots, x_{n-1}$. Then

$$F=F_0P+N+x_n^{r-1}\sum_{i=1}^{l} f_iM_i, \quad x_n^{r-1}M_i=F_i-F_0P_i-N_i,$$

$$F=F_0P'+\sum_{i=1} f_iF_i+R', \quad P'\equiv P-\Sigma f_iP_i, \quad R'\equiv N-\Sigma f_iN_i.$$

Each exponent of x_n in R' is $\leqq r-2$. We segregate its terms in which this exponent is exactly $r-2$ and have

$$F=F_0P'+\sum_{i=1}^{l} f_iF_i+M'x_n^{r-2}+N',$$

where M' is a form in $x_1, \ldots, x_{n-1}$, and N' a form in $x_1, \ldots, x_n$ whose order in x_n is $\leqq r-3$.

The theorem is applicable to the set of forms M', so that each is a linear combination of $M'_1, \ldots, M'_m$, corresponding to $F_{l+1}, \ldots, F_{l+m}$, say. As before, F differs from a linear combination of $F_0, \ldots, F_{l+m}$ by

$$M''x_n^{r-3}+N'',$$

where M'' is a form in $x_1, \ldots, x_{n-1}$ and N'' is a form whose order in x_n is $\leqq r-4$. Proceeding in this manner, we see that F differs from a linear combination of $F_0, \ldots, F_t$ by a form $\bar{R}$ in $x_1, \ldots, x_{n-1}$. One more step leads to the theorem.

49. Finiteness of a Fundamental System of Invariants. Consider the set of all invariants of the binary forms $f_1, \ldots, f_d$, homogeneous in the coefficients of each form separately. By the preceding theorem, there is a finite number of these invariants $I_1, \ldots, I_m$ in terms of which any one of the invariants I is expressible linearly:

(1) $$I = E_1 I_1 + \ldots + E_m I_m,$$

where E_j is not necessarily an invariant, but is a polynomial homogeneous in the coefficients of each f_i separately.

Let $a_1, a_2, \ldots$ be the coefficients in any order of $f_1, \ldots, f_d$. Let $A_1, A_2, \ldots$ be the coefficients in the same order of the forms obtained from them by applying a linear transformation of determinant $(\xi\eta)$. We may write

$$I(A) = (\xi\eta)^\lambda I(a), \qquad I_j(A) = (\xi\eta)^{\lambda_j} I_j(a), \qquad E_j(A) = G_j,$$

where G_j is a function of the a's, ξ's, η's. From the identity (1) in the a's, we obtain an identity by replacing the a's by the A's. Hence

$$(\xi\eta)^\lambda I = \sum_{j=1}^{m} G_j (\xi\eta)^{\lambda_j} I_j,$$

in which the arguments of the I's are a's. Thus G_j is of order $\lambda - \lambda_j$ in ξ_1, ξ_2 and of order $\lambda - \lambda_j$ in η_1, η_2. Operate on each member by V^λ. By § 43, the left member becomes

$$(\lambda+1)(\lambda!)^2 I.$$

By the formula to be proved in § 50, the right member becomes

$$\sum_{j=1}^{m} I_j \{C_0 (\xi\eta)^{\lambda_j - \lambda} G_j + C_1 (\xi\eta)^{\lambda_j - \lambda + 1} V G_j + \ldots + C_\lambda (\xi\eta)^{\lambda_j} V^\lambda G_j\},$$

where the C's are numerical constants. Since G_j is of order $\nu \equiv \lambda - \lambda_j \geqq 0$ in ξ_1, ξ_2 and of order ν in η_1, η_2,

$$V^{\nu+1} G_j = 0, \qquad V^{\nu+2} G_j = 0, \qquad \ldots, \qquad V^\lambda G_j = 0.$$

Also $C_0, C_1, \ldots, C_{\nu-1}$ are zero since they multiply powers of $(\xi\eta)$ whose exponents $-\nu, -\nu+1, \ldots, \lambda_j - \lambda + \nu - 1 = -1$ are negative. Hence

$$(\lambda+1)(\lambda!)^2 I = \sum_{j=1}^{m} I_j C_\nu V^\nu G_j.$$

The form obtained from $f_i = \alpha_x^n$ by our linear transformation has the coefficients (1), § 45. The polynomial G_j in these coefficients is therefore a sum of terms each a product of a constant by ν factors of type α_ξ and ν factors of type α_η. Hence, by § 44, $V^\nu G_j$ is a polynomial in the determinantal factors $(\alpha\beta)$ and is consequently an invariant of the forms f_i. Thus

$$I = \sum_{j=1}^{m} I_j I'_j,$$

where I'_j is an invariant. Then, by (1),

$$I'_j = \sum_{k=1}^{m} e_{jk} I_k, \qquad I = \sum_{j,\,k=1}^{m} e_{jk} I_j I_k.$$

By repeating the former process on this I, we get

$$I = \sum_{j,\,k=1}^{m} I''_{jk} I_j I_k,$$

where the I'' are invariants of the forms f_i. Since there is a reduction of degree at each step, we ultimately obtain an expression for I as a polynomial in $I_1, \ldots, I_m$ with numerical coefficients.

50. Lemma. *If $D = \xi_1\eta_2 - \xi_2\eta_1$, and P is homogeneous (of order λ) in ξ_1, ξ_2, and homogeneous (of order μ) in η_1, η_2, then*

$$(1) \qquad V^m D^n P = \sum_{r=0}^{m} C_r D^{n-m+r} V^r P,$$

where $C_0, \ldots, C_m$ are constants.

First, we have

$$VDP = P + \eta_2 \frac{\partial P}{\partial \eta_2} + \xi_1 \frac{\partial P}{\partial \xi_1} + D \frac{\partial^2 P}{\partial \xi_1 \partial \eta_2}$$

$$- \left(-P - \xi_2 \frac{\partial P}{\partial \xi_2} - \eta_1 \frac{\partial P}{\partial \eta_1} + D \frac{\partial^2 P}{\partial \xi_2 \partial \eta_1} \right) = (2 + \lambda + \mu) P + DVP,$$

by Euler's theorem for homogeneous functions (§ 24). If P is replaced by $D^{n-1}P$, so that λ and μ are increased by $n-1$, we get

$$VD^nP = (\lambda + \mu + 2n) D^{n-1} P + DVD^{n-1} P.$$

Using this as a recursion formula, we get

$$VD^nP=\{n(\lambda+\mu)+n(n+1)\}D^{n-1}P+D^nVP,$$

which reduces to the result in § 43 if $P=1$, whence $\lambda=\mu=0$. Hence (1) holds when $m=1$. To proceed by induction from m to $m+1$, apply V to (1). Thus

$$V^{m+1}D^nP=\sum_{r=0}^{m} C_rV(D^{n-m+r}V^rP).$$

In the result for VD^nP, replace n by $n-m+r$ and P by V^rP, and therefore diminish λ and μ by r. We get

$$V(D^{n-m+r}V^rP)=t_rD^{n-m+r-1}V^rP+D^{n-m+r}V^{r+1}P,$$

where

$$t_r=(n-m+r)(\lambda+\mu-r+n-m+1).$$

Hence, changing $r+1$ to r in the second summand, we get

$$V^{m+1}D^nP=\sum_{r=0}^{m+1}(C_rt_r+C_{r-1})D^{n-m+r-1}V^rP,$$

with $C_{m+1}=0$, $C_{-1}=0$. Thus (1) is true for every m.

51. Finiteness of Syzygies. Let $I_1, \ldots, I_m$ be a fundamental system of invariants of the binary forms $f_1, \ldots, f_d$. Let $S(z_1, \ldots, z_m)$ be a polynomial with numerical coefficients such that $S(I_1, \ldots, I_m)$, when expressed as a function of the coefficients c of the f's, is identically zero in the c's. Then $S(I)=0$ is a syzygy between the invariants.

By means of a new variable z_{m+1}, construct the homogeneous form $S'(z_1, \ldots, z_{m+1})$ corresponding to S. By § 48, the forms S' are expressible linearly in terms of a finite number $S'_1, \ldots, S'_k$ of them. Take $z_{m+1}=1$. Thus

$$S=C_1S_1+\ldots+C_kS_k, \tag{1}$$

where $C_1, \ldots, C_k$ are polynomials in $z_1, \ldots, z_m$. Take $z_1=I_1, \ldots, z_m=I_m$. Hence there is a finite number of syzygies $S_1=0, \ldots, S_k=0$, such that any syzygy $S=0$ implies a relation (1) in which $C_1, \ldots, C_k$ are invariants. In particular, every syzygy is a consequence of $S_1=0, \ldots, S_k=0$.

52. Transvectants. Any two binary forms

$$f=\alpha_x^k, \qquad \phi=\beta_x^l$$

have the covariant

$$(1) \qquad (f, \phi)^r=(\alpha\beta)^r\alpha_x^{k-r}\beta_x^{l-r},$$

called the rth transvectant (Ueberschiebung) of f and ϕ, and due to Cayley. It is their product if $r=0$, their Jacobian if $r=1$, and their Hessian if $f\equiv\phi$ and $r=2$, provided numerical factors are ignored (Exs. 4, 5, § 40).

It may be obtained by differentiation and without the use of the symbolic notation. In fact, a special case of (1), § 44, is

$$V^r\alpha_\xi^k\beta_\eta^l=\frac{k!}{(k-r)!}\,\frac{l!}{(l-r)!}(\alpha\beta)^r\alpha_\xi^{k-r}\beta_\eta^{l-r},$$

so that if f is of order k and ϕ of order l,

$$(2) \qquad (f(\xi),\ \phi(\xi))^r=\frac{(k-r)!}{k!}\,\frac{(l-r)!}{l!}[V^rf(\xi)\phi(\eta)]_{\eta=\xi}.$$

After $f(\xi_1,\ \xi_2)\cdot\phi(\eta_1,\ \eta_2)$ is operated on by V^r, we set $\eta_1=\xi_1$, $\eta_2=\xi_2$.

For example, let $f(\xi)=\alpha_\xi\beta_\xi$, $\phi(\xi)=\gamma_\xi^3$, $P=\alpha_\xi\beta_\xi\gamma_\eta^3$. Then

$$\frac{\partial^2P}{\partial\xi_1\partial\eta_2}=3(\alpha_\xi\beta_1+\alpha_1\beta_\xi)\gamma_\eta^2\gamma_2, \qquad \frac{\partial^2P}{\partial\xi_2\partial\eta_1}=3(\alpha_\xi\beta_2+\alpha_2\beta_\xi)\gamma_\eta^2\gamma_1.$$

The difference is VP. Taking $\eta_1=\xi_1$, $\eta_2=\xi_2$, we get

$$3\{\alpha_\xi(\beta_1\gamma_2-\beta_2\gamma_1)+\beta_\xi(\alpha_1\gamma_2-\alpha_2\gamma_1)\}\gamma_\xi^2.$$

The numerical factor in (2) is here 1/6. Hence

$$(3) \qquad (\alpha_\xi\beta_\xi,\ \gamma_\xi^3)^1=\tfrac{1}{2}(\beta\gamma)\alpha_\xi\gamma_\xi^2+\tfrac{1}{2}(\alpha\gamma)\beta_\xi\gamma_\xi^2.$$

In general, consider the two forms

$$f=\alpha_\xi^{(1)}\alpha_\xi^{(2)}\ \ldots\ \alpha_\xi^{(k)}, \qquad \phi=\beta_\xi^{(1)}\beta_\xi^{(2)}\ \ldots\ \beta_\xi^{(l)}.$$

Then by (1), § 44, and the Corollary, and by (2),

$$(4) \qquad (f,\ \phi)^r=\frac{1}{r!\binom{k}{r}\binom{l}{r}}\sum\frac{(\alpha^{(1)}\beta^{(1)})\ \ldots\ (\alpha^{(r)}\beta^{(r)})f\phi}{\alpha_\xi^{(1)}\ \ldots\ \alpha_\xi^{(r)}\beta_\xi^{(1)}\ \ldots\ \beta_\xi^{(r)}},$$

where the summation extends over all the combinations of the

α's r at a time, and over all the permutations of the β's r at a time. Thus the number of terms in the sum is the reciprocal of the factor preceding Σ.

If the α's are identified and also the β's, (4) becomes (1). If $k=2$, $l=3$, $r=1$, we have one-sixth of a sum of six terms; then if the β's are identified we have two sets of three equal terms and obtain (3).

Since V is a differential operator, (2) gives

(5) $$(\Sigma c_i f_i,\ \Sigma k_j \phi_j)^r = \Sigma\Sigma c_i k_j (f_i,\ \phi_j)^r.$$

Apolarity; Rational Curves, §§ 53–57

53. Binary Forms Apolar to a Given Form. Two binary quadratic forms are called apolar if their lineo-linear invariant is zero; then they are harmonic (Ex. 3, § 11). In general, the binary forms

$$f=\alpha_x^n=\sum_{i=0}^{n}\binom{n}{i}a_i x_1^{n-i}x_2^i, \qquad \phi=\beta_x^n=\sum_{i=0}^{n}\binom{n}{i}b_i x_1^{n-i}x_2^i,$$

of the same order, are called apolar if

(1) $$(\alpha\beta)^n=\sum_{i=0}^{n}(-1)^i\binom{n}{i}a_i b_{n-i}=0.$$

In particular, f is apolar to itself if n is odd (Ex. 4, § 38).

Let the actual linear factors of ϕ be $\beta_x^{(1)}, \ldots, \beta_x^{(n)}$. By (1), (4), § 52,

$$(\alpha\beta)^n=(\alpha_x^n,\ \beta_x^{(1)} \ldots \beta_x^{(n)})^n=(\alpha\beta^{(1)}) \ldots (\alpha\beta^{(n)}).$$

But $\beta_x^{(r)}$ vanishes if x_1 and x_2 equal respectively

$$y_1^{(r)}=\beta_2^{(r)}, \qquad y_2^{(r)}=-\beta_1^{(r)}.$$

Thus

$$(\alpha\beta^{(r)})=\alpha_1 y_1^{(r)}+\alpha_2 y_2^{(r)}=\alpha_{y^{(r)}}.$$

Hence if ϕ vanishes for $x_1=y_1^{(r)}$, $x_2=y_2^{(r)}$ $(r=1, \ldots, n)$, it is apolar to f if and only if

$$\alpha_{y^{(1)}}\,\alpha_{y^{(2)}} \ldots \alpha_{y^{(n)}}=0.$$

Thus f is apolar to an actual nth power $(y_2x_1-y_1x_2)^n$ if and only if $\alpha_y^n=0$, i.e., if y_1, y_2 is a pair of values for which $f=0$.

If no two of the actual linear factors l_i of f are proportional, f is apolar to n actual nth powers l_i^n and these are readily seen to be linearly independent. Then their linear combinations give all the forms apolar to f. For, if f is apolar to $\phi_1, \ldots, \phi_n$, it is apolar to $k_1\phi_1+\ldots+k_n\phi_n$, where $k_1, \ldots, k_n$ are constants, since, by (5), § 52,

$$(f, k_1\phi_1+\ldots+k_n\phi_n)^n=k_1(f, \phi_1)^n+\ldots+k_n(f, \phi_n)^n=0.$$

Moreover, f is not apolar to $n+1$ linearly independent forms

$$\phi_1, \phi_2, \ldots, \phi_{n+1}.$$

For, if so, we have $n+1$ equations like (1), in which the determinant of the coefficients of $a_0, \ldots, a_n$ is therefore zero. But this implies a linear relation between the ϕ's. *If f is the product of n distinct linear factors l_i, a form ϕ can be represented as a linear combination of $l_1^n, \ldots, l_n^n$ if and only if ϕ is apolar to f.* In particular, if r and s are the distinct roots of $f\equiv ax^2+2bx+c=0$, the only quadratics harmonic to f are $g(x-r)^2+h(x-s)^2$.

In case $l_1, \ldots, l_r$ are identical, while $l_1\neq l_i (i>r)$, we may replace $l_1^n, \ldots, l_r^n$ in the above discussion by $l_1^n, l_1^{n-1}\lambda, \ldots, l_1^{n-r+1}\lambda^{r-1}$, where λ is any linear function of x_1 and x_2 which is linearly independent of l_1. In fact, after a linear transformation of variables, we may set $l_1=x_2$, $\lambda=x_1$. Then the above r forms have the factor x_2^{n-r+1} and hence are of type ϕ with $b_i=0 (i\leqq n-r)$. Also, f now has the factor x_2^r, so that $a_i=0 (i<r)$. Hence every term of (1) is zero.

For example, $f=x_1^2x_2(x_1-x_2)^2$ is apolar to

$$x_1^5, x_1^4x_2; \quad x_2^5; \quad (x_1-x_2)^5, (x_1-x_2)^4x_1,$$

which give five linearly independent quintics.

In general, when there are multiple factors of f, the n forms apolar to f obtained above can be proved to be linearly independent. This fact is not presupposed in what follows.

54. Binary Forms Apolar to Several Given Forms. From the list of the given forms we may drop any one linearly dependent on the others, since a form apolar to several forms is apolar to any linear combination of them. In the resulting linearly independent forms

$$f_r = \sum_{i=0}^{n} \binom{n}{i} a_{ir} x_1^{n-i} x_2^i \qquad (r=1, \ldots, g),$$

the g-rowed determinants in the rectangular array of the coefficients are not all zero. For, if so, there are solutions $k_1, \ldots, k_g$, not all zero, of

$$k_1 a_{i1} + k_2 a_{i2} + \ldots + k_g a_{ig} = 0 \quad (i=0, 1, \ldots, n),$$

which would give, contrary to hypothesis, the identity

$$k_1 f_1 + k_2 f_2 + \ldots + k_g f_g \equiv 0.$$

If $b_0 x_1^n + \ldots$ is apolar to each f_r, then

$$\sum_{i=0}^{n} (-1)^i \binom{n}{i} a_{ir} b_{n-i} = 0 \qquad (r=1, \ldots, g).$$

These determine g of the b's as linear functions of the remaining b's, which are arbitrary. Hence there are exactly $n+1-g$ linearly independent forms apolar to each of the g given linearly independent forms.

In particular, apart from a constant factor, there is a single form apolar to each of n given linearly independent forms of order n.

Consider three binary cubic forms

$$f_1 = \alpha_x^3 = a_0 x_1^3 + 3a_1 x_1^2 x_2 + 3a_2 x_1 x_2^2 + a_3 x_2^3,$$

$$f_2 = \beta_x^3 = b_0 x_1^3 + 3b_1 x_1^2 x_2 + 3b_2 x_1 x_2^2 + b_3 x_2^3,$$

$$f_3 = \gamma_x^3 = c_0 x_1^3 + 3c_1 x_1^2 x_2 + 3c_2 x_1 x_2^2 + c_3 x_2^3.$$

Each is apolar to the cubic form

$$\phi = (\alpha\beta)(\alpha\gamma)(\beta\gamma)\alpha_x \beta_x \gamma_x.$$

For, by (4), § 52, and the removal of a constant factor by (5),

$$(\phi, \delta_x^3)^3 = (\alpha\beta)(\alpha\gamma)(\beta\gamma)(\alpha\delta)(\beta\delta)(\gamma\delta),$$

which is changed in sign if δ is interchanged with α, β, or γ, and hence is zero if δ_x^3 is one of the f_i. Hence each f_i is apolar to ϕ. Now

$$(\alpha\beta)(\alpha\gamma)(\beta\gamma)=\begin{vmatrix}\alpha_1^2 & \alpha_1\alpha_2 & \alpha_2^2\\ \beta_1^2 & \beta_1\beta_2 & \beta_2^2\\ \gamma_1^2 & \gamma_1\gamma_2 & \gamma_2^2\end{vmatrix}.$$

In fact, the determinant vanishes if $(\alpha\beta)=0$ as may be seen by setting $\beta_1=c\alpha_1$, $\beta_2=c\alpha_2$. Moreover, the two members are of total degree six and the diagonal term of the determinant equals the product of the first terms $\alpha_1\beta_2$, etc., on the left.

Since $\alpha_1^2\alpha_x=\alpha_1^3x_1+\alpha_1^2\alpha_2x_2=a_0x_1+a_1x_2$, etc., we find, by multiplying the members of the last equation by $\alpha_x\beta_x\gamma_x$,

$$\phi=\begin{vmatrix}a_0x_1+a_1x_2 & a_1x_1+a_2x_2 & a_2x_1+a_3x_2\\ b_0x_1+b_1x_2 & b_1x_1+b_2x_2 & b_2x_1+b_3x_2\\ c_0x_1+c_1x_2 & c_1x_1+c_2x_2 & c_2x_1+c_3x_2\end{vmatrix}$$

$$=[012]x_1^3+[013]x_1^2x_2+[023]x_1x_2^2+[123]x_2^3,$$

where

$$[ijk]=\begin{vmatrix}a_i & a_j & a_k\\ b_i & b_j & b_k\\ c_i & c_j & c_k\end{vmatrix}.$$

If ϕ is identically zero, the four three-rowed determinants in the rectangular array of the coefficients of f_1, f_2, f_3 are all zero, and the f's are linearly dependent.

Apart from a constant factor, ϕ is the unique form apolar to three linearly independent cubic forms f_1, f_2, f_3.

The extension to n binary n-ics is readily made.

55. Rational Plane Cubic Curves. The homogeneous coördinates ξ, η, ζ of a point on such a curve are cubic functions of a parameter t. We may take $t=x_1/x_2$ and write

$$\rho\xi=f_1, \qquad \rho\eta=f_2, \qquad \rho\zeta=f_3,$$

where ρ is a factor of proportionality and the f's are the cubic forms in § 54.

We may assume that the f's are linearly independent, since otherwise all of the points $(\xi,\ \eta,\ \zeta)$ would lie on a straight line.

There is a unique cubic form ϕ apolar to f_1, f_2, f_3 (§ 54). This cubic form, denoted by $\phi=\phi_x^3$, is fundamental in the theory of the cubic curve.

Three points determined by the pairs of parameters x_1, x_2; y_1, y_2; *and* z_1, z_2, *are collinear if and only if*

(1) $$\phi_x\phi_y\phi_z=0.$$

For, if the three points lie on the straight line

(2) $$l\xi+m\eta+n\zeta=0,$$

the three pairs of parameters are pairs of values for which

(3) $$C(x_1,\ x_2)\equiv lf_1+mf_2+nf_3=0.$$

Since C is apolar to ϕ, (1) follows from the first italicized theorem in § 53. Conversely, (1) implies that the cubic C which vanishes for the three pairs of parameters is apolar to ϕ and hence (§ 53) is a linear combination of f_1, f_2, f_3, say (3); the corresponding three points lie on the straight line (2).

Since (2) meets the curve in three points the ratios x_1/x_2 of whose parameters are the roots of (3), the curve is of the third order.

We restrict attention to the case in which the actual linear factors α_x, β_x, γ_x of ϕ are distinct. Since any cubic apolar to ϕ is a linear combination of their cubes (§ 53),

$$f_i=c_{i1}\alpha_x^3+c_{i2}\beta_x^3+c_{i3}\gamma_x^3 \qquad (i=1,\ 2,\ 3).$$

Since the determinant $|\,c_{ij}\,|$ is not zero, suitable linear combinations of the f's give α_x^3, β_x^3, γ_x^3. Hence by a linear transformation on ξ, η, ζ (i. e., by choice of a new triangle of reference), we may take *

$$\rho\xi=\alpha_x^3, \qquad \rho\eta=\beta_x^3, \qquad \rho\zeta=\gamma_x^3.$$

The line $\xi=0$ is an inflexion tangent, likewise $\eta=0$ and $\zeta=0$. In addition to the resulting three inflexion points, there are no others. For, at an inflexion point three consecutive points are collinear, so that (1) gives $\phi=\phi_x^3=0$. In the present

* We now have the formulas in the second part of § 54, where now α_x^3 is the actual, not a symbolic, expression of f_1, etc.

case there are therefore exactly three inflexion points and they are collinear.

56. Any Rational Plane Cubic Curve has a Double Point. Let P_x denote the point (ξ, η, ζ) determined by the pair of parameters x_1, x_2. If the ratios x_1/x_2 and y_1/y_2 are distinct and yet P_x coincides with P_y, then P_x is a double point. For, any straight line (2), § 55, through P_x meets the curve in only the three points whose pairs of parameters satisfy the cubic equation (3), and since two of these pairs give the same point P_x, the line meets the curve in a single further point. Hence there is a double point $P_x = P_y$ if and only if there are two distinct ratios x_1/x_2 and y_1/y_2 such that (1) holds identically in z_1, z_2.

Let Q be the quadratic form which vanishes for the pairs of parameters x_1, x_2 and y_1, y_2 giving a double point. By (1), and the first theorem in § 53, Q is apolar to $\phi_x{}^2\phi_z$ for z_1, z_2 arbitrary. Write $\phi'_x{}^3$ as a symbolic notation for ϕ, alternative to $\phi_x{}^3$. Applying the argument made in § 54 for three cubics to two quadratics, we see that the unique quadratic (apart from a constant factor) which is apolar to both $\phi_x{}^2\phi_z$ and $\phi'_x{}^2\phi'_w$ is their Jacobian

$$J = (\phi\phi')\phi_x\phi'_x \cdot \phi_z\phi'_w.$$

Since ϕ and ϕ' are equivalent symbols, their interchange must leave J unaltered. Hence

$$J = \tfrac{1}{2}(\phi\phi')\phi_x\phi'_x\{\phi_z\phi'_w - \phi'_z\phi_w\}.$$

The quantity in brackets equals $(\phi\phi')(zw)$ by (1), § 40. Discarding the constant factor $\frac{1}{2}(zw)$, we may take

$$Q = (\phi\phi')^2\phi_x\phi'_x$$

as the desired quadratic form. This is the Hessian of ϕ. Conversely, the pairs of values for which Q vanishes are the pairs of parameters of the unique double point of the curve.

57. Rational Space Quartic Curve. Such a curve is given by

$$\rho\xi = \alpha_x{}^4, \qquad \rho\eta = \beta_x{}^4, \qquad \rho\zeta = \gamma_x{}^4, \qquad \rho\omega = \delta_x{}^4,$$

where the four binary quartics are linearly independent. By § 54, there is a unique quartic ϕ apolar to each of the four. As in § 55, four points P_x, P_y, P_z, P_w on the curve are coplanar if and only if

$$\phi_x\phi_y\phi_z\phi_w=0.$$

Thus $\phi=0$ gives the four points at which the osculating plane meets the curve in four consecutive points. It may be shown that the values $x_1^{(i)}$, $x_2^{(i)}$ for which the Hessian of ϕ vanishes give the four points $P_{x^{(i)}}$ on the curve the tangents at which meet the curve again.

Fundamental Systems of Covariants of Binary Forms §§ 58–63

58. Linear Forms. A linear form α_x is its own symbolic representation. If $\alpha_x=\beta_x$, then $(\alpha\beta)=0$. Hence the only covariants of α_x are products of its powers by constants. A fundamental system of covariants of n linear forms is evidently given by the forms and the $\frac{1}{2}n(n-1)$ invariants of type $(\alpha\beta)$, where α_x and β_x are two of the forms.

59. Quadratic Form. A covariant K of a single quadratic

$$f=\alpha_x^2=\beta_x^2=\ldots$$

may have no factor of type $(\alpha\beta)$ and then it is

$$\alpha_x^2\beta_x^2\gamma_x^2\ \ldots=f^k,$$

or may have the factor $(\alpha\beta)$ and hence the further factor $(\alpha\beta)$, $(\alpha\gamma)(\beta\delta)$, $(\alpha\gamma)\beta_x$, or $\alpha_x\beta_x$, including the possibility $\delta=\gamma$. In the first case, $K=(\alpha\beta)^2K_1$, where K_1 is a covariant to which the same argument may be applied. Now $(\alpha\gamma)=\alpha_y$ if $y_1=\gamma_2$, $y_2=-\gamma_1$. Hence in the last three cases, K has a factor of the type

$$\theta=(\alpha\beta)\alpha_y\beta_z,$$

where α_y is either α_x or a new mode of writing $(\alpha\gamma)$, and similarly β_z is either β_x or a new mode of writing $(\beta\delta)$.

Interchanging the equivalent symbols α and β, we get

$$\theta=(\beta\alpha)\beta_y\alpha_z=\tfrac{1}{2}(\alpha\beta)(\alpha_y\beta_z-\beta_y\alpha_z)=\tfrac{1}{2}(\alpha\beta)^2(yz),$$

by (1), § 40. We are thus led to the first case. Hence the fundamental system of covariants of f is composed of f and its discriminant.

EXERCISES

1. The fundamental system for $f=a_x^2=b_x^2$ and $l=\alpha_x=\beta_x$ is f, l, $(ab)^2$, $(a\alpha)^2$, $(a\alpha)a_x$.

2. The fundamental system for $f=a_x^2=b_x^2$ and $\phi=\alpha_x^2=\beta_x^2$ is f, ϕ, $(ab)^2$, $(\alpha\beta)^2$, $(a\alpha)^2$, $(a\alpha)a_x\alpha_x$. Hint:

$$(a\alpha)(a\beta)\alpha_z\beta_y=(a\alpha)^2\beta_y\beta_z-\tfrac{1}{2}(\alpha\beta)^2a_ya_z,$$

as proved by multiplying together the identities (Ex. 6, § 40)

$$(\alpha\beta)a_y\equiv(a\beta)\alpha_y-(a\alpha)\beta_y,\quad(\alpha\beta)a_z\equiv(a\beta)\alpha_z-(a\alpha)\beta_z,$$

and noting that α and β are equivalent symbols.

60. Theorems on Transvectants. In the expression (4), § 52, for a transvectant, each summand taken without the prefixed numerical factor is called a *term* of the transvectant. In the first transvectant (3), § 52, the difference of the two terms is

$$\{(\beta\gamma)\alpha_\xi-(\alpha\gamma)\beta_\xi\}\gamma_\xi^2=\{(\beta\alpha)\gamma_\xi\}\gamma_\xi^2,$$

by Ex. 6, § 40, and is the negative of the 0th transvectant (viz., product) of $(\alpha\beta)$ and γ_ξ^3. The act of removing a factor α_ξ and a factor β_ξ from a product and multiplying by the factor $(\alpha\beta)$ is called a *convolution* (*Faltung*). We have therefore an illustration of the following

LEMMA. *The difference between any two terms of a transvectant equals a sum of terms each a term of a lower transvectant of forms obtained by convolution* from the two given forms.*

Consider the rth transvectant of

$$f=P\alpha_\xi^{(1)}\ \dots\ \alpha_\xi^{(k)},\qquad\phi=Q\beta_\xi^{(1)}\ \dots\ \beta_\xi^{(l)},$$

where P and Q are products of determinantal factors. Then PQ is a factor of each term of the transvectant. Any two terms T and T' differ only as to the arrangements of the α's and the β's. Hence T' can be derived from T by a permuta-

* Including the case of no convolution, as γ_ξ^3 from itself, in the above example.

tion on the α's and one on the β's, and hence by successive interchanges of two α's and successive interchanges of two β's. Any such interchange is said to replace a term by an adjacent term. For example, the two terms of (3), § 52, are adjacent, each being derived from the other by the interchange of α with β. Between T and T' we may therefore insert terms $T_1, \ldots, T_n$ such that any term of the series $T, T_1, T_2, \ldots, T_n, T'$ is adjacent to the one on either side of it. Since

$$T-T'=(T-T_1)+(T_1-T_2)+\ldots+(T_{n-1}-T_n)+(T_n-T'),$$

it suffices to prove the lemma for adjacent terms.

The interchange of two α's or two β's affects just two factors of a term of (4), § 52. The types of adjacent terms are *

$$C(\alpha'\beta')(\alpha''\beta''), \qquad C(\alpha'\beta'')(\alpha''\beta');$$
$$C(\alpha'\beta')\beta''_\xi, \qquad C(\alpha'\beta'')\beta'_\xi;$$

where β' and β'' were interchanged. The difference of the last two terms is seen to equal $C(\beta''\beta')\alpha'_\xi$ by the usual identity. The latter is evidently a term of the $(r-1)$th transvectant of f and $(\beta''\beta')\phi/\{\beta''_\xi\beta'_\xi\}$, which is obtained from ϕ by one convolution.

The difference of the first two adjacent terms equals $C(\alpha'\alpha'')(\beta'\beta'')$, since

$$(\alpha'\alpha'')(\beta'\beta'')-(\alpha'\beta')(\alpha''\beta'')+(\alpha'\beta'')(\alpha''\beta')\equiv\tfrac{1}{2}\begin{vmatrix}\alpha'_1 & \alpha''_1 & \beta'_1 & \beta''_1\\ \alpha'_2 & \alpha''_2 & \beta'_2 & \beta''_2\\ \alpha'_1 & \alpha''_1 & \beta'_1 & \beta''_1\\ \alpha'_2 & \alpha''_2 & \beta'_2 & \beta''_2\end{vmatrix}=0,$$

as shown by Laplace's development. The same relation follows also from the identity just used by taking $\xi_1=-\alpha''_2$, $\xi_2=\alpha''_1$. The resulting difference is a term of the $(r-2)$th transvectant of

$$(\alpha'\alpha'')\frac{f}{\alpha'_\xi\alpha''_\xi}, \qquad (\beta'\beta'')\frac{\phi}{\beta'_\xi\beta''_\xi},$$

which are derived from f and ϕ by a convolution.

* A pair $C(\alpha'\beta')\alpha''_\xi$, $C(\alpha''\beta')\alpha'_\xi$, obtained by interchanging α' and α'', is essentially of the second type.

The Lemma leads to a more important result. By the proof leading to (4), § 52, the coefficient of each term of a transvectant is $1/N$, if N is the number of terms. Just as $S=\frac{1}{2}(T_1+T_2)$ implies $S-T_1=\frac{1}{2}(T_2-T_1)$, so

$$S=\frac{1}{N}(T_1+\ldots+T_N)$$

implies

$$S-T_1=\frac{1}{N}\{(T_2-T_1)+\ldots+(T_n-T_1)\}.$$

Hence *the difference between a transvectant and any one of its terms equal a sum of terms each a term of a lower transvectant of forms obtained by convolution from the two given forms.*

Each term of a lower transvectant may be expressed, by the same theorem, as the sum of that transvectant and terms of still lower transvectants, etc. Finally, when we reach a 0th transvectant, i.e., the product of the two forms, the only term is that product. Hence we have the fundamental

THEOREM. *The difference between any transvectant and any one of its terms is a linear function of lower transvectants of forms obtained by convolution from the two given forms.*

For example, from (3), § 52, and the result preceding the Lemma, we have

$$(\beta\gamma)\alpha_\xi\gamma_\xi^2=(\alpha_\xi\beta_\xi,\ \gamma_\xi^3)^1-\tfrac{1}{2}((\alpha\beta),\ \gamma_\xi^3)^0,$$

and $(\alpha\beta)$ is derived from $\alpha_\xi\beta_\xi$ by one convolution.

61. Irreducible Covariants of Degree m Found by Induction. Let

$$f=\alpha_x^n=\beta_x^n=\ldots=\lambda_x^n$$

be the binary n-ic whose fundamental system of covariants is desired. Since a term with the factor $(\alpha\beta)$ is of degree at least two in the coefficients of f, the only covariants of degree unity are kf, where k is a numerical constant. We shall say that f is the only irreducible covariant of degree unity, and that f, $K_1, \ldots, K_s$ form a complete set of irreducible covariants of degrees $<m$ if every covariant of degree $<m$ is a poly-

nomial in $f, \ldots, K_s$ with numerical coefficients. Given the latter, we seek the irreducible covariants of degree m.

A covariant of degree m is a polynomial in the $(\alpha\beta)$ and the α_x such that each term contains m letters $\alpha, \beta, \gamma, \ldots$. Let T_m be one of the terms with its numerical factor suppressed. Let $\alpha, \beta, \ldots, \kappa, \lambda$ be the m letters occurring in T_m, so that

$$T_m = P(\alpha\lambda)^a(\beta\lambda)^b \ldots (\kappa\lambda)^k\lambda_x^l \qquad (a+b+\ldots+k+l=n),$$

where P involves only $\alpha, \beta, \ldots, \kappa$. Then

$$T_{m-1} = P\alpha_x^a\beta_x^b \ \ldots \ \kappa_x^k$$

is a covariant of degree $m-1$. Evidently T_m is a term of

$$(T_{m-1}, \ \lambda_x^n)^r \qquad (r=n-l),$$

since it is obtained by $r=a+b+\ldots+k$ convolutions from $T_{m-1}\lambda_x^n$. By the final theorem in § 60,

$$T_m = (T_{m-1}, f)^r + \sum_{j=0}^{r-1} c_j(\overline{T}_{m-1}, f)^j,$$

where the c_j are numerical constants, and each $\overline{T}_{m-1}$ is derived from T_{m-1} by convolutions and hence is a covariant of degree $m-1$. But the covariant of degree m was a linear function of the various T_m. Hence *every covariant of degree m of f is a linear function of transvectants $(C_{m-1}, f)^k$ of covariants C_{m-1} of order $m-1$ with f.* Such a transvectant is zero if $k>n$, in view of the order of f. Moreover, it suffices by (5), § 52, to employ the C_{m-1} which are products of powers of $f, K_1, \ldots, K_s$. Hence the covariants of degree m are linear functions of a finite number of transvectants.

In the examination of these transvectants $(C_{m-1}, f)^k$, we first consider those with $k=1$, then those with $k=2$, etc. *We may discard any $(C_{m-1}, f)^k$ for which C_{m-1} has a factor, ϕ, of order $\geqq k$, which is a product of powers of $f, K_1, \ldots, K_s$, and of degree $<m-1$.* For, if T is a term of $(\phi, f)^k$, and if $C_{m-1}=q\phi$, then T is obtained by k convolutions of ϕf, and qT by the same k convolutions of $q\phi f$, not affecting q. Hence qT is a term of $(q\phi, f)^k$. Hence

$$(C_{m-1}, f)^k = qT + \sum_{j=0}^{k-1} c_j(\overline{C}_{m-1}, f)^j.$$

But the terms of the last sum have by hypothesis been considered previously, while the covariants q and T are of degree * $<m$ and hence are expressible in terms of f, $K_1, \ldots, K_s$.

62. Binary Cubic Form. The only irreducible covariant of degree one of

$$f=\alpha_x^3=\beta_x^3=\gamma_x^3$$

was shown to be f. The only covariants of degree two are

$$(\alpha\beta)^r\alpha_x^{3-r}\beta_x^{3-r} \qquad (r=0,\ 1,\ 2,\ 3).$$

This vanishes identically if r is odd. If $r=0$, we have f^2, which is reducible. Hence the only irreducible covariant of degree two is

$$(\alpha\beta)^2\alpha_x\beta_x=(f,f)^2=\text{Hessian } H \text{ of } f.$$

To find the irreducible covariants of degree $m=3$, we have $C_{m-1}=H$ or f^2. In the second case, C_{m-1} has the factor f of degree $<m-1$ and order $3\geqq k$ (since we cannot remove by convolution more than three factors from the second function f in the transvectant). Hence we may discard $C_{m-1}=f^2$. It remains to consider $(H,f)^k$, $k=1,2$. Now

$$(H,f)=(\alpha\beta)^2(\alpha\gamma)\beta_x\gamma_x^2=\text{Jacobian } J \text{ of } H \text{ and } f$$

is irreducible, being of order and degree three and hence not a polynomial in f and H. Next,

$$(H,f)^2=(\alpha\beta)^2(\alpha\gamma)(\beta\gamma)\gamma_x=P(\alpha\beta)\gamma_x, \qquad P=(\alpha\beta)(\alpha\gamma)(\beta\gamma).$$

Interchanging α with γ, we get $P(\beta\gamma)\alpha_x$. Interchanging β with γ, we get $P(\gamma\alpha)\beta_x$. Hence

$$(H,f)^2=\tfrac{1}{3}P\{(\alpha\beta)\gamma_x+(\beta\gamma)\alpha_x+(\gamma\alpha)\beta_x\}=0.$$

The irreducible covariants of degree three or less are therefore f, H, J.

To find those of degree $m=4$, we have $C_{m-1}=f^3$, fH, J,

* This is evident for the factor q of C_{m-1}. Since ϕ is of degree $<m-1$, the term T of $(\phi,f)^k$ involves fewer than $m-1+1$ symbols $\alpha, \beta, \ldots$, and hence is of degree $<m$.

of which the first two may be discarded as before. It remains to consider $(J, f)^k$, for $k=1, 2, 3$. By § 52,

$$(J, f)=(\alpha\beta)^2(\alpha\gamma)(\beta_x\gamma_x^2,\ \delta_x^3)$$
$$=(\alpha\beta)^2(\alpha\gamma)\{\tfrac{1}{3}(\beta\delta)\gamma_x^2\delta_x^2+\tfrac{2}{3}(\gamma\delta)\beta_x\gamma_x\delta_x^2\}.$$

Replacing $(\beta\delta)\gamma_x$ by $(\gamma\delta)\beta_x+(\beta\gamma)\delta_x$, and noting that

$$(\alpha\beta)^2(\alpha\gamma)(\beta\gamma)\gamma_x\delta_x^3=(H, f)^2\cdot f=0,$$

we get

$$(J, f)=(\alpha\beta)^2(\alpha\gamma)(\gamma\delta)\beta_x\gamma_x\delta_x^2.$$

Interchange γ and δ. Hence

$$(J, f)=\tfrac{1}{2}(\alpha\beta)^2(\gamma\delta)\beta_x\gamma_x\delta_x\{(\alpha\gamma)\delta_x+(\delta\alpha)\gamma_x\}.$$

The quantity in brackets equals $-(\gamma\delta)\alpha_x$. Hence

$$(J, f)=-\tfrac{1}{2}(\alpha\beta)^2(\gamma\delta)^2\alpha_x\beta_x\gamma_x\delta_x=-\tfrac{1}{2}H^2.$$

Denoting H by $h_x^2=h'^2_x$, we have

$$J=(h_x^2,\ \alpha_x^3)=(h\alpha)h_x\alpha_x^2, \qquad f=\beta_x^3,$$
$$(J, f)^2=(h\alpha)(h\beta)(\alpha\beta)\alpha_x\beta_x+c((h\alpha)^2\alpha_x, f),$$

by the theorem in § 60. Here $\overline{J}=(h\alpha)^2\alpha_x=(H, f)^2=0$. Since the first term is changed in sign when α and β are interchanged, we have $(J, f)^2=0$.

For the third case,

$$(J, f)^3=((\alpha\beta)^2(\alpha\gamma)\beta_x\gamma_x^2,\ \delta_x^3)^3=(\alpha\beta)^2(\alpha\gamma)(\beta\delta)(\gamma\delta)^2=D,$$

an invariant, evidently equal to $(H, H)^2$, the discriminant of H. Thus D is the discriminant of f (§§ 8, 30) and is not identically zero. Hence D is the only irreducible covariant of degree four.

We can now prove by induction that f, H, J and D form a complete set of irreducible covariants of degree $\leqq m \geqq 5$. Let this be true for covariants C_{m-1} of degree $\leqq m-1$. We may discard $(C_{m-1}, f)^k$ if C_{m-1} has the factor f or J, each of which is of order $3 \geqq k$ and of degree (1 or 3) less than $m-1$; and evidently also if it has the factor D. Hence $C_{m-1}=H^e$, $e \geqq 2$. If $k \leqq 2$, it has the factor H of order $2 \geqq k$ and degree $2<m-1$. It remains to consider $(H^e, f)^3$. If $e>2$, H^e has the factor

H^2 of order $4 \geqq 3$ and degree $4 < m-1$, since H^e is of degree $\geqq 6$. Finally,

$$(H^2, f)^3 = (h_x^2 h'^2_x,\ \alpha_x^3)^3 = (h\alpha)^2 (h'\alpha) h'_x = (h'^2_x,\ (h\alpha)^2 \alpha_x) = 0.$$

Hence f, H, J, D form a fundamental system of covariants (*cf.* § 30).

63. Higher Binary Forms. The concepts introduced by Gordan in his proof of the finiteness of the fundamental system of covariants of the binary p-ic enabled him to find * the system of 23 forms for the quintic, the system of 26 forms for the sextic, as well as to obtain in a few lines the system for the cubic (§ 62) and the quartic (§ 31). Fundamental systems for the binary forms of orders 7 and 8 have been determined by von Gall.†

Gordan's method yields a set of covariants in terms of which all of the covariants are expressible rationally and integrally, but does not show that a smaller set would not serve similarly. The method is supplemented by Cayley's theory ‡ of generating functions, which gives a lower limit to the number of covariants in a fundamental system.

64. Hermite's Law of Reciprocity. This law (§ 27) can be made self-evident by use of the symbolic notation. Let the form

$$\phi = \alpha_x^p = \beta_x^p = \ldots = a_0(x_1 - \rho_1 x_2)(x_1 - \rho_2 x_2) \ldots (x_1 - \rho_p x_2)$$

have a covariant of degree d,

$$K = a_0^d \Sigma (\rho_1 - \rho_2)^i (\rho_1 - \rho_3)^j (\rho_2 - \rho_3)^k \ldots (x_1 - \rho_1 x_2)^{l_1} \ldots (x_1 - \rho_p x_2)^{l_p},$$

so that each of the roots $\rho_1, \ldots, \rho_p$ occurs exactly d times in each product. Consider the binary d-ic

$$f = a_x^d = b_x^d = \ldots = c_0(x_1 - r_1 x_2) \ldots (x_1 - r_d x_2).$$

* Gordan, *Invariantentheorie*, vol. 2 (1887), p. 236, p. 275. *Cf.* Grace and Young, *Algebra of Invariants*, 1903, p. 122, p. 128, p. 150.

† *Mathematische Annalen*, vol. 17 (1880), vol. 31 (1888).

‡ For an introduction to it, see Elliott, *Algebra of Quantics*, 1895, p. 165, p. 247.

To the various powers, whose product is any one term of K,

$$(\rho_1-\rho_2)^i,\quad (\rho_1-\rho_3)^j,\quad (\rho_2-\rho_3)^k,\quad \ldots,\quad (x_1-\rho_1x_2)^{l_1},\quad (x_1-\rho_2x_2)^{l_2},\ \ldots,$$

we make correspond the symbolic factors

$$(ab)^i,\quad (ac)^j,\quad (bc)^k,\quad \ldots,\quad a_x^{l_1},\quad b_x^{l_2},\quad \ldots$$

of the corresponding covariant of f:

$$C=(ab)^i(ac)^j(bc)^k\ \ldots\ a_x^{l_1}b_x^{l_2}c_x^{l_3}\ \ldots,$$

of degree p (since there are p symbols a, b, c, . . . , corresponding to ρ_1, . . . , ρ_p) and having the same order $l_1+l_2+l_3+$. . . as K. Conversely, C determines K.

EXAMPLES

Let $p=2$. To $K=a_0^{2s}(\rho_1-\rho_2)^{2s}$ corresponds the invariant $C=(ab)^{2s}$ of degree 2 of $f=a_x^{2s}=b_x^{2s}$. Again, to the covariant $K\phi^t$ of ϕ corresponds the covariant $(ab)^{2s}\,a_x^tb_x^t$ of the form $a_x^{2s+t}=b_x^{2s+t}$.

CONCOMITANTS OF TERNARY FORMS IN SYMBOLIC NOTATION, §§ 65–67

65. Ternary Form in Symbolic Notation. The general ternary form is

$$f=\Sigma\frac{n!}{r!s!t!}\,a_{rst}x_1^rx_2^sx_3^t,$$

where the summation extends over all sets of integers r, s, t, each $\geqq 0$, for which $r+s+t=n$.

We represent f symbolically by

$$f=\alpha_x^n=\beta_x^n\ \ldots,\qquad \alpha_x=\alpha_1x_1+\alpha_2x_2+\alpha_3x_3,\ \ldots.$$

Only polynomials in α_1, α_2, α_3 of total degree n have an interpretation and

$$\alpha_1^r\alpha_2^s\alpha_3^t=a_{rst}.$$

Just as $\alpha_1\beta_2-\alpha_2\beta_1$ was denoted by $(\alpha\beta)$ in § 39, we now write

$$(\alpha\beta\gamma)=\begin{vmatrix}\alpha_1 & \alpha_2 & \alpha_3\\ \beta_1 & \beta_2 & \beta_3\\ \gamma_1 & \gamma_2 & \gamma_3\end{vmatrix}.$$

Under any ternary linear transformation

$$T: \qquad x_i = \xi_i X_1 + \eta_i X_2 + \zeta_i X_3 \qquad (i=1, 2, 3)$$

α_x becomes $\alpha_\xi X_1 + \alpha_\eta X_2 + \alpha_\zeta X_3$, and f becomes

$$\Sigma \frac{n!}{r!s!t!} A_{rst} X_1^r X_2^s X_3^t = (\alpha_\xi X_1 + \alpha_\eta X_2 + \alpha_\zeta X_3) .$$

Thus α_x behaves like a covariant of index zero of f. Also

$$A_{rst} = \alpha_\xi^r \alpha_\eta^s \alpha_\zeta^t,$$

$$\begin{vmatrix} \alpha_\xi & \alpha_\eta & \alpha_\zeta \\ \beta_\xi & \beta_\eta & \beta_\zeta \\ \gamma_\xi & \gamma_\eta & \gamma_\zeta \end{vmatrix} = (\alpha\beta\gamma)(\xi\eta\zeta),$$

so that $(\alpha\beta\gamma)$ behaves like an invariant of index unity of f.

EXERCISES

1. The discriminant of a ternary quadratic form α_x^2 is $\frac{1}{6}(\alpha\beta\gamma)^2$.
2. The Jacobian of α_x^l, β_x^m, γ_x^n is $lmn\,(\alpha\beta\gamma)\alpha_x^{l-1}\beta_x^{m-1}\gamma_x^{n-1}$.
3. The Hessian of α_x^n is the product of $(\alpha\beta\gamma)^2\alpha_x^{n-2}\beta_x^{n-2}\gamma_x^{n-2}$ by a constant.
4. A ternary cubic form $\alpha_x^3 = \beta_x^3 = \ldots$ has the invariants

$$(\alpha\beta\gamma)(\alpha\beta\delta)(\alpha\gamma\delta)(\beta\gamma\delta), \quad (\alpha\beta\gamma)(\alpha\beta\delta)(\alpha\gamma\epsilon)(\beta\gamma\phi)(\delta\epsilon\phi)^2.$$

66. Concomitants of Ternary Forms. If u_1, u_2, u_3 are constants,

$$u_x = u_1x_1 + u_2x_2 + u_3x_3 = 0$$

represents a straight line in the point-coördinates x_1, x_2, x_3. Since u_1, u_2, u_3 determine this line, they are called its line-coördinates. If we give fixed values to x_1, x_2, x_3 and let the line-coördinates u_1, u_2, u_3 take all sets of values for which $u_x = 0$, we obtain an infinite set of straight lines through the point (x_1, x_2, x_3). Thus, for fixed x's, $u_x = 0$ is the equation of the point (x_1, x_2, x_3) in line-coördinates.

Under the linear transformation T, of § 65, whose determinant $(\xi\eta\zeta)$ is not zero, the line $u_x = 0$ is replaced by

$$U_X = U_1X_1 + U_2X_2 + U_3X_3 = 0,$$

in which

$$U_1 = \sum_{i=1}^{3} \xi_i u_i, \qquad U_2 = \sum_{i=1}^{3} \eta_i u_i, \qquad U_3 = \sum_{i=1}^{3} \zeta_i u_i.$$

The equations obtained by solving these define a linear transformation T_1 which expresses u_1, u_2, u_3 as linear functions of U_1, U_2, U_3 and which is uniquely determined * by the transformation T. Two sets of variables x_1, x_2, x_3 and u_1, u_2, u_3, transformed in this manner, are called *contragredient.*

A polynomial $P(c, x, u)$ in the two sets of contragredient variables and the coefficients c of certain forms $f_i(x_1, x_2, x_3)$ is called a *mixed concomitant* of index λ of the f's if, for every linear transformation T of determinant $\Delta\neq 0$ on x_1, x_2, x_3 and the above defined transformation T_1 on u_1, u_2, u_3, the product of $P(c, x, u)$ by Δ^λ equals the same polynomial $P(C, X, U)$ in the new variables and coefficients C of the forms derived from the f's by the first transformation. For example, u_x is a concomitant of index zero of any set of forms.

In particular, if P does not involve the u's, it is a covariant (or invariant) of the f's. If it involves the u's, but not the x's, it is called a *contravariant* of the f's.

Since $U_1=u_\xi$, $U_2=u_\eta$, $U_3=u_\zeta$, we see by the last formula in § 65, with γ replaced by u, that $(\alpha\beta u)$ behaves like a contravariant of index unity of α_x^n, and also like one of α_x^n, β_x^m.

For the linear forms α_x and β_x, $(\alpha\beta u)$ has an actual interpretation. For $f=\alpha_x^2=\beta_x^2$, where

$$f=a_{200}x_1^2+a_{020}x_2^2+a_{002}x_3^2+2a_{110}x_1x_2+2a_{101}x_1x_3+2a_{011}x_2x_3,$$

it may be shown that

$$\begin{vmatrix} a_{200} & a_{110} & a_{101} & u_1 \\ a_{110} & a_{020} & a_{011} & u_2 \\ a_{101} & a_{011} & a_{002} & u_3 \\ u_1 & u_2 & u_3 & 0 \end{vmatrix} = (\alpha\beta u)^2.$$

By equating to zero this determinant (the bordered discriminant of f), we obtain the line equation of the conic $f=0$.

67. Theorem. *Every concomitant of a system of ternary forms is a polynomial in* u_x *and expressions of the types* $\alpha_x, (\alpha\beta\gamma), (\alpha\beta u)$.

* We have only to interchange the rows and columns in the matrix of T and then take the inverse of the new matrix to obtain the matrix of the transformation T_1. Similarly, x_1, x_2 are contragredient with u_1, u_2, if we have T, § 40, and $u_1=(\eta_2 U_1-\xi_2 U_2)/(\xi\eta)$, $u_2=(-\eta_1 U_1+\xi_1 U_2)/(\xi\eta)$.

A concomitant of the forms $f_i(x_1, x_2, x_3)$ is evidently a covariant of the enlarged system of forms f_i and u_x. We may therefore restrict attention to covariants. In the proof of the corresponding theorem for binary forms, we used the operator (1), § 42. Here we employ an operator V composed of six terms each a partial differentiation of the third order:

$$V = \begin{vmatrix} \frac{\partial}{\partial \xi_1} & \frac{\partial}{\partial \xi_2} & \frac{\partial}{\partial \xi_3} \\ \frac{\partial}{\partial \eta_1} & \frac{\partial}{\partial \eta_2} & \frac{\partial}{\partial \eta_3} \\ \frac{\partial}{\partial \zeta_1} & \frac{\partial}{\partial \zeta_2} & \frac{\partial}{\partial \zeta_3} \end{vmatrix} = \frac{\partial^3}{\partial \xi_1 \partial \eta_2 \partial \zeta_3} - \cdots,$$

the determinant being symbolic. It may be shown as in § 43 that

$$V(\xi\eta\zeta)^n = n(n+1)(n+2)(\xi\eta\zeta)^{n-1}.$$

As in § 44, the result of applying V^r to a product of k factors of the type α_ξ, l factors of the type β_η, and m factors of the type γ_ζ, is a sum of terms each containing $k-r$ factors α_ξ, $l-r$ factors β_η, $m-r$ factors γ_ζ, and r factors of the type $(\alpha\beta\gamma)$.

For the case of an invariant I, the theorem can be proved without a device. In the notations of § 65, we have

$$I(A) = (\xi\eta\zeta)^\lambda I(a).$$

Each A is a product of factors α_ξ, α_η, α_ζ. Hence $I(A)$ equals a sum of terms each with λ factors of the type α_ξ, λ of type α_η, and λ of type α_ζ. Operate on each member of the equation with V^λ. The left member becomes a sum of terms each a product of a constant and factors of type $(\alpha\beta\gamma)$. The right member becomes the product of $I(a)$ by a number not zero. Hence I equals a polynomial in the $(\alpha\beta\gamma)$.

For a covariant K, we have, by definition,

$$K(A, X) = (\xi\eta\zeta)^\lambda K(a, x).$$

Solving the equations of our transformation T in § 65, we get

$$(\xi\eta\zeta)X_1 = x_1(\eta_2\zeta_3 - \eta_3\zeta_2) + x_2(\eta_3\zeta_1 - \eta_1\zeta_3) + x_3(\eta_1\zeta_2 - \eta_2\zeta_1),$$

etc. Replacing x_1 by $y_2z_3-y_3z_2$, x_2 by $y_3z_1-y_1z_3$, and x_3 by $y_1z_2-y_2z_1$, we get

$$(\xi\eta\zeta)X_1=y_\eta z_\zeta-y_\zeta z_\eta,$$

$$(\xi\eta\zeta)X_2=y_\zeta z_\xi-y_\xi z_\zeta,$$

$$(\xi\eta\zeta)X_3=y_\xi z_\eta-y_\eta z_\xi.$$

Our relation for a covariant K of order ω now becomes

$$\Sigma(\text{product of factors } \alpha_\xi,\, y_\xi,\, z_\xi,\, \alpha_\eta,\, \ldots,\, z_\zeta)=(\xi\eta\zeta)^{\lambda+\omega}K(a,\, x),$$

each term on the left having $\lambda+\omega$ factors with the subscript ξ, etc. Apply the operator V to the left member. We obtain a sum of terms with one determinantal factor $(\alpha\beta\gamma)$, $(\alpha\beta y)$ or $(\alpha yz)\equiv\alpha_x$, and with $\lambda+\omega-1$ factors with the subscript ξ, etc. The result may be modified so that the undesired factor $(\alpha\beta y)$ shall not occur. For, it must have arisen by applying V to a term with a factor like $\alpha_\xi\beta_\eta y_\zeta$ and hence (by the formulas for the X_i) with a further factor z_η or z_ξ. Consider therefore the term $C\alpha_\xi\beta_\eta y_\zeta z_\eta$ in the initial result. Then the term $-C\alpha_\xi\beta_\eta y_\eta z_\zeta$ must occur. By operating on these with V, we get $C(\alpha\beta y)z_\eta$, $-C(\alpha\beta z)y_\eta$, respectively, whose sum equals

$$C\{(\beta yz)\alpha_\eta-(\alpha yz)\beta_\eta\}\equiv C(\beta_x\alpha_\eta-\alpha_x\beta_\eta),$$

as shown by expanding, according to the elements of the last row,

$$\begin{vmatrix} \alpha_1 & \beta_1 & y_1 & z_1 \\ \alpha_2 & \beta_2 & y_2 & z_2 \\ \alpha_3 & \beta_3 & y_3 & z_3 \\ \alpha_\eta & \beta_\eta & y_\eta & z_\eta \end{vmatrix}\equiv 0.$$

The modified result is therefore a sum of terms each with one factor of type $(\alpha\beta\gamma)$ or α_x and with $\lambda+\omega-1$ factors with subscript ξ, etc.

Applying V in succession $\lambda+\omega$ times and modifying the result at each step as before, we obtain as a new left member a sum of terms each with $\lambda+\omega$ factors of the types $(\alpha\beta\gamma)$ and α_x only. From the right member we obtain nK, where n is a number $\neq 0$. Hence the theorem is proved.

68. Quaternary Forms. For $\alpha_x = \alpha_1 x_1 + \ldots + \alpha_4 x_4$,

$$f = \alpha_x^n = \beta_x^n = \gamma_x^n = \delta_x^n$$

has the determinant $(\alpha\beta\gamma\delta)$ of order 4 as a symbolic invariant of index unity. Any invariant of f can be expressed as a polynomial in such determinantal factors; any covariant as a polynomial in them and factors of type α_x. In the equation $u_x = 0$ of a plane, $u_1, \ldots, u_4$ are called plane-coördinates. The mixed concomitants defined as in § 66 are expressible in terms of u_x and factors like α_x, $(\alpha\beta\gamma\delta)$, $(\alpha\beta\gamma u)$. For geometrical reasons, we extend that definition of mixed concomitants to polynomials $P(c, x, u, v)$, where $v_1, \ldots, v_4$ as well as $u_1, \ldots, u_4$ are contragredient to $x_1, \ldots, x_4$. There may now occur the additional type of factor

$$(\alpha\beta uv) = (\alpha_1\beta_2 - \alpha_2\beta_1)(u_3v_4 - u_4v_3) + \ldots + (\alpha_3\beta_4 - \alpha_4\beta_3)(u_1v_2 - u_2v_1).$$

These six combinations of the u's and v's are called the line-coördinates of the intersection of the planes $u_x = 0$, $v_x = 0$. For instance, $(\alpha\beta uv)^2 = 0$ is the condition that this line of intersection shall touch the quadric surface $\alpha_x^2 = 0$.

We have not considered concomitants involving also a third set of variables $w_1, \ldots, w_4$, contragredient with the x's. For, in

$$u_1x_1 + \ldots + u_4x_4 = 0, \qquad v_1x_1 + \ldots + v_4x_4 = 0,$$
$$w_1x_1 + \ldots + w_4x_4 = 0,$$

$x_1, \ldots, x_4$ are proportional to the three-rowed determinants of the matrix of coefficients, so that (αuvw) is essentially α_x.