10. Theorems on Multiplication and Addition.—We can now establish two theorems with regard to the combination of inclusions and equalities by addition and multiplication:

(Th. I) $(a < b) < (ac < bc), \quad | \quad (a < b) < (a + c < b + c).$

Demonstration:

1 (Simpl.) $ac < c,$

(Syll.) $(ac < a)\ (a < b) < (ac < b),$

(Comp.) $(ac < b)\ (ac < c) < (ac < bc).$

2 (Simpl.) $c < b + c,$

(Syll.) $(a < b)\ (b < b + c) < (a < b + c),$

(Comp.) $(a < b + c)\ (c < b + c) < (a + c < b + c).$

This theorem may be easily extended to the case of equalities:

$$(a = b) < (ac = bc), \quad | \quad (a = b) < (a + c = b + c).$$

(Th. II) $(a < b)\ (c < d) < (ac < bd),$

$(a < b)\ (c < d) < (a + c < b + d).$

Demonstration:

1 (Syll.) $(ac < a)\ (a < b) < (ac < b),$

(Syll.) $(ac < c)\ (c < d) < (ac < d),$

(Comp.) $(ac < b)\ (ac < d) < (ac < bd).$

2 (Syll.) $(a < b)\ (b < b + d) < (a < b + d),$

(Syll.) $(c < d)\ (d < b + d) < (c < b + d),$

(Comp.) $(a < b + d)\ (c < b + d) < (a + c < b + d).$

This theorem may easily be extended to the case in which one of the two inclusions is replaced by an equality:

$$(a = b)\ (c < d) < (ac < bd),$$
$$(a = b)\ (c < d) < (a + c < b + d).$$

When both are replaced by equalities the result is an equality:

$$(a = b)\ (c = d) < (ac = bd),$$
$$(a = b)\ (c = d) < (a + c = b + d).$$

To sum up, two or more inclusions or equalities can be added or multiplied together member by member; the result will not be an equality unless all the propositions combined are equalities.