

ALGEBRAIC INVARIANTS

PART I

ILLUSTRATIONS, GEOMETRICAL INTERPRETATIONS AND APPLICATIONS OF INVARIANTS AND COVARIANTS.

1. Illustrations from Plane Analytics. If x and y are the coördinates of a point in a plane referred to rectangular axes, while x' and y' are the coördinates of the same point referred to axes obtained by rotating the former axes counter-clockwise through an angle θ , then

$$T: \quad x = x' \cos \theta - y' \sin \theta, \quad y = x' \sin \theta + y' \cos \theta.$$

Substituting these values into the linear function

$$l = ax + by + c,$$

we get $a'x' + b'y' + c$, where

$$a' = a \cos \theta + b \sin \theta, \quad b' = -a \sin \theta + b \cos \theta.$$

It follows that

$$a'^2 + b'^2 = a^2 + b^2.$$

Accordingly, $a^2 + b^2$ is called an *invariant* of l under every transformation of the type T .

Similarly, under the transformation T let

$$L = Ax + By + C = A'x' + B'y' + C,$$

so that

$$A' = A \cos \theta + B \sin \theta, \quad B' = -A \sin \theta + B \cos \theta.$$

By the multiplication * of determinants, we get

$$\begin{vmatrix} a' & b' \\ A' & B' \end{vmatrix} = \begin{vmatrix} a & b \\ A & B \end{vmatrix} \cdot \begin{vmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{vmatrix} = aB - bA,$$

$$\begin{vmatrix} a' & -b' \\ B' & A' \end{vmatrix} = \begin{vmatrix} a & -b \\ B & A \end{vmatrix} \cdot \begin{vmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{vmatrix} = aA + bB.$$

The expressions at the right are therefore invariants of the pair of linear functions l and L under every transformation of type T . The straight lines represented by $l=0$ and $L=0$ are parallel if and only if $aB-bA=0$; they are perpendicular if and only if $aA+bB=0$. Moreover, the quotient of $aB-bA$ by $aA+bB$ is an invariant having an interpretation; it is the tangent of one of the angles between the two lines.

As in the first example, A^2+B^2 is an invariant of L . Between our four invariants of the pair l and L the following identity holds:

$$(aA+bB)^2+(aB-bA)^2=(a^2+b^2)(A^2+B^2).$$

The equation of any conic is of the form $S=0$, where

$$S=ax^2+2bxy+cy^2+2kx+2ly+m.$$

Under the transformation T , S becomes a function of x' and y' , in which the part of the second degree

$$F=a'x'^2+2b'x'y'+c'y'^2$$

is derived solely from the part of S of the second degree:

$$f=ax^2+2bxy+cy^2.$$

The coefficient a' of x'^2 is evidently obtained by replacing x by $\cos \theta$ and y by $\sin \theta$ in f , while c' is obtained by replacing x by $-\sin \theta$ and y by $\cos \theta$ in f . It follows at once that

$$a'+c'=a+c.$$

Using also the value of b' , we can show that

$$a'c'-b'^2=ac-b^2,$$

* We shall always employ the rule which holds also for the multiplication of matrices: the element in the r th row and s th column of the product is found by multiplying the elements of the r th row of the first determinant by the corresponding elements of the s th column of the second determinant, and adding the products.

but a more general fact will be obtained in § 4 without tedious multiplications. Thus $a+c$ and $d=ac-b^2$ are invariants of f , and also of S , under every transformation of type T . When $S=0$ represents a real conic, not a pair of straight lines, the conic is an ellipse if $d>0$, an hyperbola if $d<0$, and a parabola if $d=0$. When homogeneous coördinates are used, the classifications of conics is wholly different (§ 13).

If x and y are the coördinates of a point referred to rectangular axes and if x' and y' are the coördinates of the same point referred to new axes through the new origin (r, s) and parallel to the former axes, respectively, then

$$t: \quad x = x' + r, \quad y = y' + s.$$

All of our former expressions which were invariant under the transformations T are also invariant under the new transformations t , since each letter a, b, \dots involved is invariant under t . But not all of our expressions are invariant under a larger set of transformations to be defined later.

We shall now give an entirely different interpretation to the transformations T and t . Instead of considering (x, y) and (x', y') to be the same point referred to different pairs of coördinate axes, we now regard them as different points referred to the same axes. In the case of t , this is accomplished by translating the new axes, and each point referred to them, in the direction from (r, s) to $(0, 0)$ until those axes coincide with the initial axes. Thus any point (x, y) is translated to a new point (x', y') , where

$$x' = x - r, \quad y' = y - s,$$

both points being now referred to the same axes. Thus each point is translated through a distance $\sqrt{r^2 + s^2}$ and in a direction parallel to the directed line from $(0, 0)$ to $(-r, -s)$.

In the case of T , we rotate the new axes about the origin clockwise through angle θ so that they now coincide with the initial axes. Then any point (x, y) is moved to a new point (x', y') by a clockwise rotation about the origin through angle θ . By solving the equations of T , we get

$$x' = x \cos \theta + y \sin \theta, \quad y' = -x \sin \theta + y \cos \theta.$$

These rigid motions (translations, rotations, and combinations of them) preserve angles and distances. But the transformation $x' = 2x$, $y' = 2y$ is a stretching in all directions from the origin in the ratio 2 : 1; while $x' = 2x$, $y' = y$ is a stretching perpendicular to the y -axis in each direction in the ratio 2 : 1.

From the multiplicity of possible types of transformations, we shall select as the basis of our theory of invariants the very restricted set of transformations which have an interpretation in projective geometry and which suffice for the ordinary needs of algebra.

2. Projective Transformations. All of the points on a straight line are said to form a *range* of points. Project the the

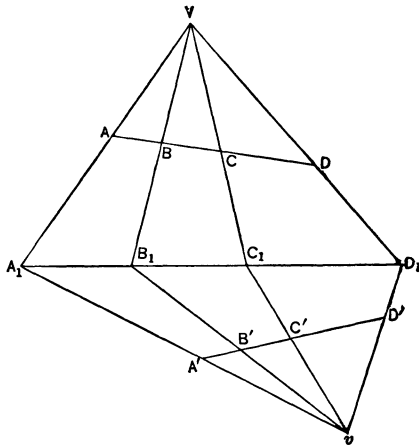


Fig. 1.

points A, B, C, \dots of a range from a point V , not on their line, by means of a pencil of straight lines. This pencil is cut by a new transversal in a range A_1, B_1, C_1, \dots , said to be *perspective* with the range A, B, C, \dots . Project the points A_1, B_1, C_1, \dots from a new vertex v by a new pencil and cut it by a new transversal. The resulting range of points A', B', C', \dots is said to be *projective* with the range A, B, C, \dots . Likewise, the range obtained by any number of projections and sections is called projective with the given range, and

the one-to-one correspondence thus established between corresponding points of the two ranges is called a *projectivity*.

To obtain an analytic property of a projectivity, we apply the sine proportion to two triangles in Fig. 1 and get

$$\frac{AC}{AV} = \frac{\sin AVC}{\sin ACV}, \quad \frac{BC}{BV} = \frac{\sin BVC}{\sin ACV}.$$

From these and the formulas with D in place of C , we get

$$\frac{AC}{BC} = \frac{AV}{BV} \cdot \frac{\sin AVC}{\sin BVC}, \quad \frac{AD}{BD} = \frac{AV}{BV} \cdot \frac{\sin AVD}{\sin BVD}.$$

Hence, by division

$$\frac{AC}{BC} \div \frac{AD}{BD} = \frac{\sin AVC}{\sin BVC} \div \frac{\sin AVD}{\sin BVD}.$$

The left member is denoted by $(ABCD)$ and is called the *cross-ratio* of the four points taken in this order. Since the right member depends only on the angles at V , it follows that

$$(ABCD) = (A_1B_1C_1D_1),$$

if A_1, \dots, D_1 are the intersections of the four rays by a second transversal. Hence if two ranges are projective, the cross-ratio of any four points of one range equals the cross-ratio of the corresponding points of the other range.

Let each point of the line AB be determined by its distance and direction from a fixed initial point of the line; let a be the resulting coördinate of A , and b, c, x those of B, C, D , respectively. Similarly, let A', B', C', D' have the coördinates a', b', c', x' , referred to a fixed initial point on their line. Then

$$(ABCD) = \frac{c-a}{c-b} \div \frac{x-a}{x-b} = \frac{c'-a'}{c'-b'} \div \frac{x'-a'}{x'-b'} = (A'B'C'D').$$

Hence

$$\frac{x'-b'}{x'-a'} = k \frac{x-b}{x-a}, \quad k = \frac{c-a}{c-b} \div \frac{c'-a'}{c'-b'}.$$

so that k is a finite constant $\neq 0$, if C is distinct from A and B , and hence C' distinct from A' and B' . Solving for x' , we obtain a relation

$$L: \quad x' = \frac{\alpha x + \beta}{\gamma x + \delta}, \quad \Delta = \begin{vmatrix} \alpha & \beta \\ \gamma & \delta \end{vmatrix} \neq 0.$$

In fact,

$$\alpha = b' - ka', \quad \beta = ka'b - ab', \quad \gamma = 1 - k, \quad \delta = bk - a.$$

If we multiply the elements of the first column of Δ by b and add the products to the elements of the second column, we get

$$\Delta = \begin{vmatrix} b' - ka' & b'(b-a) \\ 1 - k & b - a \end{vmatrix} = (b-a) \begin{vmatrix} -ka' & b' \\ -k & 1 \end{vmatrix} = k(b-a)(b' - a') \neq 0,$$

if B and A are distinct, so that B' and A' are distinct.

Hence a projectivity between two ranges defines a linear fractional transformation L between the coördinate x of a general point of one range and the coördinate x' of the corresponding point of the other range. The transformation is uniquely determined by the coördinates of three distinct points of one range and those of the corresponding points of the other range. If the ranges are on the same line and if $A' = A$, $B' = B$, $C' = C$, then $k = 1$, $\alpha = \delta$, $\beta = \gamma = 0$, and $x' = x$. Thus $(ABCD) = (ABCD')$ implies $D' = D$.

Conversely, if L is any given linear fractional transformation (of determinant $\neq 0$) and if each value of x is interpreted as the coördinate of a point on any given straight line l and the value of x' determined by L as the coördinate of a corresponding point on any second given straight line l' , the correspondence between the resulting two ranges is a projectivity. This is proved as follows:

Let A, B, C, D be the four points of l whose respective coördinates are four distinct values x_1, x_2, x_3, x_4 of x such that $\gamma x_i + \delta \neq 0$. The corresponding values x_1', x_2', x_3', x_4' of

x' determine four distinct points A', B', C', D' of l' . For, if $i \neq j$,

$$x'_i - x'_j = \frac{\alpha x_i + \beta}{\gamma x_i + \delta} - \frac{\alpha x_j + \beta}{\gamma x_j + \delta} = \frac{\Delta(x_i - x_j)}{(\gamma x_i + \delta)(\gamma x_j + \delta)} \neq 0,$$

$$(A'B'C'D') = \frac{x'_3 - x'_1}{x'_3 - x'_2} \div \frac{x'_4 - x'_1}{x'_4 - x'_2} = r \frac{x_3 - x_1}{x_3 - x_2} \div \frac{x_4 - x_1}{x_4 - x_2} = (ABCD)$$

since, if l_i denotes $\gamma x_i + \delta$,

$$r = \left(\frac{\Delta}{l_3 l_1} / \frac{\Delta}{l_3 l_2} \right) \div \left(\frac{\Delta}{l_4 l_1} / \frac{\Delta}{l_4 l_2} \right) = 1.$$

If $A' \neq A$, project the points A', B', C', D' from any convenient vertex V' on to any line AB_1 through A and distinct

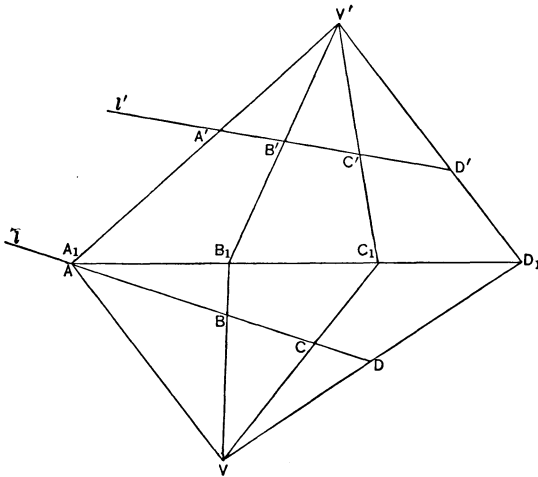


Fig. 2.

from l , obtaining the points $A_1 = A, B_1, C_1, D_1$ of Fig. 2. Let V be the intersection of BB_1 with CC_1 and let VD_1 meet l at P . Then

$$(ABCP) = (A_1 B_1 C_1 D_1) = (A' B' C' D') = (ABCD).$$

From the first and last we have $P = D$, as proved above. Holding x_1, x_2, x_3 fixed, but allowing x_4 to vary, we obtain two projective ranges on l and l' . If $A' = A$, we use l' itself as AB_1 and see that the ranges are projective.

If l and l' are identical, we first project the range on l' on to a new line ($A'B'$ in Fig. 2) and proceed as before.

Any linear fractional transformation L is therefore a projective transformation of the points of a line or of the points of one line into those of another line. The cross-ratio of any four points is invariant.

3. Homogeneous Coördinates of a Point in a Line. They are introduced partly for the sake of avoiding infinite coördinates. In fact, if $\gamma \neq 0$, the value $-\delta/\gamma$ of x makes x' infinite. We set $x = x_1/x_2$, thereby defining only the ratio of the homogeneous coördinates x_1, x_2 of a point. Let $x' = x'_1/x'_2$. Then, if ρ is a factor of proportionality, L may be given the homogeneous form

$$\rho x'_1 = \alpha x_1 + \beta x_2, \quad \rho x'_2 = \gamma x_1 + \delta x_2, \quad \alpha\delta - \beta\gamma \neq 0.$$

The nature of homogeneous coördinates of points in a line is brought out more clearly by a more general definition. We employ two fixed points A and B of the line as points of reference. We define the homogeneous coördinates of a point P of the line to be any two numbers x, y such that

$$\frac{x}{y} = c \frac{AP}{PB},$$

where c is a constant $\neq 0$, the same for all points P , while AP is a directed segment, so that $AP = -PA$. We agree to take $y = 0$ if $P = B$. Given P , we have the ratio of x to y . Conversely, given the latter ratio, we have the ratio of AP to PB , as well as their sum $AP + PB = AB$, and hence can find AP and therefore locate the point P .

Just as we obtained in plane analytics (*cf.* § 1) the relations between the coördinates of the same point referred to two pairs of axes, so here we desire the values of x and y expressed in terms of the coördinates ξ and η of the same point P referred to new fixed points of reference A', B' . By definition, there is a certain new constant $k \neq 0$ such that

$$\frac{\xi}{\eta} = k \frac{A'P}{P'B'}.$$

Since $A'P + PB' = A'B'$, we may replace $A'P$ by $A'B' - PB'$ and get

$$PB' = \frac{k\eta \cdot A'B'}{\xi + k\eta}.$$

Let A have the coördinates ξ', η' , referred to A', B' . Then

$$PA = PB' - AB' = PB' - \frac{k\eta' \cdot A'B'}{\xi' + k\eta'} = \frac{(\eta\xi' - \xi\eta')k \cdot A'B'}{(\xi + k\eta)(\xi' + k\eta')}.$$

Similarly, if B has the coördinates ξ_1, η_1 , referred to A', B' ,

$$PB = \frac{(\eta\xi_1 - \xi\eta_1)k \cdot A'B'}{(\xi + k\eta)(\xi_1 + k\eta_1)}.$$

Hence, by division,

$$\frac{x}{y} = \frac{r(\eta\xi' - \xi\eta')}{s(\eta\xi_1 - \xi\eta_1)}, \quad \frac{r}{s} = \frac{-c(\xi_1 + k\eta_1)}{\xi' + k\eta'}.$$

Since we are concerned only with the ratio of x to y , we may set

$$x = r\eta'\xi - r\xi'\eta, \quad y = s\eta_1\xi - s\xi_1\eta.$$

Since the location of A and B with reference to A' and B' is at our choice, as also the constant c (and hence r and s), the values of $r\eta'$ and $-r\xi'$ are at our choice, likewise $s\eta_1$ and $-s\xi_1$. There is, however, the restriction $A \neq B$, whence $\eta'\xi_1 \neq \eta_1\xi'$. Thus a change of reference points and constant multiplier c gives rise to a linear transformation

$$T: \quad x = \alpha\xi + \beta\eta, \quad y = \gamma\xi + \delta\eta, \quad \Delta = \begin{vmatrix} \alpha & \beta \\ \gamma & \delta \end{vmatrix} \neq 0,$$

of coördinates, and conversely every such transformation can be interpreted as the formulas for a change of reference points and constant multiplier.

4. Examples of Invariants. The linear functions

$$l = ax + by, \quad L = Ax + By$$

become, under the preceding linear transformation T ,

$$a(\alpha\xi + \beta\eta) + b(\gamma\xi + \delta\eta) = a'\xi + b'\eta, \quad A'\xi + B'\eta,$$

where

$$a' = a\alpha + b\gamma, \quad b' = a\beta + b\delta, \quad A' = A\alpha + B\gamma, \quad B' = A\beta + B\delta.$$

Hence the resultant of the new linear functions is

$$\begin{vmatrix} a' & b' \\ A' & B' \end{vmatrix} = \begin{vmatrix} a & b \\ A & B \end{vmatrix} \cdot \begin{vmatrix} \alpha & \beta \\ \gamma & \delta \end{vmatrix} = \Delta \begin{vmatrix} a & b \\ A & B \end{vmatrix},$$

and equals the product of the resultant $r = aB - bA$ of the given functions by Δ . Since this is true for every linear homogeneous transformation of determinant Δ , we call r an *invariant* of l and L of *index unity*, the factor which multiplies r being here the first power of Δ .

Employing homogeneous coördinates for points on a line, we see that l vanishes at the single point $(b, -a)$ and that $L=0$ only at $(B, -A)$. These two points are identical if and only if $b : a = B : A$, i.e., if $r=0$. The vanishing of the invariant r thus indicates a geometrical property which is independent of the choice of the points of reference used in defining coördinates on the line; moreover, the property is not changed by a projection of this line from an outside point and a section by a new line. Thus $r=0$ gives a projective property.

Among the present transformations T are the very special transformations given at the beginning of § 1. Of the four functions there called invariants of l and L under those special transformations, r alone is invariant under all of the present transformations. Henceforth the term invariant will be used only when the property of invariance holds for all linear homogeneous transformations of the variables considered.

Our next example deals with the function

$$f = ax^2 + 2bxy + cy^2.$$

The transformation T (end of § 3) replaces f by

$$F = A\xi^2 + 2B\xi\eta + C\eta^2,$$

in which

$$A = a\alpha^2 + 2b\alpha\gamma + c\gamma^2,$$

$$B = a\alpha\beta + b(\alpha\delta + \beta\gamma) + c\gamma\delta,$$

$$C = a\beta^2 + 2b\beta\delta + c\delta^2.$$

If the discriminant $d = ac - b^2$ of f is zero, f is the square of a linear function of x and y , so that the transformed function

F is the square of a linear function of ξ and η , whence the discriminant $D=AC-B^2$ of F is zero. In other words, $d=0$ implies $D=0$. By inspection, the coefficient of $-b^2$, the highest power of b , in the expansion of D is

$$(\alpha\delta + \beta\gamma)^2 - 4\alpha\gamma\beta\delta = (\alpha\delta - \beta\gamma)^2 = \Delta^2.$$

Thus $D - \Delta^2 d$ is a linear function $bq + r$ of b , where q and r are functions of $a, c, \alpha, \beta, \gamma, \delta$. Let a and c remain arbitrary, but give to b the values \sqrt{ac} and $-\sqrt{ac}$ in turn. Since $d=0$ and $D=0$, we have

$$0 = \sqrt{ac}q + r, \quad 0 = -\sqrt{ac}q + r,$$

whence $r=q=0$, $D=\Delta^2 d$. Thus d is an invariant of f of index 2. Another proof is as follows:

$$\begin{aligned} \Delta^2 d &= \begin{vmatrix} \alpha & \gamma \\ \beta & \delta \end{vmatrix} \cdot \begin{vmatrix} a & b \\ b & c \end{vmatrix} \begin{vmatrix} \alpha & \beta \\ \gamma & \delta \end{vmatrix} \\ &= \begin{vmatrix} \alpha & \gamma \\ \beta & \delta \end{vmatrix} \cdot \begin{vmatrix} a\alpha + b\gamma & a\beta + b\delta \\ b\alpha + c\gamma & b\beta + c\delta \end{vmatrix} = \begin{vmatrix} A & B \\ B & C \end{vmatrix} = D. \end{aligned}$$

We just noted that $d=0$ expresses an algebraic property of f , that of being a perfect square. To give the related geometrical property, employ homogeneous coördinates for the points in a line. Then $f=0$ represents two points which coincide if and only if $d=0$. Thus the vanishing of the invariant d of f expresses a projective property of the points represented by $f=0$.

5. Examples of Covariants. The *Hessian* (named after Otto Hesse) of a function $f(x, y)$ of two variables is defined to be

$$h = \begin{vmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{vmatrix}.$$

Let f become $F(\xi, \eta)$ under the transformation

$$T: \quad x = \alpha\xi + \beta\eta, \quad y = \gamma\xi + \delta\eta, \quad \Delta = \begin{vmatrix} \alpha & \beta \\ \gamma & \delta \end{vmatrix} \neq 0.$$

Multiplying determinants according to the rule in § 1, we have

$$h\Delta = \begin{vmatrix} \alpha \frac{\partial^2 f}{\partial x^2} + \gamma \frac{\partial^2 f}{\partial x \partial y}, & \beta \frac{\partial^2 f}{\partial x^2} + \delta \frac{\partial^2 f}{\partial x \partial y} \\ \alpha \frac{\partial^2 f}{\partial x \partial y} + \gamma \frac{\partial^2 f}{\partial y^2}, & \beta \frac{\partial^2 f}{\partial x \partial y} + \delta \frac{\partial^2 f}{\partial y^2} \end{vmatrix} = \begin{vmatrix} \frac{\partial v}{\partial x} & \frac{\partial w}{\partial x} \\ \frac{\partial v}{\partial y} & \frac{\partial w}{\partial y} \end{vmatrix},$$

where, by T ,

$$(1) \quad v = \alpha \frac{\partial f}{\partial x} + \gamma \frac{\partial f}{\partial y} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial \xi} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial \xi} = \frac{\partial F}{\partial \xi}, \quad w = \beta \frac{\partial f}{\partial x} + \delta \frac{\partial f}{\partial y} = \frac{\partial F}{\partial \eta}.$$

By the same rule of multiplication of determinants,

$$\begin{vmatrix} \alpha & \gamma \\ \beta & \delta \end{vmatrix} \cdot h\Delta = \begin{vmatrix} \left(\alpha \frac{\partial}{\partial x} + \gamma \frac{\partial}{\partial y}\right) \frac{\partial F}{\partial \xi}, & \left(\alpha \frac{\partial}{\partial x} + \gamma \frac{\partial}{\partial y}\right) \frac{\partial F}{\partial \eta} \\ \left(\beta \frac{\partial}{\partial x} + \delta \frac{\partial}{\partial y}\right) \frac{\partial F}{\partial \xi}, & \left(\beta \frac{\partial}{\partial x} + \delta \frac{\partial}{\partial y}\right) \frac{\partial F}{\partial \eta} \end{vmatrix}.$$

Applying (1) with f replaced by $\partial F/\partial \xi$ for the first column and by $\partial F/\partial \eta$ for the second column, we get

$$\Delta^2 h = \begin{vmatrix} \frac{\partial^2 F}{\partial \xi^2} & \frac{\partial^2 F}{\partial \xi \partial \eta} \\ \frac{\partial^2 F}{\partial \eta \partial \xi} & \frac{\partial^2 F}{\partial \eta^2} \end{vmatrix}.$$

Hence the Hessian of the transformed function F equals the product of the Hessian h of the given function f by the square of the determinant of the linear transformation. Consequently, h is called a *covariant of index 2* of f .

For an interpretation of $h \equiv 0$, see Exs. 4, 5, § 7. In case f is the quadratic function f of § 4, h reduces to $4d$, where d is the invariant $ac - b^2$.

The *functional determinant* or *Jacobian* (named after C. G. J. Jacobi) of two functions $f(x, y)$ and $g(x, y)$ is defined to be

$$\frac{\partial(f, g)}{\partial(x, y)} = \begin{vmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{vmatrix}.$$

Let the above transformation T replace f by $F(\xi, \eta)$, and g by $G(\xi, \eta)$. By means of (1), we get

$$\begin{aligned} \frac{\partial(F, G)}{\partial(\xi, \eta)} &= \begin{vmatrix} \alpha \frac{\partial f}{\partial x} + \gamma \frac{\partial f}{\partial y} & \beta \frac{\partial f}{\partial x} + \delta \frac{\partial f}{\partial y} \\ \alpha \frac{\partial g}{\partial x} + \gamma \frac{\partial g}{\partial y} & \beta \frac{\partial g}{\partial x} + \delta \frac{\partial g}{\partial y} \end{vmatrix} \\ &= \begin{vmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{vmatrix} \cdot \begin{vmatrix} \alpha & \beta \\ \gamma & \delta \end{vmatrix} = \Delta \frac{\partial(f, g)}{\partial(x, y)}. \end{aligned}$$

Hence the Jacobian of f and g is a covariant of index unity of f and g . For example, the Jacobian of the linear functions l and L in § 4 is their resultant r ; they are proportional if and only if the invariant r is zero. The last fact is an illustration of the

THEOREM. *Two functions f and g of x and y are dependent if and only if their Jacobian is identically zero.*

First, if $g = \phi(f)$, the Jacobian of f and g is

$$\begin{vmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \phi'(f) \frac{\partial f}{\partial x} & \phi'(f) \frac{\partial f}{\partial y} \end{vmatrix} \equiv 0.$$

Next, to prove the second or converse part of the theorem, let the Jacobian of f and g be identically zero. If g is a constant, it is a (constant) function of f . In the contrary case, the partial derivatives of g are not both identically zero. Let, for example, $\partial g / \partial x$ be not zero identically. Consider g and y as new variables in place of x and y . Thus $f = F(g, y)$ and the Jacobian is

$$\begin{vmatrix} \frac{\partial F}{\partial g} \frac{\partial g}{\partial x} & \frac{\partial F}{\partial g} \frac{\partial g}{\partial y} + \frac{\partial F}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{vmatrix} = \begin{vmatrix} 0 & \frac{\partial F}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{vmatrix}.$$

Hence $\partial F/\partial y$ is identically zero, so that F does not involve y explicitly and is a function of g only.

6. Forms and their Classification. A function like ax^3+bx^2y , every term of which is of the same total degree in x and y , is called *homogeneous* in x and y .

A homogeneous rational integral function of x, y, \dots is called a *form* (or *quantic*) in x, y, \dots . According as the number of variables is 1, 2, 3, \dots , or q , the form is called *unary, binary, ternary, \dots*, or *q -ary*, respectively. According as the form is of the first, second, third, fourth, \dots , or p th *order* in the variables, it is called *linear, quadratic, cubic, quartic, \dots*, or *p -ic*, respectively.

For the present we shall deal with binary forms. It is found to be advantageous to prefix binomial coefficients to the literal coefficients of the form, as in the binary quadratic and quartic forms

$$ax^2+2bxy+cy^2, \quad a_0x^4+4a_1x^3y+6a_2x^2y^2+4a_3xy^3+a_4y^4.$$

7. Definition of Invariants and Covariants of Binary Forms. Let the general binary form f of order p ,

$$a_0x^p+p a_1x^{p-1}y+\frac{p(p-1)}{1 \cdot 2}a_2x^{p-2}y^2+\dots+a_px^py^p,$$

be replaced by

$$A_0\xi^p+p A_1\xi^{p-1}\eta+\frac{p(p-1)}{1 \cdot 2}A_2\xi^{p-2}\eta^2+\dots+A_px^py^p$$

by the transformation T (§ 5) of determinant $\Delta \neq 0$. If, for every such transformation, a polynomial $I(a_0, \dots, a_p)$ has the property that

$$I(A_0, \dots, A_p) \equiv \Delta^\lambda I(a_0, \dots, a_p),$$

identically in a_0, \dots, a_p , after the A 's have been replaced by their values in terms of the a 's, then $I(a_0, \dots, a_p)$ is called an *invariant of index λ* of the form f .

If, for every linear transformation T of determinant $\Delta \neq 0$, a polynomial K in the coefficients and variables in f is such that *

$$K(A_0, \dots, A_p; \xi, \eta) \equiv \Delta^\lambda K(a_0, \dots, a_p; x, y),$$

identically in $a_0, \dots, a_p, \xi, \eta$, after the A 's have been replaced by their values in terms of the a 's, and after x and y have been replaced by their values in terms of ξ and η from T , then K is called a *covariant of index* λ of f .

The definitions of invariants and covariants of several binary forms are similar.

These definitions are illustrated by the examples in §§ 4, 5. Note that f itself is a covariant of index zero of f ; also that invariants are covariants of order zero.

EXERCISES

1. The Jacobian of $f = ax^2 + 2bxy + cy^2$ and $L = rx + sy$ is

$$J = 2(as - br)x + 2(bs - cr)y.$$

If J is identically zero, $f = tL^2$, where t is a constant. How does this illustrate the last result in § 5? Next, let J be not identically zero. Let k and l be the values of x/y for which $f=0$; m that for which $L=0$ and n that for which $J=0$. Prove that the cross-ratio $(k, m, l, n) = -1$. Thus the points represented by $f=0$ are separated harmonically by those represented by $L=0, J=0$.

2. If J is the Jacobian of two binary quadratic forms f and g , the points represented by $J=0$ separate harmonically those represented by $f=0$ and also those represented by $g=0$. Thus $J=0$ represents the pair of double points of the involution defined by the pairs of points represented by $f=0$ and $g=0$.

3. If $f(x, y)$ is a binary form of order n , then (Euler)

$$x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} = nf.$$

Hint: Prove this for $f = ax^k y^{n-k}$ and for $f = f_1 + f_2$.

4. The Hessian of $(ax + by)^n$ is identically zero.

Hint: It is sufficient to prove this for x^n . Why?

* The factor can be shown to be a power of Δ if it is merely assumed to be a function only of the coefficients of the transformation.

5. Conversely, if the Hessian of a binary form $f(x, y)$ of order n is identically zero, f is the n th power of a linear function.

Hints: The Hessian of f is the Jacobian of $\partial f/\partial x, \partial f/\partial y$. By the last result in § 5, these derivatives are dependent:

$$b \frac{\partial f}{\partial x} - a \frac{\partial f}{\partial y} \equiv 0,$$

where a and b are constants. Solving this with Euler's relation in Ex. 3, we get

$$\begin{aligned} (ax+by) \frac{\partial f}{\partial x} &= naf, & (ax+by) \frac{\partial f}{\partial y} &= nbf, \\ \frac{\partial \log f}{\partial x} &= \frac{na}{ax+by}, & \frac{\partial \log f}{\partial y} &= \frac{nb}{ax+by}. \end{aligned}$$

Integrating,

$$\log f - n \log(ax+by) = \phi(y) = \psi(x).$$

Hence $\phi = \psi = \text{constant}$, say $\log c$. Thus $f = c(ax+by)^n$.

8. Invariants of Covariants. The binary cubic form

$$(1) \quad f(x, y) = ax^3 + 3bx^2y + 3cxy^2 + dy^3$$

has as a covariant of index 2 its Hessian $36h$:

$$(2) \quad h = rx^2 + 2sxy + ty^2, \quad r = ac - b^2, \quad 2s = ad - bc, \quad t = bd - c^2.$$

Under any linear transformation of determinant Δ , let f become

$$(3) \quad F = A\xi^3 + 3B\xi^2\eta + 3C\xi\eta^2 + D\eta^3.$$

Let H denote the Hessian of F . Then the covariance of h gives

$$(4) \quad H = R\xi^2 + 2S\xi\eta + T\eta^2 = \Delta^2 h, \quad R = AC - B^2, \quad \dots$$

Hence $\Delta^2 r, 2\Delta^2 s, \Delta^2 t$ are the coefficients of a binary quadratic form which our transformation replaces by one with the coefficients $R, 2S, T$. Since the discriminant of a binary quadratic form is an invariant of index 2,

$$RT - S^2 = \Delta^2 \{ \Delta^2 r \cdot \Delta^2 t - (\Delta^2 s)^2 \} = \Delta^6 (rt - s^2).$$

Hence $rt - s^2$ is an invariant of index 6 of f .

A like method of proof shows that *any invariant of a covariant of a system of forms is an invariant of the forms.*

As an example in the use of the concepts invariants and covariants in demonstrations, we shall prove that the invariant *

$$(5) \quad -4(rt - s^2) = (ad - bc)^2 - 4(ac - b^2)(bd - c^2)$$

is zero if and only if $f(x/y, 1) = 0$ has a multiple root, i.e., if $f(x, y)$ is divisible by the square of a linear function of x and y . If the latter be the case, we can transform f into a form (3) with the factor ξ^2 ; then $C = D = 0$ and the function (5) written in capitals is zero, so that the invariant (5) itself is zero. Conversely, if (5) is zero, $f = 0$ has a multiple root. For, the Hessian (2) is then a perfect square and hence can be transformed into ξ^2 , which, by the covariance of h , differs only by a constant factor from the Hessian $R\xi^2$ of the transformed cubic (3). Thus $S = T = 0$. If $D = 0$, then $C = 0$ (by $T = 0$) and (3) has the factor ξ^2 , as affirmed. If $D \neq 0$,

$$B = \frac{C^2}{D}, \quad A = \frac{C^3}{D^2}, \quad F \equiv D \left(\eta + \frac{C}{D} \xi \right)^3.$$

9. Canonical Form of a Binary Cubic; Solution of Cubic Equations. We shall prove that *every binary cubic form whose discriminant is not zero † can be transformed into $X^3 + Y^3$.*

For, if the discriminant (5) of the binary cubic (1) is not zero, the Hessian (2) is the product of two linear functions which are linearly independent. Hence the cubic form f can be transformed into a form F whose Hessian (4) reduces to $2S\xi\eta$, and hence has $R = 0, T = 0, S \neq 0$. If $C = 0$, then $B = 0$ (by $R = 0$) and $F = A\xi^3 + D\eta^3, AD \neq 0$ (by $S \neq 0$). Taking

$$\xi = A^{-\frac{1}{3}}X, \quad \eta = D^{-\frac{1}{3}}Y,$$

we get $F = X^3 + Y^3$, as desired. The remaining case $C \neq 0$ is readily excluded; for, then $B \neq 0$ (by $T = 0$) and

$$A = \frac{B^2}{C}, \quad D = \frac{C^2}{B}, \quad AD = BC, \quad S = 0.$$

* It is often called the discriminant of f . It equals $-a^4P/27$, where P is the product of the squares of the differences of the roots of $f(x/y, 1) = 0$. Other writers call a^4P the discriminant of f .

† If zero, f has a square factor and hence can be transformed into X^2Y or X^3 .

To solve a cubic equation without a multiple root, we have merely to introduce as new variables the factors ξ and η of the Hessian. For, then, the new cubic is $A\xi^3 + D\eta^3 = 0$.

To treat an example, consider $f = x^3 + 6x^2y + 12xy^2 + dy^3 = 0$. The Hessian is $(d-8)(xy + 2y^2)$. Hence we take $\xi = x + 2y$ and $\eta = y$ as new variables. We get $f = \xi^3 + (d-8)\eta^3$. If $d = 9$, we have $\xi^3 + \eta^3 = 0$, whence $\xi/\eta = -1, -\omega$ or $-\omega^2$, where ω is an imaginary cube root of unity. But $x/y + 2 = \xi/\eta$. Hence $x/y = -3, -\omega - 2, -\omega^2 - 2$.

10. Covariants of Covariants. *Any covariant of a system of covariants of a system of forms is a covariant of the forms.*

The proof of this theorem is similar to that used in the following illustrations. We first show that the Jacobian of a binary cubic form f and its Hessian h is a covariant of index 3 of f . We have

$$\frac{\partial(F, H)}{\partial(\xi, \eta)} = \Delta \frac{\partial(f, \Delta^2 h)}{\partial(x, y)} = \Delta^3 \frac{\partial(f, h)}{\partial(x, y)}.$$

As the second illustration we consider the forms f, L in Ex. 1, § 7. Their Jacobian is the double of the covariant $K = vx + wy$ of index unity, where

$$v = as - br, \quad w = bs - cr.$$

Thus K and L are covariants of the system of forms f, L . These two linear covariants have as an invariant their resultant

$$I = \begin{vmatrix} v & w \\ r & s \end{vmatrix} = as^2 - 2brs + cr^2.$$

Under a linear transformation of determinant Δ , let f become $A\xi^2 + \dots$, and L become $R\xi + S\eta$. By the covariance of K ,

$$V\xi + W\eta = \Delta(vx + wy), \quad V = AS - BR, \quad W = BS - CR.$$

Thus our transformation replaces the linear form having the coefficients Δv and Δw by one having the coefficients V and W . The resultant

$$E = \begin{vmatrix} \Delta v & \Delta w \\ r & s \end{vmatrix}$$

of this linear form and L is an invariant of index unity. Hence

$$\begin{vmatrix} V & W \\ R & S \end{vmatrix} = \Delta E, \quad \begin{vmatrix} V & W \\ R & S \end{vmatrix} = \Delta^2 \begin{vmatrix} v & w \\ r & s \end{vmatrix},$$

so that $I = vs - wr$ is an invariant of index 2 of f and L .

From the earlier expression for I , we see that it is the resultant of f and L . We have therefore illustrated also the theorem that the resultant of any two binary forms is an invariant of those forms.

11. Intermediate Invariants and Covariants. From the invariant $ac - b^2$ of the binary quadratic form

$$f = ax^2 + 2bxy + cy^2$$

we may derive an invariant of the system of forms f and f' , where

$$f' = a'x^2 + 2b'xy + c'y^2.$$

Let any linear transformation replace f and f' by

$$F = A\xi^2 + 2B\xi\eta + C\eta^2, \quad F' = A'\xi^2 + 2B'\xi\eta + C'\eta^2.$$

If t is any constant, the form $f + tf'$ is transformed into $F + tF'$. By the invariance of the discriminant of $f + tf'$,

$$(A + tA')(C + tC') - (B + tB')^2 \equiv \Delta^2 \{(a + ta')(c + tc') - (b + tb')^2\},$$

identically in t . The equality of the terms free of t states only the known fact that $ac - b^2$ is an invariant of f . Similarly the equality of the terms involving t^2 states merely that $a'c' - b'^2$ is an invariant of f' . But from the terms multiplied by t , we see that

$$(1) \quad ac' + a'c - 2bb'$$

is an invariant of index 2 of the system of forms f, f' . It is said to be the invariant intermediate between their discriminants. It was discovered by Boole in 1841.

The method is a general one. Let K be any covariant of a form $f(x, y, \dots)$. Let a, b, \dots be the coefficients of f . Let $f'(x, y, \dots)$ be a form of the same order with the coefficients a', b', \dots . If in K we replace a by $a + ta'$, b by $b + tb'$, \dots , and expand in powers of t , we obtain as the

coefficient of any power t^r of t a covariant of the system f, f' . By Taylor's theorem, this covariant is

$$(2) \quad \frac{1}{r!} \left(a' \frac{\partial}{\partial a} + b' \frac{\partial}{\partial b} + \dots \right)^r K,$$

in which the symbolic r th power of $\partial/\partial a$ is to be replaced by $\partial^r/\partial a^r$, etc.

EXERCISES

- For $r=1$, $K=ac-b^2$, (2) becomes (1).
- Taking as K the Hessian (2) of cubic (1) in § 8, obtain the covariant

$$(ac' + a'c - 2bb')x^2 + (ad' + a'd - bc' - b'c)xy + (bd' + b'd - 2cc')y^2$$

of index 2 of a pair of binary cubic forms.

- If (1) is zero, the pair of points given by $f=0$ is harmonic with the pair given by $f'=0$.

12. Homogeneous Coördinates of Points in a Plane. Let

$$L_i: \quad a_i x + b_i y + c_i = 0 \quad (i=1, 2, 3)$$

be any three linear equations in x, y , such that

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \neq 0.$$

Interpret x and y as the Cartesian coördinates of a point referred to rectangular axes. Then the equations represent three straight lines L_i forming a triangle. Choose the sign before the radical in

$$p_i = \frac{a_i x + b_i y + c_i}{\pm \sqrt{a_i^2 + b_i^2}}$$

so that p_i is positive for a point (x, y) inside the triangle and hence is the length of the perpendicular from that point to L_i . The homogeneous (or trilinear) coördinates of a point (x, y) are three numbers x_1, x_2, x_3 such that

$$\rho x_1 = k_1 p_1, \quad \rho x_2 = k_2 p_2, \quad \rho x_3 = k_3 p_3,$$

where k_1, k_2, k_3 are constants, the same for all points. In view of the undetermined common factor ρ , only the ratios of x_1, x_2, x_3 are defined.

For example, let the triangle be an equilateral one with sides of length 2, base on the x -axis and vertex on the y -axis. The equations of the sides L_1, L_2, L_3 are, respectively,

$$\frac{y}{\sqrt{3}} + x = 1, \quad \frac{y}{\sqrt{3}} - x = 1, \quad y = 0.$$

Take each $k_i = 1$. Then

$$\rho x_1 = \frac{y + \sqrt{3}(x-1)}{-2}, \quad \rho x_2 = \frac{y - \sqrt{3}(x+1)}{-2}, \quad \rho x_3 = y.$$

The curve $x_1 x_2 = x_3^2$ is evidently tangent to L_1 (i.e., $x_1 = 0$) at $Q = (010)$, and tangent to L_2 at $P = (100)$. Substituting for the x_i their values, we see that the Cartesian equation of the curve is

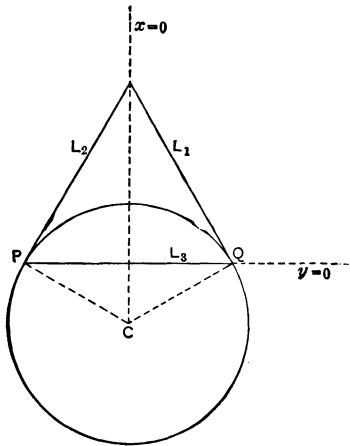


Fig. 3.

$$\frac{1}{4} \{ (y - \sqrt{3})^2 - 3x^2 \} = y^2 \text{ or } x^2 + \left(y + \frac{1}{\sqrt{3}} \right)^2 = \frac{4}{3}.$$

Hence it is a circle with radius CP and center at the intersection C of the normal to L_2 at P with the normal to L_1 at Q .

Changing the notation for the coefficients of $k_i \rho_i$, call them a_i, b_i, c_i . Then we have

$$(H) \quad \rho x_i = a_i x + b_i y + c_i, \quad \Delta \neq 0 \quad (i = 1, 2, 3).$$

Multiply the i th equation by the cofactor A_i of a_i in the determinant Δ and sum for $i=1, 2, 3$. Next use as multiplier the cofactor B_i of b_i ; finally, the cofactor C_i of c_i . We get

$$\Delta x = \rho \Sigma A_i x_i, \quad \Delta y = \rho \Sigma B_i x_i, \quad \Delta = \rho \Sigma C_i x_i.$$

Hence x and y are rational functions of x_1, x_2, x_3 :

$$(C) \quad x = \frac{A_1 x_1 + A_2 x_2 + A_3 x_3}{C_1 x_1 + C_2 x_2 + C_3 x_3}, \quad y = \frac{B_1 x_1 + B_2 x_2 + B_3 x_3}{C_1 x_1 + C_2 x_2 + C_3 x_3}.$$

Any equation $f(x, y) = 0$ in Cartesian coördinates becomes, by use of (C), a homogeneous equation $\phi(x_1, x_2, x_3) = 0$ in homogeneous coördinates. The reverse process is effected by use of (H). In particular, since any straight line is represented by an equation of the first degree in x and y , it is also represented by a homogeneous equation of the first degree in x_1, x_2, x_3 . For example, the sides of the triangle of reference are $x_1 = 0, x_2 = 0, x_3 = 0$. Conversely, any homogeneous equation of the first degree in x_1, x_2, x_3 represents a straight line.

The degree of ϕ is always that of f .

Take the y -axis as L_1 , the x -axis as L_2 , and let L_3 recede to infinity by making a_3 and b_3 approach zero. Then (H) and (C) become

$$\rho x_1 = x, \quad \rho x_2 = y, \quad \rho x_3 = 1; \quad x = \frac{x_1}{x_3}, \quad y = \frac{x_2}{x_3}.$$

We are thus led to a very special, but much used, method of passing from homogeneous to Cartesian coördinates and conversely.

For a new triangle of reference, let the homogeneous coördinates of (x, y) be y_1, y_2, y_3 . Then, as in (H),

$$\rho y_i = a'_i x + b'_i y + c'_i, \quad (i=1, 2, 3).$$

Inserting the values of x and y from (C), we get relations like

$$t: \quad \tau y_i = e_i x_1 + f_i x_2 + g_i x_3 \quad (i=1, 2, 3).$$

Hence a change of triangle of reference and constants k_1, k_2, k_3 gives rise to a linear homogeneous transformation t of coördinates. The determinant of the coefficients in t is not

zero, since $y_1=0$, $y_2=0$, $y_3=0$ represent the sides of the new triangle. Conversely, any such transformation t may be interpreted as a change of triangle of reference and constants k_i .

Instead of regarding t as a set of relations between the coördinates of the same point referred to two triangles of reference, we may regard it as defining a correspondence between the points (x_1, x_2, x_3) and (y_1, y_2, y_3) of two different planes, each referred to any chosen triangle of reference in its plane. This correspondence is projective; for, it can be effected by a series of projections and sections, each projection being that of the points of a plane from a point outside of the plane and each section being the cutting of such a bundle of projecting lines by a new plane. Proof will not be given here, nor is the theorem assumed in what follows. It is stated here to show that if I is any invariant of a ternary form f under all linear transformations t , then $I=0$ gives a projective property of the curve $f=0$. It is true conversely that any projective transformation between two planes can be effected by a linear homogeneous transformation on the homogeneous coördinates. Thus for three variables, just as for two (§§ 2, 3), the investigation of the invariants of a form under all linear homogeneous transformations is of especial importance.

13. Properties of the Hessian. Let $f(x_1, \dots, x_n)$ be a form in the independent variables x_1, \dots, x_n . The Hessian h of f is a determinant of order n in which the elements of the i th row are

$$\frac{\partial^2 f}{\partial x_i \partial x_1}, \quad \frac{\partial^2 f}{\partial x_i \partial x_2}, \quad \dots, \quad \frac{\partial^2 f}{\partial x_i \partial x_n}.$$

Let f become $\phi(y_1, \dots, y_n)$ under the transformation

$$T: \quad x_i = c_{i1}y_1 + c_{i2}y_2 + \dots + c_{in}y_n \quad (i = 1, \dots, n),$$

of determinant $\Delta = |c_{ij}|$. The product $h\Delta$ is a determinant of order n in which the element in the i th row and j th column is the sum of the products of the above elements of the i th

row of h by the corresponding elements of the j th column of Δ , and hence is

$$\begin{aligned} & \frac{\partial^2 f}{\partial x_i \partial x_1} c_{1j} + \frac{\partial^2 f}{\partial x_i \partial x_2} c_{2j} + \dots + \frac{\partial^2 f}{\partial x_i \partial x_n} c_{nj} \\ &= \frac{\partial}{\partial x_i} \left(\frac{\partial f}{\partial x_1} \frac{\partial x_1}{\partial y_j} + \frac{\partial f}{\partial x_2} \frac{\partial x_2}{\partial y_j} + \dots + \frac{\partial f}{\partial x_n} \frac{\partial x_n}{\partial y_j} \right) = \frac{\partial}{\partial x_i} \frac{\partial \phi}{\partial y_j}. \end{aligned}$$

Let Δ' be the determinant obtained from Δ by interchanging its rows and columns. In the product $\Delta' \cdot h \Delta$, the element in the r th row and j th column is therefore

$$c_{1r} \frac{\partial}{\partial x_1} \frac{\partial \phi}{\partial y_j} + \dots + c_{nr} \frac{\partial}{\partial x_n} \frac{\partial \phi}{\partial y_j} = \frac{\partial}{\partial y_r} \frac{\partial \phi}{\partial y_j},$$

since c_{ir} is the partial derivative of x_i with respect to y_r . Hence

$$\Delta^2 h = \begin{vmatrix} \frac{\partial^2 \phi}{\partial y_r \partial y_j} \end{vmatrix}_{r,j=1,\dots,n} = \text{Hessian of } \phi.$$

Thus h is a covariant of index 2 of f .

To make an application to conics, let f be a ternary quadratic form. Then h is an invariant called the discriminant of f . Let (a_1, a_2, a_3) be a point on $f=0$ (for example, one with $x_3=0$). For $c_{41}=a_1$ and c_{42}, c_{43} chosen so that $\Delta \neq 0$, transformation T makes $(x)=(a)$ correspond to $(y)=(100)$. Hence we may assume that (100) is a point on $f=0$, so that the term in x_1^2 is lacking. Consider the terms $x_1 l$ with the factor x_1 . If $l \equiv 0$, f involves only x_2 and x_3 and hence is a product of two linear functions, while $h \equiv 0$. In the contrary case, we may introduce l as a new variable in place of x_2 . This amounts to setting $l = x_2$,

$$f = x_1 x_2 + a x_2^2 + b x_2 x_3 + c x_3^2.$$

Replacing x_1 by $x_1 - a x_2 - b x_3$, we get $x_1 x_2 - k x_3^2$, whose Hessian is $2k$. Hence $f=0$ represents two (distinct or coincident) straight lines if and only if the Hessian (discriminant) of f is zero.

Moreover, if the discriminant is not zero, then $k \neq 0$ and we may replace $\sqrt{k} x_3$ by x_3 and get $x_1 x_2 - x_3^2$. Hence all conics, which do not degenerate into straight lines, are equivalent

under projective transformation. If the triangle of reference is equilateral and the coördinates are proportional to the perpendiculars upon its sides, $x_1x_2 - x_3^2 = 0$ is a circle (§ 12).

On the contrary, if we employ only translations and rotations, as in plane analytics, there are infinitely many non-equivalent conics; we saw in § 1 that there are then two invariants besides the discriminant.

Next, to make an application to plane cubic curves, let $f(x_1, x_2, x_3)$ be a ternary cubic form. A triangle of reference can be chosen so that $P = (001)$ is a point of the curve $f = 0$. Then the term in x_3^3 is lacking, so that

$$f = x_3^2 f_1 + x_3 f_2 + f_3,$$

where f_i is a homogeneous function of x_1 and x_2 of degree i . We assume that P is not a singular point, so that the partial derivatives of f with respect to x_1 , x_2 , and x_3 are not all zero at P . Hence f_1 is not identically zero and can be introduced as a new variable in place of x_1 . Thus, after a preliminary linear transformation, we have

$$x_3^2 x_1 + x_3(ax_1^2 + bx_1x_2 + cx_2^2) + \bar{f}_3.$$

Replace x_3 by $x_3 - \frac{1}{2}(ax_1 + bx_2)$. We get

$$F = x_3^2 x_1 + ex_3 x_2^2 + C,$$

where C is a cubic function of x_1, x_2 , whose second partial derivative with respect to x_i and x_j will be denoted by C_{ij} . The Hessian of F is

$$H = \begin{vmatrix} C_{11} & C_{12} & 2x_3 \\ C_{12} & C_{22} + 2ex_3 & 2ex_2 \\ 2x_3 & 2ex_2 & 2x_1 \end{vmatrix}.$$

If the transformation which replaced f by F is of determinant Δ , it replaces the Hessian h of f by $H = \Delta^2 h$. Thus $H = 0$ represents the same curve as $h = 0$, but referred to the same new triangle of reference as $F = 0$. We may therefore speak of a definite Hessian curve of the given curve $f = 0$. In investigating the properties of these curves we may therefore

refer them to the triangle of reference for which their equations are $H=0$, $F=0$.

The coefficient of x_3^3 in H is evidently $-8e$. Thus P is on the Hessian curve if and only if $e=0$. If d is the coefficient of x_2^3 in C , $x_1=0$ meets $F=0$ at the points for which $x_2^2(ex_3+dx_2)=0$ and these points coincide (at P) if and only if $e=0$. In that case, P is called a *point of inflexion* of $F=0$ and $x_1=0$ the *inflexion tangent* at P . For a cubic curve $f=0$ without a singular point, every point of inflexion is a point of intersection of the curve with its Hessian curve and conversely.

14. Inflexion Points and Invariants of a Cubic Curve. Eliminating x_3 between $f=0$, $h=0$, we obtain a homogeneous relation in x_1, x_2 , which has therefore at least one set of solutions x'_1, x'_2 . For the latter values of x_1 and x_2 , $f=0$ and $h=0$ are cubic equations in x_3 with at least one common root, x'_3 . Hence $f=0$ has at least one inflexion point (x'_1, x'_2, x'_3) . After a suitable linear transformation, this point becomes (001) . As in § 13, we can transform f into F , in which e is now zero. If $d=0$, then $F=x_1Q$, and the derivatives

$$\frac{\partial F}{\partial x_1} = Q + x_1 \frac{\partial Q}{\partial x_1}, \quad \frac{\partial F}{\partial x_2} = x_1 \frac{\partial Q}{\partial x_2}, \quad \frac{\partial F}{\partial x_3} = x_1 \frac{\partial Q}{\partial x_3}$$

all vanish at an intersection of $x_1=0$, $Q=0$. But we assume that there is no singular point on $f=0$ and thus none on $F=0$. Hence $d \neq 0$. Replacing x_2 by $d^{-1}x_2$, we have an F with $d=1$. Adding a multiple of x_1 to x_2 , we get

$$F = x_3^2 x_1 + C, \quad C = x_2^3 + 3bx_2 x_1^2 + ax_1^3,$$

$$H = -4x_3^2 C_{22} + 2x_1 \phi, \quad \phi = \begin{vmatrix} C_{11} & C_{12} \\ C_{12} & C_{22} \end{vmatrix},$$

so that ϕ is the Hessian of C . By § 8,

$$\phi = 36(-b^2 x_1^2 + ax_1 x_2 + bx_2^2).$$

Eliminating x_3^2 between $F=0$, $H=0$, we get

$$x_1^2 \phi + 2C_{22} C = 12(x_2^4 + 6bx_2^2 x_1^2 + 4ax_2 x_1^3 - 3b^2 x_1^4) = 0.$$

If $x_1=0$, then $x_2=0$ and we obtain the known intersection

(001). For the remaining intersections, we may set $x_1=1$ and obtain from each root r of

$$(1) \quad r^4 + 6br^2 + 4ar - 3b^2 = 0$$

two intersections $(1, r, \pm x'_3)$. For, if $x'_3=0$, then $C=0$, so that (1) would have a multiple root, whence $a^2+4b^3=0$. But the three partial derivatives of F would then all vanish at $(2b, -a, 0)$ or $(1, 0, 0)$, according as $b \neq 0$ or $b=0$. Hence *there are exactly nine distinct points of inflexion*.

For each of the four roots of (1), the three points of inflexion P and $(1, r, \pm x'_3)$ are collinear, being on $x_2=rx_1$. Since we may proceed with any point of inflexion as we did with P , we see that there are $9 \cdot 4/3$ or 12 lines each joining three points of inflexion and such that four of the lines pass through any one of the nine points. The six points of inflexion not on a fixed one of these lines therefore lie by threes on two new lines; three such lines form an *inflexion triangle*. Thus there are $\frac{1}{3}12=4$ inflexion triangles.

The fact that there are four inflexion triangles, one for each root r of (1), can also be seen as follows:

$$\frac{1}{24}H + rF = (rx_1 - x_2)\{x_3^2 - rx_2^2 - (r^2 + 3b)x_1x_2 - (r^3 + 6br + 3a)x_1^2\}.$$

The last factor equals

$$x_3^2 - \frac{1}{r}\{rx_2 + \frac{1}{2}(r^2 + 3b)x_1\}^2,$$

and hence is the product of two linear functions.

Corresponding results hold for any cubic curve $f=0$ without singular points. We have shown that f can be reduced to the special form F by a linear transformation of a certain determinant Δ . Follow this by the transformation which multiplies x_3 by Δ and x_1 by Δ^{-2} , and hence has the determinant Δ^{-1} . Thus there is a transformation of determinant unity which replaces f by a form of type F , and hence replaces the Hessian h of f by the Hessian H of F . Hence there are exactly four values of r for which $\psi=h+24rf$ has a linear factor and therefore three linear factors. These r 's are the roots of a quartic (1) in which a and b are functions of the coefficients

of f . To see the nature of these functions, let $x_1 - \lambda x_2 - \mu x_3$ be a factor of ψ . After replacing x_1 by $\lambda x_2 + \mu x_3$ in ψ , we obtain a cubic function of x_2 and x_3 whose four coefficients must be zero. Eliminating λ and μ , we obtain two conditions involving r and the coefficients of f rationally and integrally. The greatest common divisor of their left members is the required quartic function of r . Unless the coefficient of r^4 is constant, a root would be infinite for certain f 's. *The inflexion triangles of a general cubic curve $f=0$ are given by $h+24rf=0$, where h is the Hessian of f and r is a root of the quartic (1) in which a and b are rational integral invariants of f .*

The explicit expressions for these invariants are very long; they are given in Salmon's *Higher Plane Curves*, §§ 221-2, and were first computed by Aronhold. For their short symbolic expressions, see § 65, Ex. 4.

EXERCISES

1. Using the above inflexion triangle $y_1 y_2 y_3 = 0$, where

$$\begin{aligned} r x_1 - x_2 &= y_1, \quad \sqrt{r} x_3 \pm (r x_2 + k x_1) = 2 y_2, \quad 2 y_3, \\ k &= (r^2 + 3b)/2, \quad r^2 + k = \frac{3}{2}(r^2 + b) \neq 0, \end{aligned}$$

as shown by use of (1), we have the transformation

$$\sqrt{r} x_3 = y_2 + y_3, \quad (r^2 + k)x_1 = r y_1 + D, \quad (r^2 + k)x_2 = -k y_1 + r D,$$

where $D = y_2 - y_3$. Using (1) to eliminate a , show that

$$\frac{9}{8}(r^2 + b)F = \frac{1}{r}(y_2^3 - y_3^3) + 3y_1 y_2 y_3 - \frac{1}{8}(r^2 + 9b)y_1^3.$$

Adding the product of the latter by 54 to its Hessian, we get the product of $y_1 y_2 y_3$ by $3^5(r^2 + b)/r^2$. Hence the nine points of inflexion are found by setting y_1, y_2, y_3 equal to zero in turn.

2. By multiplying the y 's in Ex. 1 by constants, derive

$$f = \alpha(z_1^3 + z_2^3 + z_3^3) + 6\beta z_1 z_2 z_3,$$

called the canonical form. Its Hessian is $6^3 h$, where

$$h = -\alpha\beta^2(z_1^3 + z_2^3 + z_3^3) + (\alpha^3 + 2\beta^3)z_1 z_2 z_3.$$

Thus find the nine inflexion points and show that the four inflexion triangles are

$$z_1 z_2 z_3 = 0, \quad \Sigma z_1^3 - 3l z_1 z_2 z_3 = 0 \quad (l = 1, \omega, \omega^2),$$

where ω is an imaginary cube root of unity. Their left members are constant multiples of $3h+rf$, where $r=3\beta^2, -(\alpha-\beta)^2$ are the four roots of (1), with

$$b = \beta(\alpha^3 - \beta^3), \quad 4a = \alpha^6 - 20\alpha^3\beta^3 - 8\beta^6.$$

3. The Jacobian of $f_1(x_1, \dots, x_n), \dots, f_n(x_1, \dots, x_n)$ is

$$\begin{vmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \dots & \frac{\partial f_1}{\partial x_n} \\ \dots & \dots & \dots & \dots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \dots & \frac{\partial f_n}{\partial x_n} \end{vmatrix}.$$

Show that it is a covariant of index unity of f_1, \dots, f_n .

4. Hence the resultant of three ternary linear forms is an invariant of index unity.

5. If f_1, \dots, f_n are dependent functions, the Jacobian is zero.